A joint functional law for the Wiener process and principal value

by

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Summary. We present a joint functional iterated logarithm law for the Wiener process and the principal value of its local times.

Keywords. Wiener process, principal value of the local time, functional law of the iterated logarithm.

2000 Mathematics Subject Classification. 60F15; 60J55; 60J65.

[1]Research supported by the Hungarian National Foundation for Scientific Research, Grant Nos. T 029621 and T 037886.

[2]Research supported by the joint French–Hungarian Intergovernmental Grant "Balaton", No. F-39/2000.

[3]Research supported by a PSC CUNY grant, No. 634680032.

1 Introduction

Let $\{W(t);\ t \geqslant 0\}$ be a one-dimensional standard Wiener process with $W(0) = 0$, and let $\{L(t,x);\ t \geqslant 0,\ x \in \mathbb{R}\}$ denote its local time process, jointly continuous in t and x. For any Borel function $f \geqslant 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geqslant 0.$$

Put $L(t,0) = L(t)$ and

$$\begin{aligned} U_t(x) &:= \frac{W(xt)}{\sqrt{2t\log\log t}}, \\ V_t(x) &:= \frac{L(xt)}{\sqrt{2t\log\log t}}, \qquad x \in [0,1]. \end{aligned}$$

We consider $x \mapsto U_t(x)$ and $x \mapsto V_t(x)$ as elements of the space $\mathcal{C} = \mathcal{C}[0,1]$ of continuous functions with metric

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Recall the celebrated functional law of the iterated logarithm (FLIL) for W due to Strassen [15]:

Theorem A *With probability one, the set $\{U_t\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}$, with limit set equal to*

$$\mathcal{S} := \left\{ f \in \mathcal{C} :\ f(0) = 0,\ f \text{ is absolutely continuous, with } \int_0^1 (f'(x))^2\,\mathrm{d}x \leqslant 1 \right\}.$$

Using that $\{L(t),\ t \geqslant 0\}$ has the same distribution as $\{\sup_{s\in[0,t]} W(s),\ t \geqslant 0\}$, one can easily obtain (cf. Csáki and Révész [7], Mueller [13], Chen [3]),

Theorem B *With probability one, the set $\{V_t\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}$, with limit set equal to*

$$\mathcal{S}_M := \{g \in \mathcal{S} :\ g \text{ is non-decreasing}\}.$$

In Csáki and Révész [7] a joint FLIL was given for the vector $\{(U_t(x), V_t(x)),\ x \in [0,1]\}_{t\geqslant 1}$ on the space $\mathcal{C}^{(2)} := \mathcal{C} \times \mathcal{C}$ with metric

$$d((f_1,g_1),(f_2,g_2)) = \sup_{x\in[0,1]} \sqrt{(f_1(x) - f_2(x))^2 + (g_1(x) - g_2(x))^2}.$$

Theorem C *With probability one, the set* $\{(U_t, V_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{ (f,g) : \ f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\, \mathrm{d}x \leqslant 1, \ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

We are interested in studying similar joint FLIL for the Wiener process and the process

$$Y(t) = \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geqslant 0.$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) = \lim_{\varepsilon \to 0+} \int_0^t \frac{\mathrm{d}s}{W(s)} 1\!\!1_{\{|W(s)|\geqslant \varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x}\, \mathrm{d}x. \tag{1.1}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the right hand side of (1.1) is well-defined.

The study of Cauchy's principal value of Brownian local time goes back at least to Itô and McKean [12], and has become very active since the late 70s, due to applications in various branches of stochastic analysis. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in Yor [17], to Chapter 10 of the lecture notes by Yor [18], and to the survey paper by Yamada [16].

The process $Y(\cdot)$ defined in (1.1) is almost surely continuous, having zero quadratic variation. It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a > 0$, $t \mapsto a^{-1/2} Y(at)$ has the same law as $t \mapsto Y(t)$. Although the aforementioned zero quadratic variation property distinguishes $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that Y behaves somewhat like a Brownian motion. Hu and Shi [11] proved a law of the iterated logarithm for $Y(\cdot)$:

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{8t \log\log t}} = 1 \qquad \text{a.s.}$$

FLIL for Y was not known before. Here we show that similarly to Theorem C, a joint FLIL for W and Y holds. Introduce

$$Z_t(x) = \frac{Y(xt)}{\sqrt{8t \log\log t}}, \qquad 0 \leqslant x \leqslant 1.$$

Our main result is

Theorem 1.1 *With probability one the set* $\{(U_t, Z_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{(f,g) : f \in \mathcal{S}, g \in \mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leqslant 1,\ f(x)g'(x) = 0 \text{ a.e.}\right\}.$$

Some consequences are as follows.

Corollary 1.2 *With probability one, the set* $\{Z_t\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}$*, with limit set equal to* $\mathcal{S}$ *given in Theorem A.*

Corollary 1.3 *With probability one, the set* $\{(U_t(1), Z_t(1))\}_{t\geqslant 1}$ *is relatively compact in* $\mathbb{R}^2$ *with limit set equal to*

$$\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1\}.$$

The organization of the paper is as follows: In Section 2 we present some preliminary results for the distribution of the Wiener process and principal value, as well as certain estimates for the increments of the processes concerned. In Section 3 we prove Theorem 1.1. In Section 4 we prove the Corollaries. Some further remarks and consequences are given in Section 5.

Throughout the paper, for any $x \in \mathbb{R}$, we denote by $\mathbb{P}^x$ the probability under which the Wiener process W starts from $W(0) = x$ (thus $\mathbb{P} = \mathbb{P}^0$); unimportant constants (which are finite and positive) are denoted by the letter c with subscript.

2 Preliminaries

2.1 Distribution results for Wiener process and principal value

First recall some results for principal value. Biane and Yor [1] proved the following result: Let $\{B(s),\ 0 \leqslant s \leqslant 1\}$ be a Brownian bridge, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} < x\right) &= \frac{|x|}{2}\sum_{n=1}^{\infty} \exp\left(-\frac{(2n-1)^2x^2}{8}\right) \\ &\geqslant \frac{|x|}{2}\exp\left(-\frac{x^2}{8}\right). \end{aligned} \tag{2.1}$$

It follows that for $0 < \alpha < \beta$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} \in (\alpha,\beta)\right) \geqslant 2\left(\exp\left(-\frac{\alpha^2}{8}\right) - \exp\left(-\frac{\beta^2}{8}\right)\right). \tag{2.2}$$

It was proved in [5] (cf. (2.11), (2.14) and (2.16) there) that for any $\delta > 0$ there exists $c_1(\delta) > 0$ such that for all $s > 0$ and $x > 0$,

$$\sup_{z\in\mathbb{R}} \mathbb{P}^z(|Y(s)| > x) \leqslant c_1(\delta)\exp\left(-\frac{x^2}{(8+\delta)s}\right). \tag{2.3}$$

Lemma 2.1 *Let $s > 0$, $\lambda > 0$, $\delta > 0$ and $0 < \varepsilon < 1$. For $(a,\alpha,z) \in \mathbb{R}^3$, define*

$$I = I(a,\alpha,z) := \mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda\right). \tag{2.4}$$

Then

$$I \leqslant \frac{\lambda}{\sqrt{s}}\exp\left(-\frac{(|a-z| - 2\varepsilon\lambda)^2 - 4\varepsilon^2\lambda^2}{2s}\right). \tag{2.5}$$

Moreover, if $|\alpha| \geqslant 4\varepsilon\lambda$, then

$$I \leqslant c_1(\delta)\exp\left(-\frac{(|\alpha| - 4\varepsilon\lambda)^2}{(8+\delta)s}\right), \tag{2.6}$$

where $c_1(\delta)$ is the constant in (2.3).

Proof: Observe that

$$I \leqslant \mathbb{P}^z(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda) = \mathbb{P}\left(\frac{a-z}{\sqrt{s}} \leqslant N \leqslant \frac{a - z + 2\varepsilon\lambda}{\sqrt{s}}\right),$$

where N is a standard normal variable. Hence (2.5) follows from a straightforward Gaussian estimate.

Now for $|\alpha| \geqslant 4\varepsilon\lambda$, we have

$$I \leqslant \mathbb{P}^z(\alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \leqslant \mathbb{P}^z(|Y(s)| \geqslant |\alpha| - 4\varepsilon\lambda),$$

which implies (2.6) by means of (2.3). □

For the lower estimates we prove several lemmas.

Lemma 2.2 *For $\alpha > 0$, $\beta - \alpha > 4$, $0 < \delta < 1$ we have*

$$\mathbb{P}(|W(1)| \leqslant 1,\, \alpha \leqslant Y(1) \leqslant \beta) \geqslant c_2(\delta) \exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right), \tag{2.7}$$

where $c_2(\delta)$ is a constant depending only on δ.

Proof: Let

$$\begin{aligned} G &:= \sup\{t : \, t \leqslant 1,\, W(t) = 0\}, \\ B(s) &:= \frac{W(sG)}{\sqrt{G}}, \qquad s \in [0,1]. \end{aligned}$$

It is known that $(B(s),\; s \in [0,1])$, G and $(\frac{W(G+s(1-G))}{\sqrt{1-G}},\; s \in [0,1])$ are independent, and that $(B(s),\; s \in [0,1])$ is a (standard) Brownian bridge.

We have

$$\begin{aligned} & \mathbb{P}(|W(1)| \leqslant 1,\, \alpha \leqslant Y(1) \leqslant \beta) \\ \geqslant\; & \mathbb{P}(|W(1)| \leqslant 1,\, \alpha+1 \leqslant Y(G) \leqslant \beta - 1,\, |Y(1) - Y(G)| \leqslant 1,\, G \geqslant 1-\delta) \\ =\; & \int_{1-\delta}^{1} \mathbb{P}(|W(1)| \leqslant 1,\, \alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G = \kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa) \\ =\; & \int_{1-\delta}^{1} \mathbb{P}(\alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G = \kappa) \times \\ & \qquad \times \mathbb{P}(|W(1)| \leqslant 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G = \kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa). \end{aligned}$$

Since under the condition $G = \kappa$, $Y(\kappa)/\sqrt{\kappa}$ has the same distribution as $\int_0^1 \mathrm{d}s/B(s)$, where B is a Brownian bridge, we get from (2.2)

$$\begin{aligned} \mathbb{P}(\alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G = \kappa) &\geqslant 2\left(\exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) - \exp\left(-\frac{(\beta-1)^2}{8\kappa}\right)\right) \\ &\geqslant 2(1 - \mathrm{e}^{-1}) \exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) \\ &\geqslant 2(1 - \mathrm{e}^{-1}) \exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right). \end{aligned}$$

This gives (2.7), with

$$c_2(\delta) := 2(1 - \mathrm{e}^{-1})\mathbb{P}(|W(1)| \leqslant 1,\, |Y(1) - Y(G)| \leqslant 1,\, G \geqslant 1-\delta).$$

The lemma is proved. □

Now we introduce the notation

$$T_b := \inf\{t : t \geqslant 0, W(t) = b\}. \tag{2.8}$$

By the reflection principle, we have for all $u > 0$ and $(a, z) \in \mathbb{R}^2$,

$$\mathbb{P}^z(T_a \leqslant u) = 2\overline{\Phi}\left(\frac{|z-a|}{\sqrt{u}}\right), \tag{2.9}$$

where $\overline{\Phi}(x) := \mathbb{P}(N > x)$ is the standard Gaussian tail distribution function.

In the sequel we shall use the inequalities:

$$\overline{\Phi}(x) \geqslant \frac{1}{(2\pi)^{1/2}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right), \qquad x \geqslant 1, \tag{2.10}$$

$$\overline{\Phi}(x) \leqslant \frac{1}{2}\exp\left(-\frac{x^2}{2}\right), \qquad x > 0. \tag{2.11}$$

(For (2.11), see Proposition II.1.8 of Revuz and Yor [14].)

Lemma 2.3 *For $s > 0$, $0 < \delta < 1$, $z \in \mathbb{R}$ we have*

$$\mathbb{P}^z(T_0 \leqslant \delta s, \, |Y(T_0)| \leqslant 2\sqrt{s}) \geqslant c_3(\delta)\,\overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right). \tag{2.12}$$

Proof: By symmetry, it suffices to prove (2.12) for $z > 0$ (there is nothing to prove if $z = 0$). Assuming first $z > \sqrt{s}$, we have

$$\begin{aligned} & \mathbb{P}^z(T_0 \leqslant \delta s, \, |Y(T_0)| \leqslant 2\sqrt{s}\,) \\ \geqslant \; & \mathbb{P}^z(T_0 - T_{\sqrt{s}} \leqslant \delta(1-\delta)s, \, T_{\sqrt{s}} \leqslant \delta^2 s, \, Y(T_0) - Y(T_{\sqrt{s}}) \leqslant \sqrt{s}) \\ = \; & \mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)\,\mathbb{P}^z(T_{\sqrt{s}} \leqslant \delta^2 s), \end{aligned}$$

where we used the fact that $T_{\sqrt{s}} \leqslant \delta^2 s$ implies $Y(T_{\sqrt{s}}) \leqslant T_{\sqrt{s}}/\sqrt{s} \leqslant \delta^2\sqrt{s} < \sqrt{s}$.

By scaling, $\mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)$ is a positive constant depending only on δ. In view of (2.9), we have proved (2.12) in case $z > \sqrt{s}$.

If $0 < z \leqslant \sqrt{s}$, we have, by scaling,

$$\begin{aligned} \mathbb{P}^z(T_0 \leqslant \delta s, \, |Y(T_0)| \leqslant 2\sqrt{s}\,) \; & = \; \mathbb{P}^1\left(T_0 \leqslant \frac{\delta s}{z^2}, \, |Y(T_0)| \leqslant \frac{2\sqrt{s}}{z}\right) \\ & \geqslant \; \mathbb{P}^1\left(T_0 \leqslant \delta, \, |Y(T_0)| \leqslant 2\right) \\ & =: \; c_4(\delta), \end{aligned}$$

from which (2.12) follows. □

Lemma 2.4 *Let $s > 0$, $\varepsilon > 0$, $\lambda > 0$, $0 < \delta < 1$, $(\alpha, z) \in \mathbb{R}^2$ be such that $\varepsilon\lambda > 8\sqrt{s}$. Then we have*

$$\begin{aligned} & \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \\ \geqslant\ & c_5(\delta) \exp\left(-\frac{(|\alpha| + 2\varepsilon\lambda)^2}{8s(1-\delta)^2} \right) \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}} \right). \end{aligned} \tag{2.13}$$

Proof: Define, for $n \geqslant 1$,

$$I_{\lambda,z}(\alpha, n) := \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda).$$

Note that $I_{\lambda,z}(\alpha, n)$ is non-decreasing in n. Moreover,

$$\begin{aligned} I_{\lambda,z}(\alpha, n) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda,\, T_0 \leqslant \delta s) \\ &= \int_0^{\delta s} \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \\ &\geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \cap B_\tau(n) \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau), \end{aligned}$$

where

$$\begin{aligned} A_\tau &:= \{|Y(\tau)| \leqslant 2\sqrt{s}\}, \\ B_\tau(n) &:= \{|W(s)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s) - Y(\tau) \leqslant \alpha + n\varepsilon\lambda - 2\sqrt{s}\}. \end{aligned}$$

Under the condition $\{W(0) = z,\, T_0 = \tau\}$, A_τ and $B_\tau(n)$ are independent, so that

$$I_{\lambda,z}(\alpha, n) \geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \times \inf_{\tau \in (0,\, \delta s)} \mathbb{P}^z(B_\tau(n) \,|\, T_0 = \tau).$$

By Lemma 2.3,

$$\begin{aligned} \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) &= \mathbb{P}^z(|Y(T_0)| \leqslant 2\sqrt{s},\, T_0 \leqslant \delta s) \\ &\geqslant c_3(\delta)\, \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}} \right), \end{aligned}$$

whereas according to Lemma 2.2, and by scaling,

$$\begin{aligned} & \mathbb{P}^z(B_\tau(1) \,|\, T_0 = \tau) \\ =\ & \mathbb{P}(|W(s - \tau)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s - \tau) \leqslant \alpha + \varepsilon\lambda - 2\sqrt{s}) \\ \geqslant\ & \mathbb{P}\left(|W(1)| \leqslant 1,\, \frac{\alpha + 2\sqrt{s}}{\sqrt{s - \tau}} \leqslant Y(1) \leqslant \frac{\alpha + \varepsilon\lambda - 2\sqrt{s}}{\sqrt{s - \tau}} \right). \end{aligned}$$

Assume $\alpha \geqslant 0$ for the moment. By Lemma 2.2,

$$\mathbb{P}^z(B_\tau(1)\,|\,T_0=\tau) \geqslant c_2(\delta)\, \exp\left(-\frac{(\alpha+\varepsilon\lambda)^2}{8s(1-\delta)^2}\right),$$

which yields

$$I_{\lambda,z}(\alpha,1) \geqslant c_6(\delta)\, \exp\left(-\frac{(\alpha+\varepsilon\lambda)^2}{8s(1-\delta)^2}\right) \overline{\Phi}\left(\frac{||z|-\sqrt{s}|}{\delta\sqrt{s}}\right), \qquad \alpha \geqslant 0, \tag{2.14}$$

with $c_6(\delta) := c_3(\delta)c_2(\delta)$. Since $I_{\lambda,z}(\alpha,4) \geqslant I_{\lambda,z}(\alpha,1)$, this yields (2.13) in case $\alpha \geqslant 0$.

To treat the case $\alpha \leqslant -\varepsilon\lambda$, we observe that

$$\begin{aligned} I_{\lambda,z}(\alpha,4) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\ \alpha \leqslant Y(s) \leqslant \alpha+\varepsilon\lambda) \\ &= \mathbb{P}^{-z}(|W(s)| \leqslant \varepsilon\lambda,\ -\alpha-\varepsilon\lambda \leqslant Y(s) \leqslant -\alpha), \end{aligned}$$

the last identity following via replacing W by $-W$. This gives $I_{\lambda,z}(\alpha,4) \geqslant I_{\lambda,-z}(-\alpha-\varepsilon\lambda,1)$. Since $-\alpha-\varepsilon\lambda \geqslant 0$, we are entitled to apply (2.14) to deduce (2.13).

It remains to study the situation $\alpha \in (-\varepsilon\lambda, 0)$. In this case,

$$I_{\lambda,z}(\alpha,4) \geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\ \alpha+\varepsilon\lambda \leqslant Y(s) \leqslant \alpha+2\varepsilon\lambda) = I_{\lambda,z}(\alpha+\varepsilon\lambda,1),$$

which yields (2.13) in view of (2.14).

Lemma 2.4 is proved. □

Lemma 2.5 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $\varepsilon^2\lambda^2 \geqslant 2s$, $az>0$, and*

$$|z| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda, \quad |a| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda$$

we have

$$\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda) \geqslant \frac{1}{2}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right). \tag{2.15}$$

Proof: It suffices to prove the lemma for $z > \frac{s}{2\varepsilon\lambda} + \varepsilon\lambda$ and $a > \frac{s}{2\varepsilon\lambda} + \varepsilon\lambda$ (then by symmetry, it will also cover the case $a<0$ and $z<0$). We have,

$$\begin{aligned} &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda) \\ \geqslant\ &\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u) > \frac{s}{2\varepsilon\lambda},\ a \leqslant W(s) \leqslant a+2\varepsilon\lambda\right) \\ =\ &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda) \end{aligned}$$

$$-\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u)\leqslant \frac{s}{2\varepsilon\lambda},\ a\leqslant W(s)\leqslant a+2\varepsilon\lambda\right).$$

By the reflection principle,

$$\begin{aligned}
&\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u)\leqslant \frac{s}{2\varepsilon\lambda},\ a\leqslant W(s)\leqslant a+2\varepsilon\lambda\right)\\
&= \mathbb{P}^z\left(\frac{s}{\varepsilon\lambda}-a-2\varepsilon\lambda\leqslant W(s)\leqslant \frac{s}{\varepsilon\lambda}-a\right)\\
&\leqslant \mathbb{P}\left(\frac{W(s)}{\sqrt{s}}\leqslant -\frac{a+z-\frac{s}{\varepsilon\lambda}}{\sqrt{s}}\right)\\
&\leqslant \frac{1}{2}\exp\left(-\frac{\left(a+z-\frac{s}{\varepsilon\lambda}\right)^2}{2s}\right),
\end{aligned}$$

the last inequality following from (2.11). On the other hand,

$$\begin{aligned}
\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda) &= \mathbb{P}\left(\frac{a-z}{\sqrt{s}}\leqslant \frac{W(s)}{\sqrt{s}}\leqslant \frac{a-z+2\varepsilon\lambda}{\sqrt{s}}\right)\\
&\geqslant \frac{2\varepsilon\lambda}{\sqrt{2\pi s}}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right)\\
&\geqslant \exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right).
\end{aligned}$$

Since $a+z-s/(\varepsilon\lambda)\geqslant |a-z|+2\varepsilon\lambda$, we obtain (2.15). □

Lemma 2.6 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $az<0$, $|a|>2\varepsilon\lambda+\sqrt{s}$ and $\min(\varepsilon\lambda/2,\ |z|)>\sqrt{s}$, we have*

$$\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leqslant 2\varepsilon\lambda)\geqslant c_7(\delta)\,\overline{\Phi}\left(\frac{|a-z|+2\varepsilon\lambda}{(1-\delta)\sqrt{s}}\right). \tag{2.16}$$

Proof: First we show for $a>\sqrt{u}$, $\varepsilon\lambda>2\sqrt{u}$,

$$P(u):=\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda,\ |Y(u)|\leqslant 2\sqrt{u})\geqslant c_8(\delta)\ \exp\left(-\frac{a^2}{2(1-\delta)u}\right). \tag{2.17}$$

Define $G_{\sqrt{u}}:=\sup\{t\leqslant u:\ W(t)=\sqrt{u}\}$. Then

$$\begin{aligned}
P(u) &\geqslant \int_0^{\delta u}\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda,\ |Y(u)|\leqslant 2\sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v)\\
&\geqslant \int_0^{\delta u}\mathbb{P}(|Y(v)|\leqslant \sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(a\leqslant W(u)\leqslant a+2\varepsilon\lambda\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v).
\end{aligned}$$

Under the condition $\{G_{\sqrt{u}} = v\}$, $\{M(r) := \frac{W(v+r(u-v))-\sqrt{u}}{\sqrt{u-v}},\ r \in [0,1]\}$ is a standard Brownian meander, and from the well-known identity (Biane and Yor [1]) $\mathbb{P}(M(1) \leqslant x) = 1 - \exp(-x^2/2)$, we get that, for $v \in [0, \delta u]$, $a > \sqrt{u}$ and $\varepsilon\lambda > 2\sqrt{u}$,

$$
\begin{aligned}
\mathbb{P}(a \leqslant W(u) \leqslant a + 2\varepsilon\lambda \,|\, G_{\sqrt{u}} = v) &= \mathbb{P}\left(\frac{a-\sqrt{u}}{\sqrt{u-v}} \leqslant M(1) \leqslant \frac{a - \sqrt{u} + 2\varepsilon\lambda}{\sqrt{u-v}}\right) \\
&= \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right) - \exp\left(-\frac{(a-\sqrt{u}+2\varepsilon\lambda)^2}{2(u-v)}\right) \\
&\geqslant c_9 \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right) \\
&\geqslant c_9 \exp\left(-\frac{a^2}{2(1-\delta)u}\right),
\end{aligned}
$$

where $c_9 > 0$ is an absolute constant. Hence

$$
P(u) \geqslant c_{10}(\delta) \exp\left(-\frac{a^2}{2(1-\delta)u}\right),
$$

with

$$
\begin{aligned}
c_{10}(\delta) &:= c_9 \int_0^{\delta u} \mathbb{P}(|Y(v)| \leqslant \sqrt{u} \,|\, G_{\sqrt{u}} = v)\, \mathbb{P}(G_{\sqrt{u}} \in \mathrm{d}v) \\
&= c_9\, \mathbb{P}(G_{\sqrt{u}} \leqslant \delta u,\ |Y(G_{\sqrt{u}})| \leqslant \sqrt{u}\,),
\end{aligned}
$$

which, by scaling, does not depend on u. This yields (2.17).

We now start proving (2.16). Let $\varepsilon\lambda > 2\sqrt{s}$. Let T_0 and $T_{-\sqrt{s}}$ be as in (2.8). It suffices to prove (2.16) for $z < -\sqrt{s}$ and $a > \sqrt{s}$ (then by symmetry, it will also cover the case $z > \sqrt{s}$, $a < -2\varepsilon\lambda - \sqrt{s}$). Since $|Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}$ under $\mathbb{P}^z$ (recalling that $z < -\sqrt{s}$), we have

$$
\begin{aligned}
&\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda\right) \\
\geqslant\ &\mathbb{P}^z\big(T_0 - T_{-\sqrt{s}} \leqslant \delta s,\ |Y(T_0) - Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}, \\
&\qquad a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s) - Y(T_{-\sqrt{s}})| \leqslant 2\sqrt{s}\big).
\end{aligned}
$$

By the strong Markov property at times $T_{-\sqrt{s}}$ and T_0, we get:

$$
\begin{aligned}
&\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda\right) \\
\geqslant\ &\int_0^{\delta s} \left(\int_0^{s-y} P(s-h-y)\, \mathbb{P}^z(T_{-\sqrt{s}} \in \mathrm{d}h)\right) \mathbb{P}^{-\sqrt{s}}\left(T_0 \in \mathrm{d}y,\ |Y(T_0)| \leqslant \sqrt{s}\right).
\end{aligned}
\tag{2.18}
$$

By (2.17),

$$
P(s-h-y) \geqslant c_8(\delta) \exp\left(-\frac{a^2}{2(1-\delta)(s-h-y)}\right)
$$

$$
\begin{aligned}
&\geqslant 2c_8(\delta)\,\overline{\Phi}\left(\frac{a}{\sqrt{(1-\delta)(s-h-y)}}\right)\\
&= 2c_8(\delta)\,\mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y) > \frac{a}{\sqrt{1-\delta}} - z - \sqrt{s}\right),
\end{aligned}
$$

the second inequality being a consequence of (2.11). Therefore, for $y \in [0, \delta s]$,

$$
\begin{aligned}
&\int_0^{s-y} P(s-h-y)\,\mathbb{P}^z(T_{-\sqrt{s}} \in \mathrm{d}h)\\
\geqslant\ & 2c_8(\delta)\int_0^{s-y} \mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y) > \frac{a}{\sqrt{1-\delta}} - z - \sqrt{s}\right)\mathbb{P}^z\left(T_{-\sqrt{s}} \in \mathrm{d}h\right)\\
=\ & 2c_8(\delta)\,\mathbb{P}\left(W(s-y) > \frac{a}{\sqrt{1-\delta}} - z - \sqrt{s}\right)\\
=\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{\frac{a}{\sqrt{1-\delta}} - z - \sqrt{s}}{\sqrt{s-y}}\right)\\
\geqslant\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}
$$

(recalling that $z < 0$). Plugging this into (2.18), we get

$$
\begin{aligned}
&\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda\right)\\
\geqslant\ & 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right)\mathbb{P}^{-\sqrt{s}}\left(T_0 \leqslant \delta s,\ |Y(T_0)| \leqslant \sqrt{s}\right)\\
=\ & c_{11}(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}
$$

where $c_{11}(\delta) := 2c_8(\delta)\,\mathbb{P}^{-1}(T_0 \leqslant \delta,\ |Y(T_0)| \leqslant 1)$. This yields (2.16). □

Lemma 2.7 *For $s > 0$, $\varepsilon > 0$, $\lambda > 0$, $(a, z) \in \mathbb{R}^2$ such that $\varepsilon\lambda > 2\sqrt{s}$ and $|a| > 2\varepsilon\lambda + \sqrt{s}$, we have*

$$
\begin{aligned}
&\mathbb{P}^z(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leq 2\varepsilon\lambda)\\
\geqslant\ & c_{12}(\delta)\,\exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right)\overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right),
\end{aligned}
\tag{2.19}
$$

with a constant $c_{12}(\delta) > 0$.

Proof: Again, it suffices to treat the case $a > \sqrt{s}$. In this case, we have

$$
\mathbb{P}^z(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ |Y(s)| \leq 2\varepsilon\lambda)
$$

$$\geqslant \int_0^{\delta s} \mathbb{P}^z(|Y(T_0)| \le 2\sqrt{s},\, T_0 \in \mathrm{d}h)\, \mathbb{P}(a \le W(s-h) \le a + 2\varepsilon\lambda,\, |Y(s-h)| \le 2\sqrt{s-h}\,)$$
$$\geqslant c_{13}(\delta) \exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right) \mathbb{P}^z(|Y(T_0)| \le 2\sqrt{s},\, T_0 \le \delta s),$$

hence (2.19) follows from Lemma 2.3. □

2.2 Increments

Recall the results for the increments of Wiener process (cf. [9]) and principal value (cf. [4]).

As $T \to \infty$, we have almost surely

$$\sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} |X(t+s) - X(t)| = \mathcal{O}\left(\sqrt{a_T(\log(T/a_T) + \log\log T)}\right), \tag{2.20}$$

and for fixed T, as $\delta \to 0$ we have almost surely

$$\sup_{0\leqslant t\leqslant T} \sup_{0\leqslant s\leqslant \delta} |X(t+s) - X(t)| = \mathcal{O}(\sqrt{\delta \log(1/\delta)}). \tag{2.21}$$

Here in (2.20) and (2.21) X can be either W or Y.

3 Proof of Theorem 1.1

According to (2.20) for Y,

$$\lim_{\delta\to 0} \sup_{t\geqslant 1} \sup_{0\leqslant x,x'\leqslant 1,\, |x-x'|\leqslant\delta} |Z_t(x) - Z_t(x')| \to 0, \qquad \text{a.s.}$$

Now the relative compactness of $\{Z_t\}_{t\geqslant 1}$ in $\mathcal{C}$ follows from the Arzelà–Ascoli theorem. This fact and Theorem A imply that $\{(U_t, Z_t)\}$ is relatively compact in $\mathcal{C}^{(2)}$. Our further proof will consist of two steps:

(1) With probability one any $(f,g) \notin \widetilde{\mathcal{S}}_J^{(2)}$ is not a limit point.

(2) With probability one every $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point.

Proof of (1): Obviously, if either $f \notin \mathcal{S}$, or $g(0) \neq 0$, then (f,g) cannot be a limit point almost surely. So from now on we assume that $f \in \mathcal{S}$ and $g(0) = 0$. Let $x_0 \in (0,1]$ be a point, where $f(x_0) \neq 0$. Since f is continuous, there exists an interval $(x_1, x_2) \subset [0,1]$ such that $x_0 \in (x_1, x_2]$ and $f(x) \neq 0$ for all $x \in (x_1, x_2)$. We show that if (f, g) is a limit point, then g is constant in (x_1, x_2). Since (f, g) is a limit point, there exists a sequence $\{t_n\}_{n\geqslant 1}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$|W(xt_n)| \geqslant c_{14} \sqrt{2t_n \log\log t_n}, \quad x \in (x_1, x_2)$$

for some $c_{14} > 0$ and for every $x \in (x_1, x_2)$

$$\begin{aligned} \left| \frac{Y(xt_n) - Y(x_0 t_n)}{\sqrt{8t_n \log\log t_n}} \right| &= \frac{1}{\sqrt{8t_n \log\log t_n}} \left| \int_{x_0 t_n}^{xt_n} \frac{\mathrm{d}s}{W(s)} \right| \\ &\leqslant \frac{|xt_n - x_0 t_n|}{4c_{14}\, t_n \log\log t_n} \to 0 = g(x) - g(x_0), \end{aligned}$$

as $n \to \infty$. So $g(x) = g(x_0)$ for every $x \in (x_1, x_2)$. So if (f, g) is a limit point and g is absolutely continuous (which is not guaranteed so far), then we must have $f(x)g'(x) = 0$ a.e.

To this end, we need a lemma.

Lemma 3.1 *Let (f, g) be such that $f \in \mathcal{S}$, $g(0) = 0$ and either g is not absolutely continuous or $f(x)g'(x) = 0$ a.e., and*

$$\int_0^1 \left((f'(x))^2 + (g'(x))^2 \right) \mathrm{d}x > 1, \tag{3.1}$$

holds. Then there exists a partition $x_0 = 0 < x_1 < \ldots < x_{k-1} < x_k = 1$ of $[0, 1]$ such that for any $\delta > 0$ small enough, we have

$$\Lambda_k := \sum_{i=1}^{k} \left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} \mathbb{1}_{\{g_i - g_{i-1} = 0\}} + \frac{8}{8+\delta} \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} \right) > 1, \tag{3.2}$$

where $f_i := f(x_i)$ and $g_i := g(x_i)$.

Proof: If g is not absolutely continuous, then we can clearly find a partition $x_0 = 0 < x_1 < \ldots < x_{j-1} < x_j = 1$ of $[0, 1]$ such that

$$\sum_{i=1}^{k} \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} > 1 + \frac{\delta}{8},$$

so we have also (3.2). If g is absolutely continuous and (3.1) holds, then we can find a partition $x_0 = 0 < x_1 < \ldots < x_{j-1} < x_j = 1$ of $[0, 1]$ such that

$$\sum_{i=1}^{j} \left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} + \frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}} \right) > 1$$

holds. Moreover, for any small enough $\delta > 0$, we have also

$$\sum_{i=1}^{j}\left(\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} + \frac{8}{8+\delta}\frac{(g_i - g_{i-1})^2}{x_i - x_{i-1}}\right) > 1.$$

For the ith interval of the above partition consider the following three cases: (i) $f_{i-1} = f_i$, (ii) $f_{i-1} \neq f_i$, and $f_{i-1}f_i \geqslant 0$, (iii) $f_{i-1} \neq f_i$, and $f_{i-1}f_i < 0$. In case (i) we can simply write

$$\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} = \frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}}\mathbb{1}_{\{g_i - g_{i-1} = 0\}}.$$

In case (ii) let $x_i' = \max\{x \leqslant x_i : f(x) = f(x_{i-1})\}$ and $x_i'' = \min\{x \geqslant x_i' : f(x) = f(x_i)\}$. (It is possible that $x_i' = x_{i-1}$ or $x_i'' = x_i$.) Consider the refinement of the partition by replacing (x_{i-1}, x_i) with (x_{i-1}, x_i'), (x_i', x_i''), (x_i'', x_i). In the interval (x_i', x_i'') $f(x)$ must strictly be between f_{i-1} and f_i, so $f(x) \neq 0$, hence $g'(x) = 0$ for all $x \in (x_i', x_i'')$, thus $g(x_i') = g(x_i'')$. Using the elementary inequality

$$\frac{(a+b)^2}{c+d} \leqslant \frac{a^2}{c} + \frac{b^2}{d},$$

we may write

$$\begin{aligned}
\frac{(f_i - f_{i-1})^2}{x_i - x_{i-1}} &\leqslant \frac{(f(x_i') - f_{i-1})^2}{x_i' - x_{i-1}} + \frac{(f(x_i'') - f(x_i'))^2}{x_i'' - x_i'} + \frac{(f_i - f(x_i''))^2}{x_i - x_i''} \\
&= \frac{(f(x_i') - f_{i-1})^2}{x_i' - x_{i-1}}\mathbb{1}_{\{g(x_i') - g_{i-1} = 0\}} + \frac{(f(x_i'') - f(x_i'))^2}{x_i'' - x_i'}\mathbb{1}_{\{g(x_i'') - g(x_i') = 0\}} \\
&+ \frac{(f_i - f(x_i''))^2}{x_i - x_i''}\mathbb{1}_{\{g_i - g(x_i'') = 0\}}.
\end{aligned} \tag{3.3}$$

In case (iii) let $x_i' = \min\{x \geqslant x_{i-1} : f(x) = 0\}$ and $x_i'' = \max\{x \leqslant x_i : f(x) = 0\}$. Consider again the refinement of the partition by replacing (x_{i-1}, x_i) with (x_{i-1}, x_i'), (x_i', x_i''), (x_i'', x_i). In the first and the last of these three intervals $f(x) \neq 0$, hence $g'(x) = 0$, thus $g(x_i') = g_{i-1}$ and $g(x_i'') = g_i$. On the other hand, $f(x_i') = f(x_i'') = 0$. So we again have (3.3).

By repeating this argument, we get finally a partition for which (3.2) holds. This completes the proof of the Lemma. □

Returning to the main course of the proof, choose $\varepsilon > 0$ such that

$$\Lambda_k - 20\varepsilon\sum_{i=1}^{k}\frac{1}{x_i - x_{i-1}} > 1,$$
$$(f_{i-1} - \varepsilon, f_{i-1} + \varepsilon) \quad \text{and} \quad (f_i - \varepsilon, f_i + \varepsilon) \quad \text{are disjoint if } f_i \neq f_{i-1},$$

$$|g_i - g_{i-1}| > 6\varepsilon \ \text{ if } \ g_i \neq g_{i-1}.$$

Here $f_i = f(x_i)$ and $g_i = g(x_i)$, $i = 1, \dots, k$. We may also assume that $|f_i - f_{i-1}| \leqslant 1$ and $|g_i - g_{i-1}| \leqslant 1$, $i = 1, \dots, k$, otherwise (f, g) cannot be a limit point by the usual law of the iterated logarithm.

Define the events

$$\begin{aligned} A_t^{(i)} &= \{f_i - \varepsilon \leqslant U_t(x_i) \leqslant f_i + \varepsilon,\ g_i - g_{i-1} - 2\varepsilon \leqslant Z_t(x_i) - Z_t(x_{i-1}) \leqslant g_i - g_{i-1} + 2\varepsilon\} \\ &= \{a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i\} \end{aligned}$$

with $s_i = x_i t$ and

$$a_i = (f_i - \varepsilon)(2t \log\log t)^{1/2}, \quad b_i = (f_i + \varepsilon)(2t \log\log t)^{1/2},$$

$$\alpha_i = (g_i - g_{i-1} - 2\varepsilon)2(2t \log\log t)^{1/2}, \quad \beta_i = (g_i - g_{i-1} + 2\varepsilon)2(2t \log\log t)^{1/2}.$$

It follows from Lemma 2.1 putting $\lambda = (2t \log\log t)^{1/2}$ there

$$\mathbb{P}(A_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \leqslant \sqrt{\frac{2 \log\log t}{x_i - x_{i-1}}} \exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t\right)$$

and if $g_i \neq g_{i-1}$, then

$$\mathbb{P}(A_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \leqslant c_{15} \exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8+\delta)(x_i - x_{i-1})} 8 \log\log t\right)$$

with some $c_{15} > 0$. So for large enough t we have

$$\begin{aligned} \mathbb{P}(A_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \ &\leqslant \ c_{16} \sqrt{\frac{\log\log t}{x_i - x_{i-1}}} \left[\exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t\right) \mathbb{1}_{\{g_i = g_{i-1}\}}\right. \\ &+ \ \left.\exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8+\delta)(x_i - x_{i-1})} 8 \log\log t\right) \mathbb{1}_{\{g_i \neq g_{i-1}\}}\right]. \end{aligned}$$

It follows that for all large t and some constants $c_{17} > 0$ and $\widetilde{\delta} > 0$,

$$\begin{aligned} \mathbb{P}(\cap_{i=1}^k A_t^{(i)}) \ &\leqslant \ c_{17} (\log\log t)^{3k/2} \exp\left(-\left(\Lambda_k - 20\varepsilon \sum_{i=1}^k \frac{1}{x_i - x_{i-1}}\right) \log\log t\right) \\ &\leqslant \ \exp(-(1 + \widetilde{\delta}) \log\log t). \end{aligned}$$

Let $t = t_n = \exp(n/(\log n))$. Then $\sum_n \mathbb{P}(A_{t_n}) < \infty$. By the Borel–Cantelli lemma,

$$\liminf_{n\to\infty} d\left((U_{t_n}, Z_{t_n}),\, (f, g)\right) \geqslant \varepsilon \qquad \text{a.s.} \tag{3.4}$$

On the other hand, we infer from the increment results in Section 2.3 that

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |U_t(x) - U_{t_n}(x)| = 0 \qquad \text{a.s.,} \tag{3.5}$$

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |Z_t(x) - Z_{t_n}(x)| = 0 \qquad \text{a.s.,} \tag{3.6}$$

Combining (3.5)–(3.6) with (3.4) gives that

$$\liminf_{t\to\infty} d\left((U_t, Z_t)\,,\,(f,g)\right) \geqslant \varepsilon \qquad \text{a.s.}$$

for some $\varepsilon > 0$.

Thus we proved that if $(f,g) \notin \widetilde{\mathcal{S}}_J^{(2)}$, then it is not a limit point with probability one, i.e. (f,g) has an open ball neighboorhood of radius ε not containing (U_t, Z_t) for large enough t. However the exceptional ω-set of probability zero may depend on (f,g). Now we prove that the totality of these exceptional ω-sets is still of probability zero. Denote the complement of $\widetilde{\mathcal{S}}_J^{(2)}$ by $\mathcal{D}$ and for each $(f,g) \in \mathcal{D}$ consider the open balls defined above. Their union covers $\mathcal{D}$ and being $\mathcal{C}^{(2)}$ separable, we can select a countable subcover (cf. e.g. [2], p. 217). The union of exceptional ω-sets belonging to this countable subcover is still of probability zero. We call the complement of this last set of probability zero as our universal ω-set. Each $(f,g) \in \mathcal{D}$ has a neighborhood which is completely contained in one of the elements of the countable subcover, hence on the universal set this neighborhood for large enough t does not contain (U_t, Z_t), i.e. (f,g) is not a limit point. This completes the proof of (1). □

Proof of (2): Assume that $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ with strict inequality in the integral criterion, i.e.

$$\int_0^1 \left((f'(x))^2 + (g'(x))^2\right) \mathrm{d}x < 1.$$

For given $\varepsilon_1 > 0$, choose a partition $x_0 = 0 < x_1 \ldots < x_k = 1$ of the interval $[0,1]$ such that

$$\begin{aligned} \sup_{1\leqslant i\leqslant k} (x_i - x_{i-1}) &\leqslant \varepsilon_1^2, \\ \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |f(x) - f_i| &\leqslant \varepsilon_1, \\ \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |g(x) - g_i| &\leqslant \varepsilon_1, \end{aligned}$$

where $f_i = f(x_i)$, $g_i = g(x_i)$. We may assume that if $g_{i-1} \neq g_i$, then $f_{i-1} = f_i = 0$. Otherwise if it happens that $g_{i-1} \neq g_i$ but either $f_{i-1} \neq 0$ or $f_i \neq 0$ (or both), then we can choose $x' = \min\{x : x > x_{i-1},\, f(x) = 0\}$, $x'' = \max\{x : x < x_i,\, f(x) = 0\}$. We must have

$g(x') = g_{i-1}$ and $g(x'') = g_i$ so by refining the original partition by inserting new points x', x'', the new partition satisfies the above assumption. Since

$$\frac{(f(x_i) - f(x_{i-1})^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (f'(x))^2 \, \mathrm{d}x, \qquad \frac{(g(x_i) - g(x_{i-1}))^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (g'(x))^2 \, \mathrm{d}x,$$

(cf. for example [10], p. 52), we have

$$\bar{\Lambda}_k := \sum_{i=1}^{k} \frac{(f_i - f_{i-1})^2 + (g_i - g_{i-1})^2}{x_i - x_{i-1}} < 1. \tag{3.7}$$

Now choose $0 < \delta < 1$ such that $\bar{\Lambda}_k < (1-\delta)^2$ and then choose $\varepsilon > 0$ such that

$$\Gamma := \frac{\bar{\Lambda}_k}{(1-\delta)^2} + \left(\frac{20\varepsilon}{(1-\delta)^2} + \frac{2\varepsilon}{\delta^2} \right) \sum_{i=1}^{k} \frac{1}{x_i - x_{i-1}} < 1 \tag{3.8}$$

and $5\varepsilon < |f_i|$, $i = 1, 2, \ldots, k$.

Introduce the notations $\lambda = (2t \log\log t)^{1/2}$, $s_i = t x_i$,

$$a_i = (f_i - \varepsilon)\lambda, \qquad b_i = (f_i + \varepsilon)\lambda,$$

$$\alpha_i = 2(g_i - g_{i-1} - \varepsilon)\lambda, \qquad \beta_i = 2(g_i - g_{i-1} + \varepsilon)\lambda.$$

By using the strong Markov property of the Wiener process, it is readily seen that

$$\begin{aligned} & \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k) \\ \geqslant\ & \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k-1) \times \\ & \times \inf_{a_{k-1} \leqslant z_{k-1} \leqslant b_{k-1}} \mathbb{P}(a_k \leqslant W(s_k) \leqslant b_k,\, \alpha_k \leqslant Y(s_k) - Y(s_{k-1}) \leqslant \beta_k \mid W(s_{k-1}) = z_{k-1}). \end{aligned}$$

Iterating this argument we can see that

$$\begin{aligned} & \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k) \\ \geqslant & \prod_{i=1}^{k} \inf_{a_{i-1} \leqslant z_{i-1} \leqslant b_{i-1}} \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i \mid W(s_{i-1}) = z_{i-1}). \end{aligned} \tag{3.9}$$

Next we show that for $i = 1, 2, \ldots, k$ we have

$$\begin{aligned} & \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i \mid W(s_{i-1}) = z_{i-1}) \geqslant \frac{c_{18}(\delta)}{(\log\log t)^{1/2}} \times \\ & \times \exp\left(- \left(\frac{(f_i - f_{i-1})^2 + (g_i - g_{i-1})^2 + 20\varepsilon}{(1-\delta)^2 (x_i - x_{i-1})} + \frac{2\varepsilon}{\delta^2 (x_i - x_{i-1})} \right) \log\log t \right) \end{aligned} \tag{3.10}$$

with some $c_{18}(\delta) > 0$. To see (3.10) we apply Lemmas 2.4–2.7 with $s = s_i - s_{i-1} = t(x_i - x_{i-1})$, $\lambda = (2t \log\log t)^{1/2}$ and t large enough and use the inequalities $|f_i - f_{i-1}| \leqslant 1$, $|g_i - g_{i-1}| \leqslant 1$, $\varepsilon < 1$.

(1) In case $f_i = f_{i-1} = 0$, apply Lemma 2.4 with $\alpha = (g_i - g_{i-1} - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$ and observe that by (2.10), $\overline{\Phi}$ gives a constant$\times(\log\log t)^{-1/2}$ factor in front of the exponent.

(2) In case $g_i = g_{i-1}$, $f_i f_{i-1} > 0$, apply Lemma 2.5 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(3) In case $g_i = g_{i-1}$, $f_i f_{i-1} < 0$, apply Lemma 2.6 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(4) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.4 with $\alpha = -2\varepsilon\lambda$, use that $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$ and replace δ by $1 - \delta$.

(5) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.7 with $a = (f_i - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$.

Assembling all these estimations, (3.10) follows. This combined with (3.9) gives

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\ \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\ i = 1, 2, \ldots, k) \\ &\geqslant \frac{(c_{18}(\delta))^k}{(\log\log t)^{k/2}} \exp(-\Gamma \log\log t), \end{aligned} \tag{3.11}$$

where $\Gamma < 1$ is given by (3.8).

Now let $t_i = \exp(7i \log i)$, $i = 1, 2, \ldots$ and define

$$\eta_0 = 0, \quad T_i = \eta_{i-1} + t_i, \quad \eta_i = \inf\{t : t > T_i,\ W(t) = 0\},\ i = 1, 2, \ldots$$

It was shown in [6] that we have almost surely for all large enough n,

$$t_n \leqslant T_n \leqslant t_n \left(1 + \frac{1}{n}\right).$$

Define

$$\begin{aligned} \widehat{W}^{(n)}(t) &= W(t + \eta_{n-1}), \quad t \geqslant 0, \\ \widehat{Y}^{(n)}(t) &= Y(t + \eta_{n-1}) - Y(\eta_{n-1}), \quad t \geqslant 0, \\ \widehat{U}^{(n)}(x) &= \frac{\widehat{W}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0, 1], \\ \widehat{Z}^{(n)}(x) &= \frac{\widehat{Y}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0, 1]. \end{aligned}$$

Now let $x_0 = 0 < x_1 < \ldots < x_k = 1$ be a partition as before and consider the events $\widehat{E}_n = \cap_{i=1}^k \widehat{E}_n^{(i)}$ with

$$\widehat{E}_n^{(i)} = \{\widehat{a}_i \leqslant \widehat{W}^{(n)}(\widehat{s}_i) \leqslant \widehat{b}_i,\ \widehat{\alpha}_i \leqslant \widehat{Y}^{(n)}(\widehat{s}_i) - \widehat{Y}^{(n)}(\widehat{s}_{i-1}) \leqslant \widehat{\beta}_i\},$$

$\widehat{s}_i = x_i t_n,$

$$\widehat{a}_i = (f_i - \varepsilon)(2t_n \log\log t_n)^{1/2}, \quad \widehat{b}_i = (f_i + \varepsilon)(2t_n \log\log t_n)^{1/2},$$

$$\widehat{\alpha}_i = (g_i - g_{i-1} - \varepsilon)^+(2t_n \log\log t_n)^{1/2}, \quad \widehat{\beta}_i = (g_i - g_{i-1} + \varepsilon)(2t_n \log\log t_n)^{1/2}.$$

It follows from (3.11) that $\sum_n \mathbb{P}(\widehat{E}_n) = \infty$ and since $\widehat{E}_n$ are independent, we have by the Borel–Cantelli lemma $\mathbb{P}(\widehat{E}_n \text{ i.o.}) = 1$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\begin{aligned} \liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{U}^{(n)}(x_i) - f(x_i)| &= 0 \qquad \text{a.s.} \\ \liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{Z}^{(n)}(x_i) - g(x_i)| &= 0 \qquad \text{a.s.} \end{aligned}$$

Again, from the increment results in Subsection 2.2 it follows that

$$\begin{aligned} \limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{U}^{(n)}(x_{i-1}) - \widehat{U}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.} \\ \limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{Z}^{(n)}(x_{i-1}) - \widehat{Z}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.} \end{aligned}$$

Since $\varepsilon_1 > 0$ is arbitrary, these yield

$$\liminf_{n\to\infty} d\left((\widehat{U}^{(n)}, \widehat{Z}^{(n)}),\ (f,g)\right) = 0 \qquad \text{a.s.}$$

On the other hand, the increment results in Subsection 2.2 once again yields that, as $n \to \infty$, $d((\widehat{U}^{(n)}, \widehat{Z}^{(n)}),\ (U_{T_n}, Z_{T_n}))$ converges to 0 almost surely. Therefore,

$$\liminf_{n\to\infty} d\left((U_{T_n}, Z_{T_n}),\ (f,g)\right) = 0 \qquad \text{a.s.,}$$

Hence, (f,g) is a limit point of (U_t, Z_t) with probability 1.

To complete the proof of Theorem 1.1, we have to show that there exists an ω-set of probability one for which every $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point.

First we show that there exists a countable dense subset $K \subset \widetilde{\mathcal{S}}_J^{(2)}$. For any $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ and $\varepsilon > 0$, as before, choose a partition $x_0 = 0 < x_1 < \ldots < x_{k-1} < x_k = 1$ such that

$$\sup_{x_{i-1}\leqslant x\leqslant x_i} |f(x) - f(x_i)| \leqslant \varepsilon, \qquad \sup_{x_{i-1}\leqslant x\leqslant x_i} |g(x) - g(x_i)| \leqslant \varepsilon$$

and $g(x_{i-1}) \neq g(x_i)$ implies $f(x_{i-1}) = f(x_i) = 0$. Define $(\widetilde{f}, \widetilde{g}) \in S_J^{(2)}$ such that $\widetilde{f}(x_i) = f(x_i)$, $\widetilde{g}(x_i) = g(x_i)$, $i = 1, 2, \ldots, k$ and let $\widetilde{f}$ and $\widetilde{g}$ be linear in between. Then

$$d((f,g),\ (\widetilde{f},\widetilde{g})) < 2\sqrt{2}\varepsilon,$$

meaning that the set of pairs (f, g), where both f and g are piecewise linear (with the same cut-off points), is dense. It can be seen that one can choose a countable dense subset $K = \{(f_n, g_n)\}_{n=1}^{\infty}$ (for example by taking all x_i, $f_n(x_i)$, $g_n(x_i)$ rational) such that

$$\int_0^1 (f_n'(x))^2 + (g_n'(x))^2 \, \mathrm{d}x < 1.$$

It follows that there exists an ω-set of probability one such that all $(f_n, g_n) \in K$ are limit points. Next we show that for this ω-set every $(f, g) \in \widetilde{\mathcal{S}}_J^{(2)}$ is a limit point. Since K is dense, for each n we find $(f_n, g_n) \in K$ such that

$$d((f,g),\ (f_n,g_n)) < \frac{1}{n}$$

and since (f_n, g_n) is a limit point, we can find t_n such that $d((f_n, g_n),\ (U_{t_n}, Z_{t_n})) < \frac{1}{n}$. Hence $d((f,g),\ (U_{t_n}, Z_{t_n})) < \frac{2}{n}$. Consequently,

$$\lim_{n\to\infty} (U_{t_n}, Z_{t_n}) = (f, g),$$

i.e., (f, g) is a limit point.

This completes the proof of Theorem 1.1. □

4 Proof of Corollaries

The proof of Corollary 1.2 is obvious. To show Corollary 1.3 we need the following lemma.

Lemma 4.1 *If f and g are absolutely continuous functions and $f(x)g'(x) = 0$ a.e., then*

$$\int_0^1 (f'(x))^2 \, \mathbb{1}_{\{g'(x)\neq 0\}} \, \mathrm{d}x = 0. \tag{4.1}$$

Proof: Let

$$\mathcal{A} = \{x \in [0,1] : \ f(x) = 0, \ f'(x) \neq 0\}.$$

For each $x \in \mathcal{A}$, there exists $\delta_x > 0$ such that $f(y) \neq 0$ for all $y \in (x - \delta_x, \ x + \delta_x) \backslash \{x\}$. The intervals $\{(x - \delta_x, \ x + \delta_x)\}_{x \in \mathcal{A}}$ being disjoint and thus containing each a different rational number, they are at most countably many. This means $\mathcal{A}$ is a countable set. Now (4.1) follows immediately.

This proof, more elegant than our original one, was kindly communicated to us by Omer Adelman. □

Now we prove Corollary 1.3. It follows from Lemma 4.1 that if $(f,g) \in \widetilde{S}_J^{(2)}$, then

$$\int_0^1 (f'(x) + g'(x))^2 \, \mathrm{d}x \leqslant 1, \qquad \int_0^1 (f'(x) - g'(x))^2 \, \mathrm{d}x \leqslant 1,$$

from which (cf. [15])

$$|f(1) + g(1)| \leqslant 1, \qquad |f(1) - g(1)| \leqslant 1$$

showing that a limit point cannot be outside the set given in the Corollary.

To show that every point is a limit point, define

$$f(u) = \frac{x(u - 1 + |x|)}{|x|} \mathbb{1}_{\{1-|x| \leq u \leq 1\}}, \qquad g(u) = \frac{yu}{|y|} \mathbb{1}_{\{0 \leq u \leq |y|\}} + y \mathbb{1}_{\{|y| \leq u \leq 1\}}.$$

It is easy to see that $(f,g) \in \widetilde{S}_J^{(2)}$ and $f(1) = x$, $g(1) = y$. So (x,y) is a limit point. □

5 Further consequences: additive functionals

Consider the additive functional

$$A(t) = \int_0^t \psi(W(s)) \, \mathrm{d}s = \int_{\mathbb{R}} \psi(x) L(t,x) \, \mathrm{d}x,$$

where ψ is an integrable function such that $\overline{\psi} := \int_{\mathbb{R}} \psi(x) \, \mathrm{d}x \neq 0$. Then by the ratio ergodic theorem (cf. [12], p. 228)

$$\lim_{t \to \infty} \frac{A(t)}{\overline{\psi} \, L(t)} = 1 \qquad \text{a.s.}$$

Hence, introducing

$$\widetilde{V}_t(x) := \frac{A(xt)}{\overline{\psi} \sqrt{2t \log \log t}},$$

Theorem C implies

Corollary 5.1 *With probability one, the set* $\{(U_t, \widetilde{V}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{ (f,g) :\ f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leq 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

On the other hand, there are additive functionals which can be approximated by the principal value $Y(t)$. Let ψ be a function as above and consider its Hilbert transform:

$$\mathcal{H}(\psi)(x) = \frac{1}{\pi}\,\text{p.v.} \int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,\mathrm{d}y,$$

where p.v. indicates that the integral should be considered as a principal value. It was shown in [8] that if ψ is a Borel function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^{\kappa} |\psi(x)|\,\mathrm{d}x < \infty,$$

for some $\kappa > 0$, then for all sufficiently small $\varepsilon > 0$, when $t \to \infty$,

$$B(t) := \int_0^t (\mathcal{H}\psi)(W(s))\,\mathrm{d}s = \frac{\overline{\psi}}{\pi} Y(t) + o(t^{1/2-\varepsilon}), \qquad \text{a.s.}$$

Introducing the notation

$$\widetilde{Z}_t(x) = \frac{\pi B(xt)}{\overline{\psi}\,\sqrt{8t\log\log t}},$$

we have

Corollary 5.2 *With probability one, the set* $\{(U_t, \widetilde{Z}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{ (f,g) :\ f \in \mathcal{S}, g \in \mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leq 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

Acknowledgements

We are grateful to Omer Adelman for helpful discussions and a referee for useful remarks.

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