SOME RESULTS ON DEFINITE QUADRATIC FORMS

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1. The quadratic forms dealt with in this paper are all of the classic type

$$f(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \quad (a_{ij} = a_{ji}),$$

with integer coefficients a_{ij} and determinant

$$D = \| a_{ij} \| \quad (i, j = 1, 2, \ldots, n).$$

A positive definite form $f(x)$ is called non-decomposable if it cannot be expressed as a sum of two positive definite or positive semi-definite forms.

Mordell† has proved that, if

$$D \geqslant (2/\pi)^n \Big(\Gamma(2+\tfrac{1}{2}n) \Big)^2,$$

$f(x)$ is decomposable. It is an interesting problem to find non-decomposable forms for which D is large. Let μ_n be the largest value of D for a non-decomposable form in n variables. Mordell‡ has proved that there exist non-decomposable forms for $n = 6$, 7, and 8. We§ have proved that there exist non-decomposable forms for every $n > 8$, and that, for $n > 189$, $\mu_n \geqslant (n-176)/13$.

In §2, we prove that *for certain sequences of* n, *there exist non-decomposable forms with* $D > (1.1)^n$. It is not difficult to show that $\mu_n > (1.1)^n$, for all sufficiently large n, but we do not give the proof here, since it is rather complicated.

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* Received 6 June, 1938; read 16 June, 1938.

† Mordell, "The representation of a definite quadratic form as a sum of two others", *Annals of Math.*, 38 (1937), 751–757.

‡ *Loc. cit.*

§ Erdös and Ko, "On definite quadratic forms which are not the sum of two definite or semi-definite forms", *Acta Arithmetica*, not yet published.

Suppose now $D = 1$. Denote by h_n the number of classes of positive definite quadratic forms in n variables with determinant unity. For $n \leqslant 7$, it is well known that $h_n = 1$. For $n = 8, 9, 10, 11$, $h_n = 2$, these results being due to Mordell*, Ko†, Ketley‡, and Ko§, respectively. Ko‖ has proved that $h_{12} = h_{13} \geqslant 3$.

For $n = 2, 3, \ldots, 7$, the forms are decomposable into an obvious sum of n squares. For $n = 8$, Mordell¶ has proved that one of the two classes is non-decomposable. Ko** has proved that all the forms are decomposable for $n = 9, 10, 11, 13$, the result for $n = 10$ being due to Ketley††. We‡‡ have proved that non-decomposable forms exist for $n > 23$ and also for

$$n = 12, 14, 15, 16, 18, 20, 22.$$

The cases $n = 17, 19, 23$ are not yet settled. The proof depends upon finding certain forms with $D = 1$ which do not represent unity. This suggests the problem of the existence of forms with $D = 1$ which do not represent any integer less than K_n, where K_n depends only upon n. We cannot even construct a form which does not represent 1 and 2, but in § 3 we prove that *if $n = 8m+4$, there exists a form with $D = 1$ which does not represent odd integers less than $2m+1$.*

2. LEMMA 1*. *The form*

$$f = \alpha x_1^2 + 2\beta x_1 x_2 + 2 \sum_{i=2}^{n} x_i^2 + 2 \sum_{i=2}^{n-1} x_i x_{i+1},$$

with determinant $D < n$, where $\alpha > 0$, $\beta \geqslant 0$ are integers satisfying the

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* Mordell, " The definite quadratic forms in eight variables with determinant unity ", *Journal de Math.*, 17 (1938), 41–46.

† Ko, " Determination of the class number of positive quadratic forms in nine variables with determinant unity ", *Journal London Math. Soc.*, 13 (1938), 102–110.

‡ Ketley, M.Sc. Dissertation of the University of Manchester, 1938.

§ Ko, " On the positive definite quadratic forms with determinant unity ", *Acta Arithmetica*, not yet published.

‖ *Loc. cit.*

¶ Mordell, " The representation of a definite quadratic form as a sum of two others " *Annals of Math.*, 38 (1937), 751–757.

** See Ko, *loc. cit.*

†† See Ketley *loc. cit.*

‡‡ See Erdös and Ko., *loc. cit.*

conditions

$$\beta^2 > \alpha > (1-1/n)\beta^2, \quad 2\beta \leqslant n,$$

is positive definite and non-decomposable.

LEMMA 2†. *Let the positive definite quadratic forms*

$$g_1 = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \quad g_2 = \sum_{i,j=m+2}^{n} a_{ij} x_i x_j,$$

$$g_3 = b x_{m+1}^2 + 2 x_{m+1} x_{m+2} + g_2,$$

having determinants $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, *respectively, be non-decomposable. Denote by B the cofactor of a_{mm} in $\mathfrak{D}_1$. If there exists a positive definite quadratic form g of determinant $\mathfrak{D} < \mathfrak{D}_1 \mathfrak{D}_2$, of the type*

$$g = g_1 + a x_{m+1}^2 + 2 x_m x_{m+1} + g_3,$$

where a is an integer and $0 < a < B/\mathfrak{D}_1$, then g is non-decomposable.

LEMMA 3‡. *The form*

$$2 \sum_{i=1}^{n} x_i^2 + 2 \sum_{i=1}^{n-1} x_i x_{i+1}$$

has determinant $n+1$.

LEMMA 4. *Let the forms*

$$g = \sum_{i,j=1}^{n} a_{ij} x_i x_j, \quad g' = \sum_{i,j=1}^{n'} a'_{ij} x_i x_j$$

have determinants D, D' respectively, and let the cofactor of a_{nn} in D be A, and that of a'_{11} in D' be A'. Then the form

$$g'' = g(x_1, \ldots, x_n) + 2 x_n x_{n+1} + 3 x_{n+1}^2 + 2 x_{n+1} x_{n+2} + g'(x_{n+2}, \ldots, x_{n+n'+1})$$

has determinant $3DD' - DA' - D'A$.

* See Erdös and Ko, *loc. cit.*

† See Erdös and Ko, *loc. cit.*

‡ See Erdös and Ko, *loc. cit.*

The determinant of g'' is of the type

$$D'' = \left|\begin{array}{ccccc} A & D & & & 0 \\ & & 1 & & \\ & 1 & 3 & 1 & \\ & & 1 & & \\ 0 & & & D' & A' \end{array}\right|,$$

By Laplace's development, D'' is equal to the sum of all the signed products $\pm MM'$, where M is an n-rowed minor having its elements in the first n columns of D'', and M' is the minor complementary to M. The sign is $+$ or $-$ according as an even or odd number of interchanges of the rows of D'' will bring M into the position occupied by the minor D whose elements lie in the first n rows and first n columns of D''. All the M's are zero except possibly D and those obtained by replacing one row of D by $(0, 0, \ldots, 0, 1)$. The complementary minor of D is $3D'-A'$. The complementary minors of the others are zero, except that of the minor obtained by replacing the last row of D by $(0, \ldots, 0, 1)$. This gives $M = A$, $M' = D'$ and the number of interchanges of the rows is 1. Hence we have

$$D'' = D(3D'-A') - AD' = 3DD' - DA' - AD'.$$

LEMMA 5. *Let*

$$f_1(x_1, \ldots, x_{m+1}) = (c^2-1)x_1^2 + 2cx_1x_2 + 2\sum_{i=2}^{m+1} x_i^2 + 2\sum_{i=2}^{m} x_i x_{i+1},$$

$$\phi(x_1, \ldots, x_{m+2}) = 3x_1^2 + 2\sum_{i=1}^{m} x_i x_{i+1} + 2\sum_{i=2}^{m+1} x_i^2 + 2cx_{m+1}x_{m+2} + (c^2-1)x_{m+2}^2.$$

Write

$$f_{t+1} = f_1(x_1, \ldots, x_{m+1}) + \sum_{k=1} \left(2x_{m+2}^{(k-1)} x_1^{(k)} + \phi(x_1^{(k)}, \ldots, x_{m+2}^{(k)})\right),$$

where $x_{m+2}^{(0)}$ *is written for* x_{m+1}, *and where* $c > 4$ *and* $m = [\frac{1}{2}c^2]$. *Then* f_{t+1} *is a form in* $m+1+t(m+2)$ *variables, with determinant not less than*

$$\left(\tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)}\right)^{t+1}$$

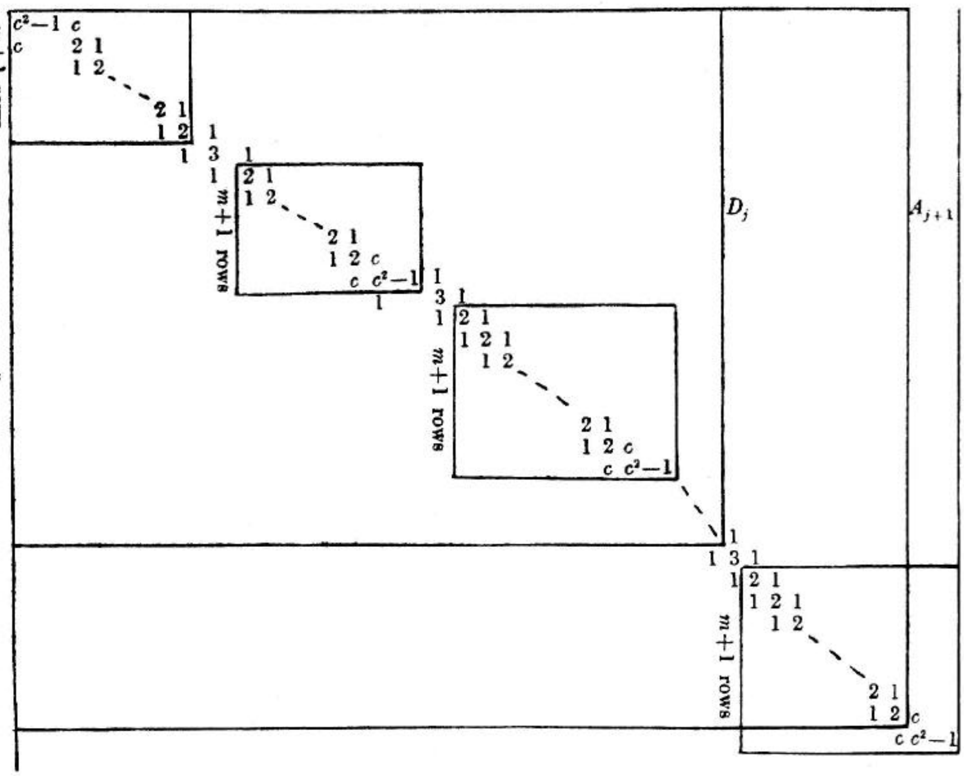
or

$$\left(\tfrac{1}{4}(c^2-5) + \tfrac{1}{4}\sqrt{(c^4 - 26c^2 + 25)}\right)^{t+1},$$

according as c *is even or odd.*

Let the determinant of f_{j+1} be D_{j+1} and the co-factor of the lower right-hand corner element of D_{j+1} be A_{j+1}. Then the determinant of f_{j+1} is of the form:

$D_{j+1} =$



$j+1$ blocks.

By using Lemma 3,

$$(1) \qquad \begin{cases} D_1 = (c^2-1)(m+1)-c^2 m = c^2-m-1, \\ A_1 = (c^2-1)\,m-c^2(m-1) = c^2-m = D_1+1. \end{cases}$$

By Lemma 4, on taking $D = D' = D_1$, $A = A' = A_1$,

$$(2) \qquad D_2 = 3D_1^2-2A_1 D_1 = D_1^2-2D_1.$$

Similarly from Lemma 4, on taking

$$D = D_j, \quad A = A_j, \quad D' = D_1, \quad A' = A_1 = D_1+1,$$

$$(3) \qquad D_{j+1} = (2D_1-1)\,D_j-A_j\,D_1.$$

By Lemmas 3 and 4, on taking $D = D_j$, $A = A_j$, $D' = m+1$, $A' = m$,

$$A_{j+1} = (2m+3)\,D_j - (m+1)\,A_j. \tag{4}$$

To solve these recurrence formulae, solve (3) for A_j, change j into $j+1$ and substitute in (4), then

$$D_{j+2} = (2D_1 - m - 2)\,D_{j+1} - (D_1 + m + 1)\,D_j, \tag{5}$$

with the initial values $D_1 = c^2 - m - 1$, $D_2 = D_1^2 - 2D_1$ from (1) and (2).

If c is even, $m = \frac{1}{2}c^2$, $D_1 = \frac{1}{2}c^2 - 1$, and from (5)

$$D_{j+2}/D_{j+1} = \tfrac{1}{2}c^2 - 4 - c^2/(D_{j+1}/D_j). \tag{6}$$

From $c > 4$, it is easily seen that $\frac{1}{4}c^4 - 8c^2 + 16 > 0$ and that

$$D_2/D_1 = D_1 - 2 = \tfrac{1}{2}c^2 - 3 \geqslant \tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)},$$

the larger root of the quadratic associated with the recurrence formula. Hence by obvious induction from (6),

$$\begin{aligned} D_{j+2}/D_{j+1} &\geqslant \tfrac{1}{4}c^2 - 4 - c^2/\left(\tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)}\right) \\ &= \tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)}. \end{aligned}$$

Hence $$D_t \geqslant \left(\tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)}\right)^t.$$

Similarly, if c is odd, $m = \frac{1}{2}(c^2 - 1)$, $D_1 = \frac{1}{2}(c^2 - 1)$, and so from (6),

$$D_{j+2}/D_{j+1} = \tfrac{1}{2}(c^2 - 5) - c^2/(D_{j+1}/D_j). \tag{7}$$

As above, by using (7), and the relation

$$D_2/D_1 = D_1 - 2 = \tfrac{1}{2}(c^2 - 5) \geqslant \tfrac{1}{4}(c^2 - 5) + \tfrac{1}{4}\sqrt{(c^4 - 26c^2 + 25)},$$

where $c^4 - 26c^2 + 25 > 0$ for $c > 4$, we obtain the required result of the lemma.

LEMMA 6. *The form f_{t+1} of Lemma 5 is positive definite.*

For $t = 0$, it is obvious that f_1 is positive definite on calculating the minors of the determinant D_1 by Lemma 3.

Suppose f_j is positive definite. Then the lemma is proved if we can prove that f_{j+1} is positive definite.

Denote the $j(m+2), \ldots, (j+1)(m+2)-1$ rowed minors of D_{j+1} by $d_1, \ldots, d_{m+2}$, where $d_{m+2} = D_{j+1} > 0$. Since f_j is positive definite, f_{j+1} is not positive definite if and only if $d_i \leqslant 0$ for certain i lying between 1 and $m+2$. Thus if f_{j+1} is not positive definite, without loss of generality, we can assume that $d_r \leqslant 0$. and $d_k > 0$ for $1 \leqslant k < r$. Write $d_0 = D_j > 0$. Then, on referring to the diagram giving the determinant D_{j+1} of Lemma 5, it is easy to see that

$$
\begin{aligned}
d_{r+1} &= 2d_r - d_{r-1}, \\
d_{r+2} &= 2d_{r+1} - d_r = 3d_r - 2d_{r-1}, \\
\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
d_{m+2} &= 2d_{m+1} - d_m = (m-r+3)\, d_r - (m-r+2)\, d_{r-1} \leqslant 0,
\end{aligned}
$$

in contradiction to $d_{m+2} = D_{j+1} > 0$. Hence the lemma is established.

LEMMA 7. *The form f_{t+1} is non-decomposable.*

By Lemma 1, it is easy to see that f_1 is non-decomposable. It suffices to suppose that f_j is non-decomposable and to prove that f_{j+1} is non-decomposable.

In Lemma 2, if we take

$$
g_1 = f_j, \quad g_2 = 2 \sum_{i=2}^{m+1} x_i^{(j)^2} + 2 \sum_{i=2}^{m} x_i^{(j)} x_{i+1}^{(j)} + 2c x_{m+1}^{(j)} x_{m+2}^{(j)} + (c^2-1)\, x_{m+2}^{(j)},
$$

$$
g_3 = 2x_1^{(j)^2} + 2x_1^{(j)} x_2^{(j)} + g_2,
$$

then $a = 1$, $g = f_{j+1}$ and $\mathfrak{D}_1 = D_j$, $\mathfrak{D}_2 = D_1$, $\mathfrak{D} = D_{j+1}$, $B = A_j$.

By Lemma 1, g_2, g_3 are non-decomposable; and by Lemma 6, f_{j+1} is positive definite. Hence, by Lemma 2, f_{j+1} is non-decomposable if

$$
A_j / D_j > 1 > 0 \quad \text{and} \quad D_{j+1} < D_1 D_j. \tag{8}
$$

Since f_j is non-decomposable, $A_j > D_j$, for otherwise

$$
f_j = x_{m+2}^{(j-1)^2} + (f_j - x_{m+2}^{(j-1)^2})
$$

is a decomposition of f_j. Next, from (5),

$$
D_{j+1} = (2D_1 - m - 2)\, D_j - (D_1 + m + 1)\, D_{j-1} < D_1\, D_j,
$$

if

$$
(D_1 - m - 2)\, D_j < (D_1 + m + 1)\, D_{j-1}.
$$

This holds, since

$$D_1 - m - 2 = c^2 - 2m - 3 = c^2 - 2[\tfrac{1}{2}c^2] - 3 < 0 \quad \text{and} \quad (D_1 + m + 1) D_{j-1} > 0.$$

Hence our lemma is proved.

From Lemmas 5 and 7, we easily deduce

THEOREM 1. *If $n = ([\frac{1}{2}c^2]+1)t - 1$, where $c > 4$, $t > 0$ are integers, then a non-decomposable form exists with determinant*

$$D \geqslant \left(\tfrac{1}{4}c^2 - 2 + \tfrac{1}{2}\sqrt{(\tfrac{1}{4}c^4 - 8c^2 + 16)}\right)^t, \quad or \quad \left(\tfrac{1}{4}(c^2-5) + \tfrac{1}{4}\sqrt{(c^4 - 26c^2 + 25)}\right)^t,$$

according as c is even or odd.

When we take $c = 5$, we have a non-decomposable form in $n = 13t - 1$ variables with determinant greater than or equal to $5^t > (1.13)^n$, since $5^{\frac{1}{13}} > 1.13$.

3. THEOREM 2. *If $n = 8m + 4$, there exists a form with $D = 1$ which does not represent odd integers less than $2m + 1$.*

The form is the extreme form given by Korkine and Zolotareff*,

$$F = \sum_{i=1}^{8n+4} x_i^2 + \left(\sum_{i=1}^{8n+4} x_i\right)^2 + (2n-1)x_{8n+4}^2 - 2x_1x_2 - 2x_2x_{8n+4}$$

$$= 2\left(x_1 + \tfrac{1}{2}\sum_{i=3}^{8n+4} x_i\right)^2 + 2\left(x_2 + \tfrac{1}{2}\sum_{i=3}^{8n+3} x_i\right)^2 + \sum_{i=3}^{8n+3}(x_i + \tfrac{1}{2}x_{8n+4})^2 + \tfrac{1}{4}x_{8n+4}^2,$$

with determinant unity. F represents odd integers only when x_{8n+4} is odd and then

$$\tfrac{1}{4}x_{8n+4}^2 \geqslant \tfrac{1}{4}, \quad (x_i + \tfrac{1}{2}x_{8n+4})^2 \geqslant \tfrac{1}{4} \quad (i = 3, \ldots, 8n+3).$$

Hence
$$F \geqslant (8n+2)/4 = 2n + \tfrac{1}{2},$$

and so $F \geqslant 2n + 1$.

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* Korkine and Zolotareff, *Math. Annalen*, 6 (1873), 366–389.

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