Almost sure limit theorems for the maximum of stationary Gaussian sequences

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Abstract. We prove an almost sure limit theorem for the maxima of stationary Gaussian sequences with covariance r_n under the condition $r_n \log n\,(\log\log n)^{1+\varepsilon} = O(1)$.

Key words: almost sure central limit theorem, logarithmic average, stationary Gaussian sequence.

Introduction. The early results on the almost sure central limit theorem (ASCLT) dealt mostly with partial sums of random variables. A general pattern of these investigations is that if $X_1, X_2, \dots$ is a sequence of random variables with partial sums $S_n = \sum_{k=1}^n X_k$ satisfying $a_n(S_n - b_n) \xrightarrow{\mathcal{D}} G$ for some numerical sequences (a_n), (b_n) and distribution function G, then under some additional mild conditions we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}\left(a_k(S_k - b_k) < x\right) = G(x) \qquad \text{a.s.}$$

for any continuity point x of G, where $\mathbf{I}$ is indicator function.

For more discussions about ASCLT we refer to the survey papers by Berkes (1998), and Atlagh and Weber (2000). Recently Fahrner and Stadtmüller (1998) and Cheng et al. (1998) have extended this principle by proving ASCLT for the maxima of independent random variables.

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THEOREM A. *Let $X_1, X_2, \ldots$ be i.i.d. random variables and $M_k = \max_{i \le k} X_i$. If $a_k(M_k - b_k) \xrightarrow{\mathcal{D}} G$ for a nondegenerate distribution G and some numerical sequences (a_k) and (b_k) , then we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}\left(a_k(M_k - b_k) < x\right) = G(x) \qquad \text{a.s.}$$

for any continuity point x of G.

Berkes and Csáki (2001) extended the ASCLT for general nonlinear functionals of independent random variables. For strong invariance principles improving Theorem A see Berkes and Horváth (2001) and Fahrner (2001).

Throughout this paper $Z_1, Z_2, \ldots$ is a stationary Gaussian sequence and we denote its covariance function by $r_n = \mathbf{Cov}(Z_1, Z_{n+1})$, and $M_n = \max_{1\le i\le n} Z_i$ and $M_{k,n} = \max_{k+1\le i\le n} Z_i$. Here $a \ll b$ and $a \sim b$ stand for $a = O(b)$ and $a/b \to 1$ respectively. $\Phi(x)$ is the standard normal distribution function and $\phi(x)$ is its density function.

For notational convenience let $R(n) = r_n \log n \,(\log\log n)^{1+\varepsilon}$.

1. Main Result. The main result is an almost sure central limit theorem for the maximum of stationary Gaussian sequences.

THEOREM 1.1. *Let $Z_1, Z_2, \ldots$ be a standardized stationary Gaussian sequence with $R(n) = O(1)$ as $n \to \infty$. Then*

(i) *If $n(1 - \Phi(u_n)) \to \tau$ for $0 \le \tau < \infty$, then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(M_k \le u_k) = e^{-\tau} \quad \text{a.s.,}$$

(ii) *If $a_n = (2\log n)^{1/2}$ and $b_n = (2\log n)^{1/2} - \frac{1}{2}(2\log n)^{-1/2}(\log\log n + \log 4\pi)$, then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(a_k(M_k - b_k) \le x) = \exp(-e^{-x}) \quad \text{a.s..}$$

2. Auxiliary Results. The main weak convergence result for the maximum of stationary Gaussian sequence is summarized in the following theorem.

THEOREM 2.1. (Theorem 4.3.3 in Leadbetter et al. (1983)). *Let $Z_1, Z_2, \dots$ be a standardized stationary Gaussian sequence with $r_n \log n \to 0$. Then*

(i) *For $0 \le \tau < \infty$, $\mathbf{P}(M_n \le u_n) \to e^{-\tau}$ if and only if $n(1-\Phi(u_n)) \to \tau$*

(ii) $\mathbf{P}(a_n(M_n - b_n) \le x) \to \exp(-e^{-x})$,

where $a_n = (2\log n)^{1/2}$ and $b_n = (2\log n)^{1/2} - \frac{1}{2}(2\log n)^{-1/2}(\log\log n + \log 4\pi)$.

We need the following lemmas for the proof of our main result.

LEMMA 2.1. *Let $Z_1, Z_2, \dots$ be a standardized stationary Gaussian sequence. Assume that $R(n) = O(1)$ and $n(1-\Phi(u_n))$ is bounded. Then*

$$\sup_{1\le k\le n} k\sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_k^2+u_n^2}{2(1+|r_j|)}\right) \ll (\log\log n)^{-(1+\varepsilon)}.$$

PROOF OF LEMMA 2.1: Under the condition $r_n \to 0$ we have $\sup_{n\ge 1}|r_n| = \sigma < 1$ (cf., Leadbetter et al., 1983). By assumption, $n(1-\Phi(u_n)) \le K$. Let the sequence (v_n) be defined by $v_n = u_n$ if $n \le K$ and $n(1-\Phi(v_n)) = K$, if $n > K$. Then clearly $u_n \ge v_n$ and hence

$$k\sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_k^2+u_n^2}{2(1+|r_j|)}\right) \le k\sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_k^2+v_n^2}{2(1+|r_j|)}\right).$$

Thus it would be enough to prove the lemma for the sequence (v_n). By the well known fact

$$1-\Phi(x) \sim \frac{\phi(x)}{x}, \quad x\to\infty$$

we can see that

$$\exp\left(-\frac{v_n^2}{2}\right) \sim \frac{K\sqrt{2\pi}v_n}{n}, \qquad v_n \sim (2\log n)^{1/2}. \tag{2.1}$$

Define α to be $0 < \alpha < (1-\sigma)/(1+\sigma)$. Note that

$$k\sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_k^2+v_n^2}{2(1+|r_j|)}\right) =$$
$$= k\sum_{1\le j\le n^\alpha} |r_j| \exp\left(-\frac{v_k^2+v_n^2}{2(1+|r_j|)}\right) + k\sum_{n^\alpha < j\le n} |r_j| \exp\left(-\frac{v_k^2+v_n^2}{2(1+|r_j|)}\right) =$$
$$=: T_1 + T_2.$$

Using (2.1)

$$T_1 \le kn^\alpha \exp\left(-\frac{v_k^2+v_n^2}{2(1+\sigma)}\right) = kn^\alpha \left(\exp\left(-\frac{v_k^2+v_n^2}{2}\right)\right)^{1/(1+\sigma)} \ll$$
$$\ll kn^\alpha \left(\frac{v_k v_n}{kn}\right)^{1/(1+\sigma)} \ll k^{1-1/(1+\sigma)} n^{\alpha-1/(1+\sigma)} (\log k \log n)^{1/2(1+\sigma)} \le$$
$$\le n^{1+\alpha-2/(1+\sigma)} (\log n)^{1/(1+\sigma)}.$$

Since $1+\alpha-2/(1+\sigma) < 0$, we get $T_1 \le n^{-\delta}$ for some $\delta > 0$, uniformly for $1 \le k \le n$. Now we estimate the second term T_2. Setting $\sigma_n = \sup_{j\ge n} |r_j|$ and counting on $R(n) = O(1)$ as $n \to \infty$

$$\sigma_n \log n(\log\log n)^{1+\varepsilon} \le \sup_{j\ge n} |r_j| \log j(\log\log j)^{1+\varepsilon} = O(1), \quad n \to \infty. \tag{2.2}$$

Set $p = [n^\alpha]$. By (2.1) and (2.2) we have

$$\sigma_p v_k v_n \ll \sigma_{[n^\alpha]} (\log k \log n)^{1/2} \ll \sigma_{[n^\alpha]} \log n^\alpha \ll$$
$$\ll (\log\log n^\alpha)^{-(1+\varepsilon)} \sim (\log\log n)^{-(1+\varepsilon)} \tag{2.3}$$

and similarly, for $1 \le k \le n$

$$\sigma_p v_k^2 \ll (\log\log n)^{-(1+\varepsilon)}. \tag{2.4}$$

Hence using (2.1), (2.3) and (2.4)

$$T_2 \le k\sigma_p \exp\left(-\frac{v_k^2+v_n^2}{2}\right) \sum_{p\le j\le n} \exp\left(\frac{(v_k^2+v_n^2)|r_j|}{2(1+|r_j|)}\right) \le$$
$$\le kn\sigma_p \exp\left(-\frac{v_k^2+v_n^2}{2}\right) \exp\left(\frac{(v_k^2+v_n^2)\sigma_p}{2}\right) \ll (\log\log n)^{-(1+\varepsilon)}.$$

The proof is completed.

LEMMA 2.2. *Let $Z_1, Z_2, \dots$ be a standard stationary Gaussian sequence. Suppose that $\sup_{n\ge1} |r_n| < 1$. Then for $k < n$*

$$|P(M_k \le u_k, M_{k,n} \le u_n) - P(M_k \le u_k)P(M_{k,n} \le u_n)| \ll$$
$$\ll k \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_k^2+u_n^2}{2(1+|r_j|)}\right).$$

PROOF OF LEMMA 2.2. We use the following

THEOREM 2.2. (Theorem 4.2.1, Normal Comparison Lemma in Leadbetter et al. (1983)). *Suppose* $\xi_1, \dots, \xi_n$ *are standard normal variables with covariance matrix* $\Lambda^1 = (\Lambda^1_{ij})$, *and* $\eta_1, \dots, \eta_n$ *with covariance matrix* $\Lambda^0 = (\Lambda^0_{ij})$, *and let* $\rho_{ij} = \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|)$. *Further, let* $u_1, \dots, u_n$ *be real numbers. Then*

$$|P(\xi_j \le u_j,\ j = 1, \dots, n) - P(\eta_j \le u_j,\ j = 1, \dots, n)| \le$$
$$\le K \sum_{1 \le i < j \le n} |\Lambda^1_{ij} - \Lambda^0_{ij}| \exp\left(-\frac{u_i^2 + u_j^2}{2(1 + \rho_{ij})} \right).$$

Apply this Theorem with ($\xi_i = Z_i$, $i = 1, \dots, n$), ($\eta_j = Z_j$, $j = 1, \dots, k$; $\eta_j = \tilde{Z}_j$, $j = k+1, \dots, n$), where $(\tilde{Z}_{k+1}, \dots, \tilde{Z}_n)$ has the same distribution as $(Z_{k+1}, \dots, Z_n)$, but it is independent of $(Z_1, \dots, Z_k)$. Further, $u_i = u_k$, $i = 1, \dots, k$ and $u_i = u_n$, $i = k+1, \dots, n$. Then $\Lambda^1_{ij} = \Lambda^0_{ij} = r_{j-i}$ if either $1 \le i < j \le k$, or $k+1 \le i < j \le n$. Otherwise $\Lambda^1_{ij} = r_{j-i}$, $\Lambda^0_{ij} = 0$. Hence we have

$$|P(M_k \le u_k, M_{k,n} \le u_n) - P(M_k \le u_k)P(M_{k,n} \le u_n)| \ll$$
$$\ll \sum_{i=1}^{k} \sum_{j=k+1}^{n} |r_{j-i}| \exp\left(-\frac{u_k^2 + u_n^2}{2(1 + |r_{j-i}|)} \right) \le k \sum_{m=1}^{n} |r_m| \exp\left(-\frac{u_k^2 + u_n^2}{2(1 + |r_m|)} \right).$$

This completes the proof of LEMMA 2.2.

LEMMA 2.3. *Let* $Z_1, Z_2, \dots$ *be a standardized stationary Gaussian sequence. Assume that* $R(n) = O(1)$ *and* $n(1 - \Phi(u_n))$ *is bounded. Then for* $1 \le k < n$

$$\mathbf{Cov}\,(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_{k,n} \le u_n)) \ll (\log\log n)^{-(1+\varepsilon)}.$$

PROOF OF LEMMA 2.3: It follows simply from LEMMA 2.1 and LEMMA 2.2.

LEMMA 2.4. *Let* $Z_1, Z_2, \dots$ *be a standardized stationary Gaussian sequence. Assume that* $R(n) = O(1)$ *and* $n(1 - \Phi(u_n))$ *is bounded, then*

$$\mathbf{E}|\mathbf{I}(M_n \le u_n) - \mathbf{I}(M_{k,n} \le u_n)| \ll \frac{k}{n} + (\log\log n)^{-(1+\varepsilon)}.$$

PROOF OF LEMMA 2.4: Note that

$$\mathbf{E}|\mathbf{I}(M_n \le u_n) - \mathbf{I}(M_{k,n} \le u_n)| = \mathbf{P}(M_{k,n} \le u_n) - \mathbf{P}(M_n \le u_n) \le$$

$$\le |\mathbf{P}(M_{k,n} \le u_n) - \Phi^{n-k}(u_n)| + |\mathbf{P}(M_n \le u_n) - \Phi^n(u_n)| +$$

$$+ |\Phi^{n-k}(u_n) - \Phi^n(u_n)| =: D_1 + D_2 + D_3.$$

From the elementary fact that

$$x^{n-k} - x^n \le \frac{k}{n}, \qquad 0 \le x \le 1$$

we have $D_3 \le (k/n)$. By Corollary 4.2.4 in Leadbetter et al. (1983), p. 84

$$D_i \ll n \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_n^2}{1+|r_j|}\right) \quad i = 1, 2.$$

Thus by LEMMA 2.1 we have $D_i \ll (\log\log n)^{-(1+\varepsilon)}$, $i = 1, 2$.

3. Proof of Main Result. We now give the proof of THEOREM 1.1. We need the following lemma for the proof.

LEMMA 3.1. *Let $\eta_1, \eta_2, \ldots$ be a sequence of bounded random variables. If*

$$\mathbf{Var}\left(\sum_{k=1}^{n} \frac{1}{k}\eta_k\right) \ll \log^2 n \, (\log\log n)^{-(1+\varepsilon)} \quad \textit{for some} \quad \varepsilon > 0,$$

then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(\eta_k - \mathbf{E}\eta_k) = 0 \quad \text{a.s.}.$$

PROOF OF LEMMA 3.1: Setting

$$\mu_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}(\eta_k - \mathbf{E}\eta_k)$$

and $n_k = \exp(\exp(k^\nu))$ for some $\frac{1}{1+\epsilon} < \nu < 1$, we have

$$\sum_{k=3}^{\infty} \mathbf{E}\mu_{n_k}^2 \ll \sum_{k=3}^{\infty} (\log\log n_k)^{-(1+\epsilon)} \ll \sum_{k=3}^{\infty} k^{-\nu(1+\epsilon)} < \infty$$

implying $\sum_{k=3}^{\infty} \mu_{n_k}^2 < \infty$ a.s. Thus

$$\mu_{n_k} \to 0 \quad \text{a.s.}.$$

Since

$$(k+1)^{\nu} - k^{\nu} \to 0 \quad \text{as} \quad k \to \infty \quad \text{if} \quad \nu < 1,$$

we have

$$\frac{\log n_{k+1}}{\log n_k} = e^{(k+1)^{\nu} - k^{\nu}} \to 1 \quad \text{as} \quad k \to \infty.$$

Obviously for any given n there is an integer k such that $n_k < n \leq n_{k+1}$. Therefore

$$\begin{aligned} |\mu_n| &\leq \frac{1}{\log n} \left| \sum_{j=1}^{n} \frac{1}{j} \left(\eta_j - \mathbf{E}\eta_j\right) \right| \leq \\ &\leq \frac{1}{\log n_k} \left| \sum_{j=1}^{n_k} \frac{1}{j} \left(\eta_j - \mathbf{E}\eta_j\right) \right| + \frac{1}{\log n_k} \sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j} \left|\eta_j - \mathbf{E}\eta_j\right| \ll \\ &\ll |\mu_{n_k}| + \frac{1}{\log n_k} \left(\log n_{k+1} - \log n_k\right) \ll |\mu_{n_k}| + \left(\frac{\log n_{k+1}}{\log n_k} - 1 \right) \end{aligned}$$

and thus

$$\lim_{n\to\infty} \mu_n = 0 \quad \text{a.s.}.$$

PROOF OF THEOREM 1.1: First, we claim that under the assumptions that $R(n) = O(1)$ and $n(1 - \Phi(u_n))$ is bounded, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left(\mathbf{I}(M_k \leq u_k) - \mathbf{P}(M_k \leq u_k)\right) = 0 \quad \text{a.s.}. \tag{3.1}$$

In order to show this, by LEMMA 3.1 it is sufficient to show

$$\mathbf{Var}\left(\sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(M_k \leq u_k) \right) \ll (\log\log n)^{-(1+\varepsilon)} \log^2 n \quad \text{for some} \quad \varepsilon > 0. \tag{3.2}$$

Let $\eta_k = \mathbf{I}(M_k \leq u_k) - \mathbf{P}(M_k \leq u_k)$. Then

$$\mathbf{Var}\left(\sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(M_k \leq u_k) \right) = \mathbf{E}\left(\sum_{k=1}^{n} \frac{1}{k} \eta_k \right)^2 =$$

$$= \sum_{k=1}^{n} \frac{1}{k^2} \mathbf{E}|\eta_k|^2 + 2 \sum_{1\leq k<l\leq n} \frac{|\mathbf{E}(\eta_k \eta_l)|}{kl} =: \mathbf{L}_1 + \mathbf{L}_2. \tag{3.3}$$

Since $|\eta_k| \le 2$, it follows that

$$\mathbf{L}_1 \ll \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty. \tag{3.4}$$

To estimate $\mathbf{L}_2$, note that for $l > k$

$$\begin{aligned}\big|\mathbf{E}(\eta_k\eta_l)\big| = \big|\mathbf{Cov}\big(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_l \le u_l)\big)\big| \le \big|\mathbf{Cov}\big(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_l \le u_l) - \\ - \mathbf{I}(M_{k,l} \le u_l)\big)\big| + \big|\mathbf{Cov}\big(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_{k,l} \le u_l)\big)\big| \ll \\ \ll \mathbf{E}|\mathbf{I}(M_l \le u_l) - \mathbf{I}(M_{k,l} \le u_l)| + \big|\mathbf{Cov}\big(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_{k,l} \le u_l)\big)\big|.\end{aligned} \tag{3.5}$$

By LEMMA 2.3 and LEMMA 2.4 we get

$$\big|\mathbf{Cov}\big(\mathbf{I}(M_k \le u_k), \mathbf{I}(M_{k,l} \le u_l)\big)\big| \ll (\log\log l)^{-(1+\varepsilon)}$$

and

$$\mathbf{E}|\mathbf{I}(M_l \le u_l) - \mathbf{I}(M_{k,l} \le u_l)| \ll \frac{k}{l} + (\log\log l)^{-(1+\varepsilon)}.$$

Hence for $l > k$

$$|\mathbf{E}(\eta_k\eta_l)| \ll \frac{k}{l} + (\log\log l)^{-(1+\varepsilon)} \tag{3.6}$$

and consequently

$$\begin{aligned}\mathbf{L}_2 \ll \sum_{1\le k<l\le n} \frac{1}{kl}\left(\frac{k}{l}\right) + \sum_{1\le k<l\le n} \frac{1}{kl(\log\log l)^{1+\varepsilon}} = \\ =: \mathbf{L}_{21} + \mathbf{L}_{22}.\end{aligned} \tag{3.7}$$

For $\mathbf{L}_{21}$ and $\mathbf{L}_{22}$ we have the following estimates:

$$\begin{aligned}\mathbf{L}_{22} \ll \sum_{l=3}^{n} \frac{1}{l(\log\log l)^{1+\varepsilon}} \sum_{k=1}^{l-1} \frac{1}{k} \ll \sum_{l=3}^{n} \frac{\log l}{l(\log\log l)^{1+\varepsilon}} \ll \\ \ll \log n \sum_{l=3}^{n} \frac{1}{l(\log\log l)^{1+\varepsilon}} \ll \log^2 n\,(\log\log n)^{-(1+\varepsilon)}\end{aligned} \tag{3.8}$$

and

$$\text{(3.9)} \qquad \mathbf{L}_{21} \leq \sum_{1\leq k<l\leq n} \frac{1}{kl}\left(\frac{k}{l}\right) \ll \log n.$$

Thus (3.3)–(3.9) together establish (3.1).

PROOF OF (i): Note that $R(n) = O(1)$ implies $r_n \log n \to 0$. By THEOREM 4.3.3(i) in Leadbetter et al. (1983), we have $P(M_n \leq u_n) \to e^{-\tau}$. Clearly this implies

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{P}(M_k \leq u_k) = e^{-\tau}$$

which is, by (3.1), equivalent to

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(M_k \leq u_k) = e^{-\tau} \quad \text{a.s..}$$

PROOF OF (ii): By THEOREM 2.1 we have $n(1-\Phi(u_n)) \to e^{-x}$ for $u_n = x/a_n + b_n$. Thus the statement of (ii) is a special case of (i).

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