Strong limit theorems for anisotropic random walks on $\mathbb{Z}^2$

Endre Csáki[1]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary. E-mail address: csaki@renyi.hu

Miklós Csörgő [2]
School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario, Canada K1S 5B6. E-mail address: mcsorgo@math.carleton.ca

Antónia Földes[3]
Department of Mathematics, College of Staten Island, CUNY, 2800 Victory Blvd., Staten Island, New York 10314, U.S.A. E-mail address: foldes@mail.csi.cuny.edu

Pál Révész[1]
Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/107 A-1040 Vienna, Austria. E-mail address: reveszp@renyi.hu

Abstract
We study the path behaviour of the anisotropic random walk on the two-dimensional lattice $\mathbb{Z}^2$. Strong approximation of its components with independent Wiener processes are proved. We also give some asymptotic results for the local time in the periodic case.

MSC: primary 60F17, 60G50, 60J65; secondary 60F15, 60J10.

Keywords: anisotropic random walk; strong approximation; 2-dimensional Wiener process; local time; laws of the iterated logarithm.

1 Introduction and main results

We consider random walks on the square lattice $\mathbb{Z}^2$ of the plane with possibly unequal symmetric horizontal and vertical step probabilities, so that these probabilities can only depend on the value of the vertical coordinate. In particular, if such a random walk is situated at a site on the horizontal line $y = j \in \mathbb{Z}$, then at the next step it moves with probability p_j to either vertical neighbour, and with probability $1/2 - p_j$ to either horizontal neighbour. The initial motivation for studying

[1]Research supported by the Hungarian National Foundation for Scientific Research, Grant No. K 67961.
[2]Research supported by an NSERC Canada Discovery Grant at Carleton University.
[3]Research supported by a PSC CUNY Grant, No. 68030-0039.

such two-dimensional random walks on anisotropic lattice has originated from transport problems of statistical physics.

More formally, consider the random walk $\{\mathbf{C}(N) = (C_1(N), C_2(N))\,;\, N = 0, 1, 2, \ldots\}$ on $\mathbb{Z}^2$ with the transition probabilities

$$\mathbf{P}(\mathbf{C}(N+1) = (k+1, j)|\mathbf{C}(N) = (k, j)) = \mathbf{P}(\mathbf{C}(N+1) = (k-1, j)|\mathbf{C}(N) = (k, j)) = \frac{1}{2} - p_j,$$

$$\mathbf{P}(\mathbf{C}(N+1) = (k, j+1)|\mathbf{C}(N) = (k, j)) = \mathbf{P}(\mathbf{C}(N+1) = (k, j-1)|\mathbf{C}(N) = (k, j)) = p_j,$$

for $(k, j) \in \mathbb{Z}^2$, $N = 0, 1, 2, \ldots$ We assume throughout the paper that $0 < p_j \le 1/2$ and $\min_{j \in \mathbb{Z}} p_j < 1/2$. Unless otherwise stated we assume also that $\mathbf{C}(0) = (0, 0)$.

The case $p_j = 1/4$, $j = 0, \pm 1, \pm 2, \ldots$ corresponds to simple symmetric random walk on the plane. For this case we refer to Erdős and Taylor [14], Dvoretzky and Erdős [13], Révész [25]. The case $p_j = 1/2$ for some j means that the horizontal line $y = j$ is missing. If all $p_j = 1/2$, then the random walk takes place on the y axis, so it is only a one-dimensional random walk, and this case is excluded from the present investigations. The case however when $p_j = 1/2$, $j = \pm 1, \pm 2, \ldots$ but $p_0 = 1/4$ is an interesting one which is the so-called random walk on the two-dimensional comb. For this model we may refer to Weiss and Havlin [33], Bertacchi and Zucca [2], Bertacchi [1], Csáki *et al.* [8]. One of the main properties of this comb model is that the scaling of the first coordinate $C_1(N)$ is of order $N^{1/4}$, so it is a so-called sub-diffusion, and can be approximated by an iterated (time changed) Wiener process, while the second coordinate is of order $N^{1/2}$ and can be approximated by a Wiener process. Horváth [20] proved a weak convergence of $C_2(\cdot)$ when p_j are small so that scaling is smaller than $N^{1/2}$ and its limiting process is a certain time changed Wiener process.

In the present paper we investigate the case when both coordinates are of order $N^{1/2}$ and can be simultaneously approximated by independent Wiener processes. First we outline some history of this problem. Note that in the literature usually horizontal and vertical lines are changed, i.e., our horizontal lines correspond to their vertical lines and vice versa. As noted already, the treatment of anisotropic random walks is motivated by transport problems in statistical physics. For early investigations of the model we refer to Silver *et al.* [29], Seshadri *et al.* [27], Shuler [28], Westcott [34], where certain properties of this random walk were studied under various conditions. Heyde [17] proved an almost sure approximation for $C_2(\cdot)$ under the condition (1.1) below. Heyde *et al.* [19] treated the case when conditions similar to (1.1) are assumed but γ can be different for the two parts of (1.1) and obtained almost sure convergence to the so-called oscillating Brownian motion. In Heyde [18] limiting distributions were given for $\mathbf{C}(\cdot)$ under the condition (1.1) but without remainder. Den Hollander [12] proved strong approximations for $\mathbf{C}(\cdot)$ in the case when p_j are random variables with values 1/4 and 1/2. Roerdink and Shuler [26] proved some asymptotic properties, including local limit theorems, under certain conditions. For more detailed history see [12].

In the sequel we restrict ourselves to the following condition of Heyde [17]:

$$n^{-1}\sum_{j=1}^{n} p_j^{-1} = 2\gamma + o(n^{-\eta}), \qquad n^{-1}\sum_{j=1}^{n} p_{-j}^{-1} = 2\gamma + o(n^{-\eta}) \tag{1.1}$$

as $n \to \infty$ for some constants γ, $1 < \gamma < \infty$ and $1/2 < \eta < \infty$.

Under this condition we will prove a joint strong approximation result for $C_1(\cdot), C_2(\cdot)$, the coordinates of the walk $\mathbf{C}(\cdot)$ by approximating them by two independent Wiener processes (Brownian motions).

Theorem 1.1 *Under the condition* (1.1) *with* $1/2 < \eta \le 1$, *on an appropriate probability space for the random walk*

$$\{\mathbf{C}(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}$$

one can construct two independent standard Wiener processes $\{W_1(t);\, t \ge 0\}$, $\{W_2(t);\, t \ge 0\}$ *so that, as* $N \to \infty$, *we have with any* $\varepsilon > 0$

$$\left|C_1(N) - W_1\left(\frac{\gamma - 1}{\gamma}N\right)\right| + \left|C_2(N) - W_2\left(\frac{1}{\gamma}N\right)\right| = O(N^{5/8-\eta/4+\varepsilon}) \quad a.s. \tag{1.2}$$

A particular case, the so called periodic case, deserves a special attention here, namely when $p_j = p_{j+L}$ for each $j \in \mathbb{Z}$, where $L \ge 1$ is a positive integer. In this case, denoting the random walk by $\mathbf{C}^{\mathbf{P}}(N) = (C_1^P(N), C_2^P(N))$, we have a better approximation and, moreover, we give some results for the local time in Section 3.

Theorem 1.2 *On an appropriate probability space for the random walk*

$$\{\mathbf{C}^{\mathbf{P}}(N) = (C_1^P(N), C_2^P(N)); N = 0, 1, 2, \ldots\}$$

one can construct two independent standard Wiener processes $\{W_1(t);\, t \ge 0\}$, $\{W_2(t);\, t \ge 0\}$ *so that, as* $N \to \infty$, *we have with any* $\varepsilon > 0$

$$\left|C_1^P(N) - W_1\left(\frac{\gamma - 1}{\gamma}N\right)\right| + \left|C_2^P(N) - W_2\left(\frac{1}{\gamma}N\right)\right| = O(N^{1/4+\varepsilon}) \quad a.s., \tag{1.3}$$

where

$$\gamma = \frac{\sum_{j=0}^{L-1} p_j^{-1}}{2L}.$$

We mention a particular periodic case, the so-called uniform case, when $p_j = 1/4$ if $|j| \equiv 0(\mathrm{mod} L)$ and $p_j = 1/2$ otherwise. Then Theorem 1.2 is true with $\gamma = (L+1)/L$.

The following corollaries are consequences of Lemma D and Theorem 1.1. Define the continuous time process $\mathbf{C}(u)$, $u \ge 0$ by linear interpolation of $\mathbf{C}(N)$. The space $C([0,1], \mathbb{R}^2)$ is the set of continuous functions defined on $[0,1]$ with values in $\mathbb{R}^2$. Recall the definition of the two dimensional Strassen class of absolutely continuous functions:

$$\mathcal{S}^{(2)} = \{(f(x), g(x)),\ 0 \le x \le 1 :\ f(0) = g(0) = 0,\ \int_0^1 (\dot{f}^2(x) + \dot{g}^2(x))\, dx \le 1\}. \tag{1.4}$$

Corollary 1.1 *Under the conditions of* Theorem 1.1 *for the random walk* $\mathbf{C}(\cdot)$ *we have*

- (i) *the sequence of random vector-valued functions*

$$\left(\sqrt{\frac{\gamma}{\gamma-1}}\frac{C_1(xN)}{(2N\log\log N)^{1/2}}, \sqrt{\gamma}\frac{C_2(xN)}{(2N\log\log N)^{1/2}},\ 0 \le x \le 1\right)_{N\ge 3}$$

  is almost surely relatively compact in the space $C([0,1],\mathbb{R}^2)$ *and its limit points is the set of functions* $\mathcal{S}^{(2)}$.

- (ii) *In particular, the vector sequence*

$$\left(\frac{C_1(N)}{(2N\log\log N)^{1/2}}, \frac{C_2(N)}{(2N\log\log N)^{1/2}}\right)_{N\ge 3}$$

  is almost surely relatively compact in the rectangle

$$\left[-\frac{\sqrt{\gamma-1}}{\sqrt{\gamma}}, \frac{\sqrt{\gamma-1}}{\sqrt{\gamma}}\right] \times \left[-\frac{1}{\sqrt{\gamma}}, \frac{1}{\sqrt{\gamma}}\right]$$

  and the set of its limit points is the ellipse

$$\left\{(x,y) : \frac{\gamma}{\gamma-1}x^2 + \gamma y^2 \le 1\right\}. \tag{1.5}$$

- (iii) *Moreover,*

$$\limsup_{N\to\infty} \frac{C_1(N)}{\sqrt{2N\log\log N}} = \frac{\sqrt{\gamma-1}}{\sqrt{\gamma}} \quad \text{and} \quad \limsup_{N\to\infty} \frac{C_2(N)}{\sqrt{2N\log\log N}} = \frac{1}{\sqrt{\gamma}} \quad a.s.$$

- (iv)

$$\liminf_{N\to\infty} \left(\frac{\log\log N}{N}\right)^{1/2} \max_{1\le k\le N} |C_1(k)| = \frac{\pi\sqrt{\gamma-1}}{\sqrt{8\gamma}} \quad a.s.$$

$$\liminf_{N\to\infty} \left(\frac{\log\log N}{N}\right)^{1/2} \max_{1\le k\le N} |C_2(k)| = \frac{\pi}{\sqrt{8\gamma}} \quad a.s.$$

Let us consider $D([0,\infty),\mathbb{R}^2)$, the space of $\mathbb{R}^2$ valued *càdlàg* functions on $[0,\infty)$. For $f(t) = (f_1(t), f_2(t))$ and $g(t) = (g_1(t), g_2(t))$ in this function space, define for all fixed $T > 0$

$$\Delta = \Delta_T(f,g) := \sup_{0\le t\le T} \|(f_1(t) - g_1(t)), (f_2(t) - g_2(t))\|, \tag{1.6}$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^2$, usually the $\|\cdot\|_p$ norm with $p = 1$ or 2 in our case. Define also the measurable space $(D([0,\infty),\mathbb{R}^2), \mathcal{D})$, where $\mathcal{D}$ is the σ-field generated by the collection of all Δ-open balls for all $T > 0$ of the function space $D([0,\infty),\mathbb{R}^2)$.

As a consequence of Theorem 1.1, we conclude a weak convergence result in terms of the following functional convergence in distribution statement.

Corollary 1.2 *Under the conditions of* Theorem 1.1, *as $N \to \infty$, we have*

$$h\left(N^{-1/2}\mathbf{C}([Nt])\right) = h\left(\frac{C_1([Nt])}{N^{1/2}}, \frac{C_2([Nt])}{N^{1/2}}\right) \xrightarrow{d} h\left(W_1\left(\frac{\gamma-1}{\gamma}t\right), W_2\left(\frac{1}{\gamma}t\right)\right)$$

for all $h : D([0,\infty),\mathbb{R}^2) \to \mathbb{R}^2$ that are $(D([0,\infty),\mathbb{R}^2), \mathcal{D})$ measurable and Δ-continuous for all $T > 0$, or Δ-continuous for all $T > 0$, except at points forming a set of measure zero on $(D[0,\infty),\mathbb{R}^2), \mathcal{D})$ with respect to the measure generated by $\{W_1(t), W_2(t);\ 0 \le t < \infty\}$, where W_1 and W_2 are two independent standard Wiener processes, and $\xrightarrow{d}$ denotes convergence in distribution.

2 Preliminaries

First we are to redefine our random walk $\{\mathbf{C}(N);\ N = 0, 1, 2, \ldots\}$. It will be seen that the process described right below is equivalent to that given in the Introduction (cf. (2.2) below).

To begin with, on a suitable probability space consider two independent simple symmetric (one-dimensional) random walks $S_1(\cdot)$, and $S_2(\cdot)$. We may assume that on the same probability space we have a double array of independent geometric random variables $\{G_i^{(j)},\ i \ge 1, j \in \mathbb{Z}\}$ which are independent from $S_1(\cdot)$, and $S_2(\cdot)$, where $G_i^{(j)}$ has the following geometric distribution

$$\mathbf{P}(G_i^{(j)} = k) = 2p_j(1 - 2p_j)^k,\ \ k = 0, 1, 2, \ldots \tag{2.1}$$

We now construct our walk $\mathbf{C}(N)$ as follows. We will take all the horizontal steps consecutively from $S_1(\cdot)$ and all the vertical steps consecutively from $S_2(\cdot)$. First we will take some horizontal steps from $S_1(\cdot)$, then exactly one vertical step from $S_2(\cdot)$, then again some horizontal steps from $S_1(\cdot)$ and exactly one vertical step from $S_2(\cdot)$, and so on. Now we explain how to get the number of horizontal steps on each occasion. Consider our walk starting from the origin proceeding first horizontally $G_1^{(0)}$ steps (note that $G_1^{(0)} = 0$ is possible with probability $2p_0$), after which it takes exactly one vertical step, arriving either to the level 1 or -1, where it takes $G_1^{(1)}$ or $G_1^{(-1)}$ horizontal steps (which might be no steps at all) before proceeding with another vertical step. If this step carries the walk to the level j, then it will take $G_1^{(j)}$ horizontal steps, if this is the first visit to level j, otherwise it takes $G_2^{(j)}$ horizontal steps. In general, if we finished the k -th vertical step and arrived to the level j for the i-th time, then it will take $G_i^{(j)}$ horizontal steps.

Let now H_N, V_N be the number of horizontal and vertical steps, respectively from the first N steps of the just described process. Consequently, $H_N + V_N = N$, and

$$\begin{aligned}\{\mathbf{C}(N);\ N=0,1,2,\ldots\} &= \{(C_1(N), C_2(N));\ N=0,1,2,\ldots\} \\ &\stackrel{d}{=} \{(S_1(H_N), S_2(V_N));\ N=0,1,2,\ldots\}, \end{aligned} \tag{2.2}$$

where $\stackrel{d}{=}$ stands for equality in distribution.

Now we list some well-known results, and some new ones which will be used in the rest of the paper. In case of the known ones we won't give the most general form of the results, just as much as we intend to use, while the exact reference will also be provided for the interested reader. Denote the simple symmetric random walk on the line by $S(n)$ and let $M(n) = \max_{0\le k\le n} |S(k)|$. Then we have the LIL and Chung [6]:

Lemma A *We have almost surely*

$$\limsup_{n\to\infty} \frac{M(n)}{\sqrt{2n\log\log n}} = 1, \qquad \liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right)^{1/2} M(n) = \frac{\pi}{\sqrt{8}}.$$

Denote by $\xi(x,n)$ the local time of the simple symmetric random walk $S(n)$ defined by

$$\xi(x,n) = \sum_{i=0}^{n} I\{S(i) = x\}, \qquad x\in\mathbb{Z},\ n=0,1,2,\ldots,$$

where $I\{\cdot\}$ is the indicator function. Let the maximal local time be

$$\xi(n) = \sup_{x\in\mathbb{Z}} \xi(x,n).$$

For the next Lemma see Kesten [21].

Lemma B *For the maximal local time we have*

$$\limsup_{n\to\infty} \frac{\xi(n)}{(2n\log\log n)^{1/2}} = 1 \quad a.s.$$

In Heyde [17] the following result was given about the uniformity of the local time (see also [9], Lemma 5).

Lemma C *For the simple symmetric walk we have for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} |\xi(x+1,n) - \xi(x,n)|}{n^{1/4+\varepsilon}} = 0 \quad a.s.$$

The next lemma is the two-dimensional version, and that of its consequence, of the celebrated functional iterated logarithm law for multidimensional Wiener process due to Strassen [31]:

Lemma D *Let $W_1(t)$ and $W_2(t)$ be independent standard Wiener processes starting from zero. Then with probability 1, the limit points for the random vector valued functions*

$$\left(\frac{W_1(xT)}{(2T\log\log T)^{1/2}}, \frac{W_2(xT)}{(2T\log\log T)^{1/2}},\ 0\le x\le 1\right)_{T\ge 3}$$

as $T\to\infty$ is $\mathcal{S}^{(2)}$ of (1.4). *In particular, the limit points of the random vectors*

$$\left(\frac{W_1(T)}{(2T\log\log T)^{1/2}}, \frac{W_2(T)}{(2T\log\log T)^{1/2}}\right)_{T\ge 3}$$

as $T\to\infty$ is the unit circle

$$\{(x,y):\ x^2+y^2\le 1\}.$$

We will need the celebrated KMT strong invariance principle (cf. Komlós *et al.* [22]).

Lemma E *On an appropriate probability space one can construct $\{S(n),\ n=1,2,\ldots\}$, a simple symmetric random walk on the line and a standard Wiener process $\{W(t),\ t\ge 0\}$ such that as $n\to\infty$,*

$$S(n)-W(n)=O(\log n)\qquad a.s.$$

Lemma 2.1 *Let $\{S(n);\ n=0,1,\ldots\}$ be a simple symmetric random walk on the line. Put*

$$\tau(i)=\min\{n>0:S(n)=i\},$$

$$\tau_L=\min\left(\tau(0),\tau(-L),\tau(L)\right).$$

Then

$$\mathbf{E}(\tau_L)=L, \tag{2.3}$$

and τ_L has finite variance, (*the value of which is unimportant in the present context*).

Proof. For $0\le a\le b\le c$ define

$$p(a,b,c):=\mathbf{P}(\min\{n:\ n>m,\ S(n)=a\}<\min\{n:\ n>m,\ S(n)=c\}\mid S(m)=b). \tag{2.4}$$

It is well-known that (cf. e.g. [25], p. 23)

$$p(a,b,c)=\frac{c-b}{c-a}.$$

Consider

$$\xi(k,\tau_L),\quad k=\pm1,\pm2,\ldots,\pm(L-1),$$

i.e. the local time of k up to time τ_L.

It is obvious that for $L=1$ or $L=2$, we have $\tau_L=L$. So assume that $L\geq 3$.

$$\begin{aligned}
\mathbf{P}(\xi(1,\tau_L)=0) &= \frac{1}{2}\\
\mathbf{P}(\xi(1,\tau_L)=j) &= \frac{1}{2}\left(\frac{1}{2}p(1,2,L)\right)^{j-1}\left(\frac{1}{2}+\frac{1}{2}(1-p(1,2,L))\right)\\
&= \frac{1}{2}\left(\frac{L-2}{2(L-1)}\right)^{j-1}\left(\frac{L}{2(L-1)}\right), \qquad j=1,2,\dots
\end{aligned}$$

For $k=2,3,\dots,L-1$ the same type of argument results in

$$\begin{aligned}
\mathbf{P}(\xi(k,\tau_L)=0) &= 1-\frac{1}{2k}\\
\mathbf{P}(\xi(k,\tau_L)=j) &= \frac{1}{2}(1-p(0,1,k))\left(\frac{1}{2}(1-p(0,1,k))+\frac{1}{2}p(k,k+1,L)\right)^{j-1}\times\\
&\quad\times\left(\frac{1}{2}p(0,k-1,k)+\frac{1}{2}(1-p(k,k+1,L))\right)\\
&= \frac{1}{2k}\left(\frac{k-1}{2k}+\frac{L-k-1}{2(L-k)}\right)^{j-1}\left(\frac{1}{2k}+\frac{1}{2(L-k)}\right), \qquad j=1,2,\dots
\end{aligned}$$

From the above distributions, which are geometric, we get by simple calculation

$$\mathbf{E}(\xi(k,\tau_L))=\frac{L-k}{L},$$

Obviously, the same is true for $k=-1,-2,\dots,-(L-1)$, with k replaced by $-k$. Consequently

$$\mathbf{E}(\tau_L)=1+2\sum_{k=1}^{L-1}\mathbf{E}(\xi(k,\tau_L))=1+2\sum_{k=1}^{L-1}\frac{L-k}{L}=L.$$

It is clear from the above calculations that τ_L has finite variance. $\square$

Let

$$E_L=\{jL; j=0,\pm1,\pm2,\dots\},$$

$$\gamma_{1,L}=\tau_L, \quad \gamma_{i,L}=\min\{j>0: S(\gamma_{i-1,L}+j)\in E_L\},$$

$$T_{N,L}=\sum_{i=1}^{N}\gamma_{i,L},$$

$$N_n=\max\{k:\, T_{k,L}\leq n\}.$$

Since $T_{N,L}$ is a sum of i.i.d. random variables, N_n is a renewal process. It follows from Gut *et al.* [16]

Lemma F *As $N \to \infty$, we have almost surely*

$$T_{N,L} = NL + O(N^{1/2+\varepsilon})$$

and as $n \to \infty$ we have almost surely

$$N_n = \frac{n}{L} + O(n^{1/2+\varepsilon}).$$

For sums of geometric random variables we need the following exponential estimation.

Lemma 2.2 *Let $\{G_i^{(j)},\ i = 1, 2, \dots, n_j,\ j = 0, \pm 1, \pm 2, \dots, \pm K\}$ be independent random variables with distribution*

$$\mathbf{P}(G_i^{(j)} = k) = \alpha_j (1 - \alpha_j)^k, \quad k = 0, 1, 2, \dots,$$

where $0 < \alpha_j \le 1$. Put

$$B_K = \sum_{j=-K}^{K} \sum_{i=1}^{n_j} G_i^{(j)}, \quad \sigma^2 = Var B_K = \sum_{j=-K}^{K} \frac{n_j (1 - \alpha_j)}{\alpha_j^2}.$$

Then, for $\lambda < -\sigma^2 \log(1 - \alpha_j)$ for each $j \in [-K, K]$, we have

$$\mathbf{P}\left(\left| \sum_{j=-K}^{K} \sum_{i=1}^{n_j} \left(G_i^{(j)} - \frac{1 - \alpha_j}{\alpha_j} \right) \right| > \lambda \right) \le 2 \exp\left(-\frac{\lambda^2}{2\sigma^2} + \sum_{\ell=3}^{\infty} \frac{\lambda^\ell}{\sigma^{2\ell}} \sum_{j=-K}^{K} \frac{n_j}{\alpha_j^\ell} \right). \tag{2.5}$$

Proof. Since $G_i^{(j)}$, $i = 1, 2, \dots, n_j$, $j = 0, \pm 1, \pm 2, \dots, \pm K$ are independent, and $G_i^{(j)}$ has moment generating function

$$\mathbf{E}\left(e^{\theta G_i^{(j)}} \right) = \frac{\alpha_j}{1 - e^\theta (1 - \alpha_j)}$$

for $e^\theta (1 - \alpha_j) < 1$, the cumulant generating function of B_K can be obtained from the series expansions of logarithmic and exponential functions as follows.

$$\log \mathbf{E}\left(e^{\theta B_K} \right) = \sum_{j=-K}^{K} n_j (\log \alpha_j - \log(1 - e^\theta (1 - \alpha_j))) = \sum_{j=-K}^{K} n_j \left(\log \alpha_j + \sum_{\ell=0}^{\infty} \frac{\theta^\ell}{\ell!} \sum_{k=1}^{\infty} (1 - \alpha_j)^k k^{\ell - 1} \right).$$

But

$$\sum_{k=1}^{\infty} (1 - \alpha_j)^k k^{-1} = -\log \alpha_j,$$

$$\sum_{k=1}^{\infty}(1-\alpha_j)^k = \frac{1-\alpha_j}{\alpha_j},$$
$$\sum_{k=1}^{\infty}(1-\alpha_j)^k k = \frac{1-\alpha_j}{\alpha_j^2},$$

and for $\ell \geq 3$

$$\sum_{k=1}^{\infty}(1-\alpha_j)^k k^{\ell-1} = \frac{\sum_{m=1}^{\ell-1} A(\ell-1,m)(1-\alpha_j)^m}{\alpha_j^{\ell}},$$

where $A(\cdot,\cdot)$ are Eulerian numbers defined by

$$A(n,m) = \sum_{j=0}^{m-1}(-1)^j\binom{n+1}{j}(m-j)^n, \qquad n=1,2,\ldots, \qquad m=1,2,\ldots,n.$$

(see e.g. Comtet [7], pp. 242-243). Since

$$\sum_{m=1}^{\ell-1} A(\ell-1,m)(1-\alpha_j)^m \leq \sum_{m=1}^{\ell-1} A(\ell-1,m) = (\ell-1)!,$$

and

$$\mathbf{E}(B_K) = \sum_{j=-K}^{K}\frac{n_j(1-\alpha_j)}{\alpha_j},$$

we have

$$\mathbf{E}\left(e^{\theta(B_K-\mathbf{E}(B_K))}\right) \leq \exp\left\{\frac{\theta^2\sigma^2}{2}+\sum_{\ell=3}^{\infty}\theta^{\ell}\sum_{j=-K}^{K}\frac{n_j}{\alpha_j^{\ell}}\right\}.$$

By Markov inequality,

$$\mathbf{P}(B_K-\mathbf{E}(B_K)\geq\lambda) \leq e^{-\lambda\theta}\mathbf{E}\left(e^{\theta(B_K-\mathbf{E}(B_K)}\right) \leq e^{-\lambda\theta}\exp\left\{\frac{\theta^2\sigma^2}{2}+\sum_{\ell=3}^{\infty}\theta^{\ell}\sum_{j=-K}^{K}\frac{n_j}{\alpha_j^{\ell}}\right\}.$$

The estimation of $\mathbf{P}(\mathbf{E}(B_K)-B_K\geq\lambda)$ is similar. By choosing $\theta=\lambda/\sigma^2$, we have the Lemma. □

3 Local times and range

Before proving our main results, Theorems 1.1 and 1.2, in Section 4, in this section we deal with the periodic case on its own. In this case, for a given positive integer $L\geq 1$, $p_{j+L}=p_j$ for all $j\in\mathbb{Z}$, and for $i=0,1,\ldots$, we have

$$\frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{p_{j+iL}} = \frac{1}{L}\sum_{j=0}^{L-1}\frac{1}{p_j} = 2\gamma, \tag{3.1}$$

with $1 < \gamma < \infty$, as in (1.1).

In this special context we conclude results of interest on their own for the local time and range of the walk and relate them to our main approximation theorems, as well as to similar ones for other walks in the literature. The topics discussed in this section are not needed in Section 4 for the proofs of Theorems 1.1 and 1.2 themselves.

First we note that in the case of $L \geq 2$ there is a relation between the periodic case and a particular case of the so-called random walk with internal states (or random walk with internal degrees of freedom). This was introduced by Sinai [30], and further investigated by Krámli and Szász [23], Telcs [32], Nándori [24] and others. Let F be a finite set. On $\mathbb{Z}^d \times F$ the Markov chain $\{\mathbf{U}(N) = (\mathbf{X}(N), Z(N))\}$ is a random walk with internal states, if for $\mathbf{x}_N, \mathbf{x}_{N+1} \in \mathbb{Z}^d$, $\ell_N, \ell_{N+1} \in F$

$$\mathbf{P}(\mathbf{U}(N+1) = (\mathbf{x}_{N+1}, \ell_{N+1}) \mid \mathbf{U}(N) = (\mathbf{x}_N, \ell_N)) = P(\mathbf{x}_{N+1} - \mathbf{x}_N, \ell_N, \ell_{N+1}).$$

In the particular case $d = 2$, $F = \{0, 1, \ldots, L-1\}$, for $(k, j) \in \mathbb{Z}^2$

$$\mathbf{P}(\mathbf{U}(N+1) = (k, j, \ell+1) \mid \mathbf{U}(N) = (k, j, \ell)) = \mathbf{P}(\mathbf{U}(N+1) = (k, j, \ell-1) \mid \mathbf{U}(N) = (k, j, \ell)) = p_\ell$$

for $\ell = 1, \ldots, L-2$,

$$\mathbf{P}(\mathbf{U}(N+1) = (k, j, 1) \mid \mathbf{U}(N) = (k, j, 0)) = \mathbf{P}(\mathbf{U}(N+1) = (k, j-1, L-1) \mid \mathbf{U}(N) = (k, j, 0)) = p_0,$$

$$\begin{aligned}&\mathbf{P}(\mathbf{U}(N+1) = (k, j+1, 0) \mid \mathbf{U}(N) = (k, j, L-1))\\ &= \mathbf{P}(\mathbf{U}(N+1) = (k, j, L-2) \mid \mathbf{U}(N) = (k, j, L-1)) = p_{L-1},\end{aligned}$$

$$\mathbf{P}(\mathbf{U}(N+1) = (k+1, j, \ell) \mid \mathbf{U}(N) = (k, j, \ell)) = \mathbf{P}(\mathbf{U}(N+1) = (k-1, j, \ell) \mid \mathbf{U}(N) = (k, j, \ell)) = \frac{1}{2} - p_\ell$$

for $\ell = 0, 1, \ldots, L-1$.

It is clear that for $\mathbf{C}^{\mathbf{P}}(N) = (C_1^P(N), C_2^P(N))$ and $\mathbf{U}(N) = (X_1(N), X_2(N), Z(N))$, we have a one-to-one correspondence, namely

$$C_1^P(N) = X_1(N), \qquad C_2^P(N) = LX_2(N) + Z(N),$$

with $0 \leq Z(N) \leq L-1$. Hence $Z(N) = \ell$ if and only if $C_2^P(N) \equiv \ell\,(\mathrm{mod} L)$ and $X_2(N) = (C_2^P(N) - Z(N))/L$.

Since $Z(N)$ is bounded, it follows that Theorem 1.2 is true with $C_1^P(N)$ replaced by $X_1(N)$, and $C_2^P(N)$ replaced by $LX_2(N)$.

To study the local times of our random walk, we need a local limit theorem for $\mathbf{C}^{\mathbf{P}}(\cdot)$.

Lemma 3.1 *In the periodic case we have*

$$\mathbf{P}(\mathbf{C}^{\mathbf{P}}(2N) = (0,0)) \sim \frac{1}{4\pi N p_0 \sqrt{\gamma - 1}} \tag{3.2}$$

with $L \geq 1$.

Proof. For the proof we could have used the local limit theorems for random walks with internal states, given as in Krámli and Szász [23]. Nándori [24] also gives a remainder term in this local limit theorem. To determine the exact constant, we take another route, namely apply (2.3.9) of Roerdink and Shuler [26] with $d = 2$. Using their notations, what we have to determine is the invariant probability measure $(\pi_0, \pi_1, \ldots, \pi_{L-1})$ and the determinants of their matrices $2D$ and A. Now $2D$, what they call diffusion matrix, whose elements are the constants in the variances and covariances of the components, can be seen in Theorems 1.1 and 1.2, i.e., we have

$$2D = \begin{pmatrix} \frac{\gamma-1}{\gamma} & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix}.$$

The matrix A has elements (cf. (2.2.25) in [26]) $A_{ki} = \mathbf{a}_k\mathbf{e}_i$ (scalar product), where $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (0, L)$, $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, hence

$$A = \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix},$$

giving $\det(2D) = (\gamma - 1)/\gamma^2$, $\det A = L$.

Now (2.3.9) of [26] with $d = 2$ reads as follows.

$$\mathbf{P}(\mathbf{C}^{\mathbf{P}}(N) = (0, \alpha)|\mathbf{C}^{\mathbf{P}}(0) = (0, \beta)) \sim \pi_\alpha(\det \mathbf{s})^{-1/2}(2\pi N)^{-1}, \tag{3.3}$$

where $0 \le \alpha, \beta \le L - 1$,

$$\det \mathbf{s} = (\det 2D)(\det A)^{-2} = \frac{\gamma - 1}{L^2\gamma^2} \tag{3.4}$$

and π_α are the components of the left eigenvector corresponding to the maximal eigenvalue $\lambda_0 = 1$ of the stochastic matrix T defined by (2.1.10) in [26].

We are to show now that

$$\pi_\alpha = \frac{1}{2p_\alpha L\gamma}, \quad \alpha = 0, \ldots, L - 1. \tag{3.5}$$

First we note that the elements of the matrix T are the transition probabilities for the internal states of the walk in hand.

Now for $L = 1$, this matrix has only one element that is equal to 1, hence $\pi_0 = 1$, in agreement with (3.5), via (3.1) with $L = 1$.

When $L = 2$, the matrix T of the transition probabilities for the internal states is the 2x2 matrix

$$T = \begin{pmatrix} 1 - 2p_0 & 2p_0 \\ 2p_1 & 1 - 2p_1 \end{pmatrix}.$$

In view of this, easy computations show that (π_0, π_1) as given by (3.5) are the components of the left eigenvector $(p_1/(p_0 + p_1),\ p_0/(p_0 + p_1))$ that corresponds to the eigenvalue $\lambda_0 = 1$ of this matrix.

Continuing along these lines, for $L > 2$, the elements of the matrix T are the transition probabilities

$$\mathbf{P}(i, i+1) = \mathbf{P}(i, i-1) = p_i, \quad \mathbf{P}(i,i) = 1 - 2p_i, \quad i = 1, 2, \ldots, L-2,$$
$$\mathbf{P}(0,1) = \mathbf{P}(0, L-1) = p_0, \quad \mathbf{P}(0,0) = 1 - 2p_0,$$
$$\mathbf{P}(L-1, L-2) = \mathbf{P}(L-1, 0) = p_{L-1}, \quad \mathbf{P}(L-1, L-1) = 1 - 2p_{L-1}.$$

Consequently, it can be seen that π_α; $\alpha = 0, 1, \ldots, L-1$ as in (3.5), are the components of the left eigenvector corresponding to the eigenvalue $\lambda_0 = 1$ of this matrix T in this case too, i.e., when $L > 2$.

Putting $\alpha = \beta = 0$ into (3.3), using (3.4), and since the probability on the left hand side is non-zero only if the number of steps are even, we have to modify (3.3) by multiplying it by $(1 + (-1)^N)$, just as in (2.3.15) or (2.3.17) of [26]. With this modification, replacing N by $2N$ in (3.3), we arrive at (3.2) of Lemma 3.1. □

Having now Lemma 3.1, we are to study the local time and range of the periodic random walk. These results are additional to that of Theorem 1.2.

It follows from Lemma 3.1 that the truncated Green function $g(\cdot)$ is given by

$$g(N) = \sum_{k=0}^{N} \mathbf{P}(\mathbf{C}^{\mathbf{P}}(k) = (0,0)) \sim \frac{\log N}{4p_0 \pi \sqrt{\gamma - 1}}, \qquad N \to \infty,$$

which implies that our anisotropic random walk in this case is recurrent and also Harris recurrent.

First, we define the local time by

$$\Xi((k,j), N) = \sum_{r=1}^{N} I\{\mathbf{C}^{\mathbf{P}}(r) = (k,j)\}, \quad (k,j) \in \mathbb{Z}^2. \tag{3.6}$$

In the case when the random walk is (Harris) recurrent, then we have (cf. e.g. Chen [3])

$$\lim_{N \to \infty} \frac{\Xi((k_1, j_1), N)}{\Xi((k_2, j_2), N)} = \frac{\mu(k_1, j_1)}{\mu(k_2, j_2)} \quad a.s.,$$

where $\mu(\cdot)$ is an invariant measure. So to obtain limit theorems for the local time as in (3.6), it suffices to find an invariant measure that, in combination with appropriate results for $\Xi((0,0), N)$, will also yield general results.

In this context, an invariant measure is defined via

$$\mu(A) = \sum_{(k,j) \in \mathbb{Z}^2} \mu(k,j) \mathbf{P}(\mathbf{C}^{\mathbf{P}}(N+1) \in A | \mathbf{C}^{\mathbf{P}}(N) = (k,j)).$$

For $(k,j) \in \mathbb{Z}^2$, in our case we have

$$\mu(k,j) = \mu(k+1, j)\left(\frac{1}{2} - p_j\right) + \mu(k-1, j)\left(\frac{1}{2} - p_j\right) + \mu(k, j+1)p_{j+1} + \mu(k, j-1)p_{j-1}.$$

It is easy to see that

$$\mu(k,j) = \frac{1}{p_j}, \quad (k,j) \in \mathbb{Z}^2,$$

satisfies this equation. So this defines an invariant measure. Hence

$$\lim_{N\to\infty} \frac{\Xi((0,0),N)}{\Xi((k,j),N)} = \frac{p_j}{p_0} \quad a.s.$$

for $(k,j) \in \mathbb{Z}^2$ fixed.

Thus, using now $g(N)$, it follows from Darling and Kac [11] that we have

Corollary 3.1

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{\Xi((0,0),N)}{g(N)} \geq x\right) = \lim_{N\to\infty} \mathbf{P}\left(\frac{4p_0\pi\sqrt{\gamma-1}\,\Xi((0,0),N)}{\log N} \geq x\right) = e^{-x}, \quad x \geq 0.$$

For a limsup result, via Chen [3] we conclude

Corollary 3.2

$$\limsup_{N\to\infty} \frac{\Xi((0,0),N)}{\log N \log\log\log N} = \frac{1}{4p_0\pi\sqrt{\gamma-1}} \quad a.s.$$

For moderate and large deviations and functional limit laws for the local time see Csáki *et al.* [10], which was extended by Gantert and Zeitouni [15]. In our case the functional limit theorem reads as follows: Let $\mathcal{M}$ be the set of functions $m(x)$, $0 \leq x \leq 1$ which are non-decreasing, right-continuous on $[0,1)$ and left-continuous at $x=1$, equipped with weak topology, induced by Lévy metric. Furthermore, let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ with $m(0)=0$ and

$$\int_0^1 \frac{dm(x)}{x} \leq 1.$$

Corollary 3.3 *Let $t(N,x) \in \mathcal{M}$ be a sequence of functions such that*

$$\lim_{N\to\infty} \frac{\log t(N,x)}{\log N} = x$$

for all $0 \leq x \leq 1$, like for example, $t(N,x) = N^x$. Put

$$f_N(x) = \frac{4p_0\pi\sqrt{\gamma-1}\,\Xi(0,t(N,x))}{\log N \log\log\log N}.$$

Then, almost surely, the set of limit points of $\{f_N(x),\ 0 \leq x \leq 1\}_{N\geq 16}$ is $\mathcal{M}^*$.

For further results, including second order limit laws, we refer to Chen [4] and [5].

The range of the random walk $\{\mathbf{C}^{\mathbf{P}}(\cdot)\}$ is defined by

$$R(N) = \sum_{(k,j)\in\mathbb{Z}^2} I\{\Xi((k,j),N) > 0\},$$

i.e. the number of distinct sites visited by the random walk up to time N. Roerdink and Shuler [26] gives

$$\mathbf{E}(R(N)) \sim \frac{2\pi\sqrt{\gamma-1}}{\gamma}\frac{N}{\log N}, \quad N\to\infty.$$

Moreover, a law of large numbers follows from Nándori [24]

Corollary 3.4

$$\lim_{N\to\infty}\frac{R(N)}{\mathbf{E}(R(N))} = \lim_{N\to\infty}\frac{\gamma\, R(N)\log N}{2\pi\sqrt{\gamma-1}\, N} = 1 \qquad a.s.$$

4 Proofs of the approximation theorems

Proof of Theorem 1.1.

The proof of Theorem 1.1 will be based on the following Proposition.

Proposition 4.1 *Assume the conditions of* Lemma 2.2 *and put* $M = \sum_{j=-K}^{K} n_j$. *For* $M\to\infty$ *and* $K\to\infty$ *assume moreover that*

$$K = K(M) = O(M^{1/2+\delta}), \quad \max_{-K\le j\le K} n_j = O(M^{1/2+\delta}), \tag{4.1}$$

for all $\delta > 0$,

$$\frac{1}{\alpha_j} \le c_1|j|^{1-\eta}, \quad j = 0, \pm1, \pm2, \dots \tag{4.2}$$

for some $1/2 < \eta \le 1$ *and* $c_1 > 0$,

$$\sum_{j=-K}^{K}\frac{1}{\alpha_j} = O(K), \qquad \frac{1}{\sigma} \le \frac{c_2}{M^{1/2}} \tag{4.3}$$

for some $c_2 > 0$. *Then we have as* $K, M\to\infty$,

$$\sum_{j=-K}^{K}\sum_{i=1}^{n_j} G_i^{(j)} = \sum_{j=-K}^{K} n_j\frac{1-\alpha_j}{\alpha_j} + O(M^{3/4-\eta/4+\varepsilon}) \quad a.s. \tag{4.4}$$

for some $\varepsilon > 0$.

Proof. By (4.1), (4.2) and (4.3) we have

$$\sum_{j=-K}^{K} \frac{n_j}{\alpha_j^\ell} = O(M^{(1/2+\delta)(1+(\ell-1)(1-\eta))}) \sum_{j=-K}^{K} \frac{1}{\alpha_j} = O(M^{(1/2+\delta)(\ell(1-\eta)+1+\eta)}).$$

For $\ell = 2$ this gives

$$\sigma^2 = \sum_{j=-K}^{K} \frac{n_j(1-\alpha_j)}{\alpha_j^2} \leq \sum_{j=-K}^{K} \frac{n_j}{\alpha_j^2} = O(M^{(1/2+\delta)(3-\eta)}).$$

Put

$$\lambda = M^\varepsilon \sigma = O(M^{(3/2-\eta/2)(1/2+\delta)+\varepsilon})$$

into (2.5) of Lemma 2.2 with $\varepsilon > 0$ small enough. This is possible, since for $|j| \leq K$ and $\varepsilon < \eta/2$ we can select $\delta > 0$ small enough, such that

$$\frac{\lambda}{\sigma^2 \log \frac{1}{1-\alpha_j}} = \frac{M^\varepsilon}{\sigma \log \frac{1}{1-\alpha_j}} \leq c\frac{M^{\varepsilon-1/2}}{\alpha_j} \leq cM^{\varepsilon-1/2}|j|^{1-\eta} \leq cM^{\varepsilon-1/2+(1/2+\delta)(1-\eta)} < 1$$

for large enough M. We get

$$\mathbf{P}\left(\left|\sum_{j=-K}^{K}\sum_{i=1}^{n_j} G_i^{(j)} - \sum_{j=-K}^{K} n_j \frac{1-\alpha_j}{\alpha_j}\right| > \lambda\right)$$

$$\leq 2\exp\left(-\frac{M^{2\varepsilon}}{2} + \sum_{\ell=3}^{\infty} \frac{M^{\ell\varepsilon}}{\sigma^\ell} O(M^{(1/2+\delta)(\ell(1-\eta)+1+\eta)})\right). \tag{4.5}$$

But using (4.3),

$$\sum_{\ell=3}^{\infty} \frac{M^{\ell\varepsilon}}{\sigma^\ell} O(M^{(1/2+\delta)(\ell(1-\eta)+1+\eta)}) \leq O(M^{(1/2+\delta)(1+\eta)}) \sum_{\ell=3}^{\infty} \left(\frac{M^{(1/2+\delta)(1-\eta)+\varepsilon}}{\sigma}\right)^\ell$$

$$= O(M^{1/2-\eta+3\varepsilon+\delta(4-2\eta)}).$$

Choosing $\delta > 0$ and $\varepsilon > 0$ small enough, the dominant term on the exponent of the right-hand side of (4.5) is $-M^{2\varepsilon}/2$, hence by Borel-Cantelli lemma

$$\sum_{j=-K}^{K}\sum_{i=1}^{n_j} G_i^{(j)} = \sum_{j=-K}^{K} n_j \frac{1-\alpha_j}{\alpha_j} + O(M^{3/4-\eta/4+\varepsilon+\delta}) \quad a.s.$$

as $M \to \infty$. This completes the proof of the Proposition 4.1. $\square$

To prove Theorem 1.1, we start with the redefined version of $\mathbf{C}(N)$ given in the Preliminaries. By Lemma E we may assume that on the same probability space where the two simple symmetric random walks $S_1(n)$ and $S_2(n)$ are already defined, there are also two independent Wiener processes $W_1(\cdot)$, $W_2(\cdot)$ such that for $i = 1, 2$ we have

$$\sup_{k \leq n} |S_i(k) - W_i(k)| = O(\log n) \tag{4.6}$$

almost surely, as $n \to \infty$.

As in Section 2, let H_N, V_N be the number of horizontal and vertical steps, respectively from the first N steps of $\mathbf{C}(\cdot)$. In view of (2.2) it is enough to show that

$$H_N = \frac{\gamma - 1}{\gamma} N + O\left(N^{5/4-\eta/2+\epsilon}\right) \tag{4.7}$$

almost surely, as $N \to \infty$.

Consider the sum

$$G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)}$$

which is the total number of horizontal steps on the level j, where $\xi_2(\cdot, \cdot)$ is the local time of the walk $S_2(\cdot)$. This statement is slightly incorrect if j happens to be the level where the last vertical step (up to the total of N steps) takes the walk. In this case the last geometric random variable might be truncated. However the error which might occur from this simplification will be part of the $O(\cdot)$ term. This can be seen as follows. Let

$$H_N^+ = \sum_j \sum_{i=1}^{\xi_2(j,V_N)} G_i^{(j)}, \quad H_N^- = \sum_j \sum_{i=1}^{\xi_2(j,V_N)-1} G_i^{(j)},$$

where $G_i^{(j)}$ has distribution (2.1). Obviously, $H_N^- \leq H_N \leq H_N^+$ and

$$H_N^+ - H_N^- = \sum_j G_{\xi_2(j,V_N)}^{(j)}.$$

Here and in the sequel

$$\sum_j = \sum_{\min_{0 \leq k \leq V_N} S_2(k) \leq j \leq \max_{0 \leq k \leq V_N} S_2(k)}.$$

Now consider the following sum:

$$\sum_{j=-K}^{K} G^{(j)},$$

where $G^{(j)}$ are independent random variables with distribution (2.1). Then

$$\mathbf{E}\left(\sum_{j=-K}^{K} G^{(j)}\right) = \sum_{j=-K}^{K} \frac{1-2p_j}{2p_j}, \qquad \sigma^2 = Var\left(\sum_{j=-K}^{K} G^{(j)}\right) = \sum_{j=-K}^{K} \frac{1-2p_j}{(2p_j)^2}.$$

From Lemma 2.2 with $n_j = 1$, we obtain

$$\mathbf{P}\left(\left|\sum_{j=-K}^{K}\left(G_i^{(j)} - \frac{1-2p_j}{2p_j}\right)\right| > \lambda\right) \le 2\exp\left(-\frac{\lambda^2}{2\sigma^2} + \sum_{\ell=3}^{\infty} \frac{\lambda^\ell}{\sigma^{2\ell}} \sum_{j=-K}^{K} \frac{1}{\alpha_j^\ell}\right). \tag{4.8}$$

Note that by using (1.1) and (4.2), with $\alpha_j = 2p_j$, we get as $K \to \infty$,

$$\sum_{j=-K}^{K} \frac{1-2p_j}{2p_j} = O(K),$$

and

$$\sum_{j=-K}^{K} \frac{1}{(2p_j)^\ell} = O(K^{\ell(1-\eta)+\eta}),$$

$$cK \le \sigma^2 = O(K^{2-\eta}).$$

Put $\lambda = \sigma K^\varepsilon$ into (4.8) with $\varepsilon > 0$ small enough, we get similarly to the proof of (4.4) of Proposition 4.1 that

$$\sum_{j=-K}^{K} G^{(j)} = \sum_{j=-K}^{K} \frac{1-2p_j}{2p_j} + O(K^{1-\eta/2+\varepsilon}) = O(K).$$

Choosing $K = \max_{1\le k\le V_N} |S_2(k)|$, this gives

$$H_N^+ - H_N \le H_N^+ - H_N^- = O(N^{1/2+\delta})$$

almost surely, as $N \to \infty$, with arbitrary small δ.

So consider

$$H_N^+ = \sum_j \left(G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)}\right).$$

We apply Proposition 4.1 with $M = V_N$, $K = \max_{0\le k\le V_N} |S_2(k)|$, $n_j = \xi_2(j, V_N)$, $\alpha_j = 2p_j$, $j = 0, \pm 1, \pm 2, \ldots$

The assumptions of this Proposition are satisfied almost surely: (4.1) follows from Lemma A and Lemma B, and for (4.2) we refer to Heyde [17]. The first part of (4.3) follows from (1.1). It remains to verify the second part of (4.3). According to a result of Heyde [17] (p. 726, formula (6))

$$\lim_{N\to\infty} \frac{1}{V_N} \sum_j \frac{\xi_2(j, V_N)}{2p_j} = \gamma \quad a.s. \tag{4.9}$$

Hence, as $N \to \infty$, we have almost surely,

$$\frac{\sigma^2}{V_N} = \frac{1}{V_N}\sum_j \frac{\xi_2(j,V_N)(1-2p_j)}{(2p_j)^2} \geq \frac{1}{V_N}\sum_j \frac{\xi_2(j,V_N)(1-2p_j)}{2p_j} = \frac{1}{V_N}\sum_j \frac{\xi_2(j,V_N)}{2p_j} - 1 \to \gamma - 1 > 0.$$

This verifies the second part of (4.3).

Hence by Proposition 4.1, since $V_N \leq N$, we have almost surely, as $N \to \infty$,

$$\begin{aligned} H_N^+ &= \sum_j \left(G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)}\right) \\ &= \sum_j \xi_2(j,V_N)\frac{1-2p_j}{2p_j} + O(N^{3/4-\eta/4+\varepsilon}) \\ &= -V_N + \frac{1}{2}\sum_j \xi_2(j,V_N)\frac{1}{p_j} + O(N^{3/4-\eta/4+\varepsilon}). \end{aligned} \tag{4.10}$$

In what follows we take some more ideas from Heyde [17] in order to prove (4.9) with an appropriate remainder term. Introduce the notation

$$\frac{1}{j}\sum_{k=1}^{j}\frac{1}{p_k} = \kappa_j, \quad \frac{1}{j}\sum_{k=1}^{j}\frac{1}{p_{-k}} = \beta_j.$$

Then

$$\sum_j \xi_2(j,V_N)\frac{1}{p_j} = \sum_{j=1}^{\infty}\xi_2(j,V_N)(j\kappa_j - (j-1)\kappa_{j-1}) + \sum_{j=1}^{\infty}\xi_2(-j,V_N)(j\beta_j - (j-1)\beta_{j-1}) + \xi_2(0,V_N)\frac{1}{p_0}$$

$$= \sum_{j=1}^{\infty} j\kappa_j(\xi_2(j,V_N) - \xi_2(j+1,V_N)) + \sum_{j=1}^{\infty} j\beta_j(\xi_2(-j,V_N) - \xi_2(-j-1,V_N)) + \xi_2(0,V_N)\frac{1}{p_0}$$

$$= \sum_{j=1}^{\infty} j(\kappa_j - 2\gamma)(\xi_2(j,V_N) - \xi_2(j+1,V_N)) + 2\gamma\sum_{j=1}^{\infty} j(\xi_2(j,V_N) - \xi_2(j+1,V_N))$$

$$+ \sum_{j=1}^{\infty} j(\beta_j - 2\gamma)(\xi_2(-j,V_N) - \xi_2(-j-1,V_N)) + 2\gamma\sum_{j=1}^{\infty} j(\xi_2(-j,V_N) - \xi_2(-j-1,V_N)) + \xi_2(0,V_N)\frac{1}{p_0}$$

$$= 2\gamma\sum_{j=-\infty}^{\infty}\xi_2(j,V_N) + \sum_{j=1}^{\infty} j(\kappa_j - 2\gamma)(\xi_2(j,V_N) - \xi_2(j+1,V_N))$$

$$+ \sum_{j=1}^{\infty} j(\beta_j - 2\gamma)(\xi_2(-j,V_N) - \xi_2(-j-1,V_N)) + \xi_2(0,V_N)\left(\frac{1}{p_0} - 2\gamma\right).$$

Observe that

$$2\gamma \sum_{j=-\infty}^{\infty} \xi_2(j, V_N) = 2\gamma V_N.$$

Applying Lemma A for $S_2(\cdot)$, Lemma C and (1.1) again we get that

$$\sum_{j=1}^{\infty} j(\kappa_j - 2\gamma)(\xi_2(j, V_N) - \xi_2(j+1, V_N)) + \sum_{j=1}^{\infty} j(\beta_j - 2\gamma)(\xi_2(-j, V_N) - \xi_2(-j-1, V_N))$$

$$= O(N^{1/4+\epsilon}) \sum_{j=1}^{\max_{k\le N}|S_2(k)|} j^{1-\eta} = O(N^{1/4+\epsilon})O(N^{1-\eta/2+\varepsilon}) = O(N^{5/4-\eta/2+\varepsilon}),$$

where here and throughout the paper the value of ε might change from line to line.

Assembling the pieces we arrive at

$$H_N = (\gamma-1)V_N + O(N^{5/4-\eta/2+\varepsilon}) + O(N^{1/2+\varepsilon}) = (\gamma-1)V_N + O(N^{5/4-\eta/2+\varepsilon}) \quad a.s. \tag{4.11}$$

as $N \to \infty$.

In the middle equation the last term is coming from the single term $\xi_2(0, V_N)/p_0$. Being $H_N + V_N = N$, we get (4.7). Consequently, we have (4.6), (2.2) and (4.7), hence

$$C_1(N) = S_1(H_N) = W_1(H_N) + O(\log H_N) = W_1\left(\frac{\gamma-1}{\gamma}N + O(N^{5/4-\eta/2+\varepsilon})\right) + O(\log N)$$

$$= W_1\left(\frac{\gamma-1}{\gamma}N\right) + O(N^{5/8-\eta/4+\varepsilon}), \tag{4.12}$$

almost surely, as $N \to \infty$. Similarly,

$$C_2(N) = W_2\left(\frac{1}{\gamma}N\right) + O(N^{5/8-\eta/4+\varepsilon}), \tag{4.13}$$

almost surely, as $N \to \infty$, which concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2 By Lemma F

$$\sum_{j\equiv 0(\mathrm{mod} L)} \xi_2(j, n) = \frac{n}{L} + O(n^{1/2+\varepsilon})$$

With a similar argument we also have

$$\sum_{j\equiv i(\mathrm{mod} L)} \xi_2(j, n) = \frac{n}{L} + O(n^{1/2+\varepsilon}), \quad i = 0, 1, \ldots, L-1. \tag{4.14}$$

Consequently, from the law of the iterated logarithm we conclude almost surely, as $N \to \infty$,

$$H_N = \sum_j \left(G_1^{(j)} + G_2^{(j)} + ... + G_{\xi_2(j,V_N)}^{(j)} \right) = \frac{V_N}{L} \sum_{j=0}^{L-1} \frac{1-2p_j}{2p_j} + O(N^{1/2+\varepsilon}) = V_N(\gamma - 1) + O(N^{1/2+\varepsilon}).$$

Thus, having

$$H_N = \frac{\gamma - 1}{\gamma} N + O(N^{1/2+\varepsilon}), \quad V_N = \frac{1}{\gamma} N + O(N^{1/2+\varepsilon}),$$

similarly to the above argument, results in

$$C_1^P(N) = S_1(H_N) = W_1(H_N) + O(\log H_N) = W_1\left(\frac{\gamma-1}{\gamma} N\right) + O(N^{1/4+\varepsilon}) \tag{4.15}$$

and

$$C_2^P(N) = S_2(V_N) = W_2(V_N) + O(\log V_N) = W_2\left(\frac{1}{\gamma} N\right) + O(N^{1/4+\varepsilon}) \tag{4.16}$$

almost surely, as $N \to \infty$, which proves Theorem 1.2. □

Acknowledgement. The authors wish to extend their sincere thanks to the referee for his/her careful reading of, and insightful comments on, this paper.

References

[1] Bertacchi, D. (2006). Asymptotic behaviour of the simple random walk on the 2-dimensional comb. *Electron. J. Probab.* **11** 1184–1203.

[2] Bertacchi, D. and Zucca, F. (2003). Uniform asymptotic estimates of transition probabilities on combs. *J. Aust. Math. Soc.* **75** 325–353.

[3] Chen, X. (1999). How often does a Harris recurrent Markov chain recur? *Ann. Probab.* **27** 1324–1346.

[4] Chen, X. (2000). On the limit laws of second order for additive functionals of Harris recurrent Markov chain. *Probab. Th. Rel. Fields* **116** 89–123.

[5] Chen, X. (2001). Moderate deviations for Markovian occupation time. *Stochastic Process. Appl.* **94** 51–70.

[6] Chung, K.L. (1948). On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** 205–233.

[7] Comtet, L. (1974). *Advanced Combinatorics. The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, enlarged edition.

[8] Csáki, E., Csörgő, M., Földes, A. and Révész, P. (2009). Strong limit theorems for a simple random walk on the 2-dimensional comb. *Electr. J. Probab.* **14** 2371–2390.

[9] Csáki, E. and Révész, P. (1983). Strong invariance for local time. *Z. Wahrsch. verw. Gebiete* **50** 5–25.

[10] Csáki, E., Révész, P. and Rosen, J. (1998). Functional laws of the iterated logarithm for local times of recurrent random walks on $\mathbb{Z}^2$. *Ann. Inst. H. Poincaré, Probab. Statist.* **34** 545–563.

[11] Darling, D.A. and Kac, M. (1957). On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* **84** 444–458.

[12] den Hollander, F. (1994). On three conjectures by K. Shuler. *J. Statist. Physics* **75** 891–918.

[13] Dvoretzky, A. and Erdős, P. (1951). Some problems on random walk in space. *Proc. Second Berkeley Symposium*, pp. 353–367.

[14] Erdős, P. and Taylor, S.J. (1960). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* **11** 137–162.

[15] Gantert, N. and Zeitouni, O. (1998). Large and moderate deviations for the local time of a recurrent Markov chain on $\mathbb{Z}^2$. *Ann. Inst. H. Poincaré, Probab. Statist.* **34** 687–704.

[16] Gut, A., Klosov, O. and Steinebach, J. (1997). Equivalences in strong limit theorems for renewal counting processes. *Statist. Probab. Lett.* **35** 381–394.

[17] Heyde, C.C. (1982). On the asymptotic behaviour of random walks on an anisotropic lattice. *J. Statist. Physics* **27** 721–730.

[18] Heyde, C.C. (1993). Asymptotics for two-dimensional anisotropic random walks. In: *Stochastic Processes.* Springer, New York, pp. 125–130.

[19] Heyde, C.C., Westcott, M. and Williams, E.R. (1982). The asymptotic behavior of a random walk on a dual-medium lattice. *J. Statist. Physics* **28** 375-380.

[20] Horváth, L. (1998). Diffusion approximation for random walks on anisotropic lattices. *J. Appl. Probab.* **35** 206–212.

[21] Kesten, H. (1965) An iterated logarithm law for the local time. *Duke Math.J.* **32** 447-456.

[22] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent rv's and the sample df. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

[23] Krámli, A. and Szász, D. (1983). Random walks with internal degrees of freedom I. Local limit theorems. *Z. Wahrsch. verw. Gebiete* **63** 85–95.

[24] Nándori, P. (2011). Number of distinct sites visited by a random walk with internal states. *Probab. Th. Rel. Fields* **150** 373–403.

[25] Révész, P. (2005). *Random Walk in Random and Non-Random Environments*, 2nd ed. World Scientific, Singapore.

[26] Roerdink, J. and Shuler, K.E. (1985). Asymptotic properties of multistate random walks. I. Theory. *J. Statist. Physics* **40** 205–240.

[27] Seshadri, V., Lindenberg, K. and Shuler, K.E. (1979). Random Walks on Periodic and Random Lattices. II. Random Walk Properties via Generating Function Techniques. *J. Statist. Physics* **21** 517–548.

[28] Shuler, K.E. (1979). Random walks on sparsely periodic and random lattices I. *Physica A* **95** 12–34.

[29] Silver, H., Shuler, K.E. and Lindenberg, K. (1977). Two-dimensional anisotropic random walks. In: *Statistical mechanics and statistical methods in theory and application (Proc. Sympos., Univ. Rochester, Rochester, N.Y., 1976)*, Plenum, New York, pp. 463–505.

[30] Sinai, Ya.G. (1981). Random walks and some problems concerning Lorentz gas. In: *Proceedings of the Kyoto Conference*, pp. 6–17.

[31] Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** 211–226.

[32] Telcs, A. (1984). Local impurities in random walk with internal states. In: *Limit Theorems in Probability and Statistics, Vol. II.* Colloquia Mathematica Societatis János Bolyai 36, Ed. P. Révész. North-Holland, Amsterdam, 1043–1068.

[33] Weiss, G.H. and Havlin, S. (1986). Some properties of a random walk on a comb structure. *Physica A* **134** 474–482.

[34] Westcott, M. (1982). Random walks on a lattice. *J. Statist. Physics* **27** 75–82.