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ON SOME ASYMPTOTIC FORMULAS IN THE THEORY OF THE "FACTORISATIO NUMERORUM"

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Let $1 < a_1 \leqq a_2 \leqq \cdots$ be a sequence of integers. Denote by $f(n)$ the number of representations of n as the product of the a's, where two representations are considered equal only if they contain the same factors in the same order. As far as I know the first papers written on the subject are those of L. Kalmár,[1] who proved by using the methods of analytic number theory that if $a_k = k + 1$ then

$$(1) \qquad F(n) = \sum_{r=1}^{n} f(r) = -\frac{n^\rho}{\rho\zeta'(\rho)}[1 + o(1)],$$

ρ is defined as the unique positive root of $\zeta(\rho) = 2$. He also gives estimates for the error term.

Another paper on this subject is that of E. Hille.[2] He obtains among others the following results: Let $p_1 < p_2 < \cdots$ be a sequence of primes and $a_1 < a_2 < \cdots$ the sequence of integers composed of these primes, then

$$(2) \qquad F(n) = cn^\rho[1 + o(1)],$$

where $\sum_i \frac{1}{a_i^\rho} = 1$, $\rho > 0$. Hille uses the theorem of Wiener and Ikehara.

In the present paper we assume that $\sum \frac{1}{a_i^{1+\epsilon}}$ converges for every ϵ and that the a's are not all powers of a_1, then we prove that

$$(3) \qquad F(n) = cn^\rho[1 + o(1)],$$

where $\sum_i \frac{1}{a_i^\rho} = 1$, $\rho > 0$. The proof will be elementary.

First we need 2 Lemmas.

Lemma 1

$$(4) \qquad F(n) = \sum_k F\left[\frac{n}{a_k}\right] + 1.^3$$

Proof. Follows immediately by considering those products in which a_k is the first factor, and summing for a_k.

[1] L. Kalmár, Acta Litt ac Scient. Szeged, Tom. 5 (1930) p. 95–107.

[2] E. Hille, Acta Arithmetica Vol. 2 (1937) p. 134–146.

[3] The use of this identity was suggested to me by L. Kalmár.

LEMMA 2.

$$0 < \underline{\lim}\, \frac{F(n)}{n^\rho} \leq \overline{\lim}\, \frac{F(n)}{n^\rho} < \infty. \tag{5}$$

PROOF. Put $F(n) = c_n n^\rho$. We have from (4)

$$c_n n^\rho < \max_{i \leq \frac{n}{2}} c_i \sum_{a_k} \frac{n^\rho}{a_k^\rho} + 1,$$

hence

$$c_n < \max_{i \leq \frac{n}{2}} c_i + \frac{1}{n^\rho}.$$

Thus by induction

$$c_n < 1 + \sum_{2^{m-1} < n} \frac{1}{2^{m\rho}} < \infty,$$

which proves the first half of (5).

The proof of the second half of (5) will be slightly more complicated. Put $F(n) = c_n'(n+1)^\rho$. It suffices to prove that $\underline{\lim}\, c_n' > 0$. From $\left[\frac{n}{a_k}\right] \geq \frac{n+1}{a_k} - 1$ we obtain by (4)

$$c_n'(n+1)^\rho > \min_{i \leq \frac{n}{2}} c_i' \sum_{a_k \leq n} \frac{(n+1)^\rho}{a_k^\rho} = \min_{i \leq \frac{n}{2}} c_i'(n+1)^\rho \left(1 - \sum_{a_k > n} \frac{1}{a_k^\rho}\right).$$

Thus

$$c_n' > \min_{i \leq \frac{n}{2}} c_i' \left(1 - \sum_{a_k > n} \frac{1}{a_k^\rho}\right).$$

Hence by induction

$$c_n' > \prod_{2^{m-1} < n} \left(1 - \sum_{a_k > 2^m} \frac{1}{a_k^\rho}\right).$$

The product on the right side (if extended to infinity) converges since

$$\sum_{m=1}^{\infty} \sum_{a_k > 2^m} \frac{1}{a_k^\rho} \leq \sum_{a_k} \frac{\log a_k}{a_k^\rho} < c \sum \frac{1}{a_k^{1+\epsilon}}$$

converges. This proves $\underline{\lim}\, c_n' > 0$, and completes the proof of Lemma 2

Now we can prove our theorem. Suppose that (3) does not hold, denote

$$0 < c = \underline{\lim}\, \frac{F(n)}{n^\rho} = \underline{\lim}\, \frac{F(n)}{(n+1)^\rho} < \overline{\lim}\, \frac{F(n)}{n^\rho} = \overline{\lim}\, \frac{F(n)}{(n+1)^\rho} = C < \infty. \tag{6}$$

Let m be sufficiently large and such that $F(m) > (C - \delta)(m + 1)^\rho$. Clearly a fixed k exists (depending only on c and C) such that for every x satisfying $m \leqq x \leqq m(1 + k)$

$$\frac{F(x)}{(x+1)^\rho} > \frac{C+c}{2}. \tag{7}$$

Now let a_i be the least a which is not a power of a_1. Consider any x satisfying $ma_1 \leqq x \leqq ma_1(1 + k)$. We have by (4), (6), (7) and $\left[\frac{x}{a_i}\right] + 1 \geqq \frac{x+1}{a_i}$

$$F(x) > \sum_{a_i \leqq x} F\left[\frac{x}{a_i}\right] > \frac{c+C}{2}\,\frac{(x+1)^\rho}{a_1^\rho} + c \sum_{a_i > a_1} \frac{(x+1)^\rho}{a_i^\rho} - o(x^\rho). \tag{8}$$

Thus

$$\frac{F(x)}{(x+1)^\rho} > c + \frac{C-c}{2a_1^\rho} - o(1). \tag{9}$$

Similarly we obtain that for the x satisfying $a_1^\alpha a_i^\beta m \leqq x \leqq a_1^\alpha a_i^\beta m(1 + k)$

$$\frac{F(x)}{(x+1)^\rho} > c + \delta_{\alpha,\beta}, \tag{10}$$

where $\delta_{\alpha,\beta}$ depends only upon α and β. It is well known that the quotient of two consecutive integers of the form $a_1^\alpha a_i^\beta$ tends to 1. Thus there exists a sequence of integers $A_1 < A_2 < \cdots < A_r$ all of the form $a_1^\alpha a_i^\beta$ and satisfying

$$\frac{A_{i+1}}{A_i} < 1 + k, \quad i = 1, 2, \cdots r - 1 \quad \text{and} \quad A_r > a_1 A_1.$$

Thus by (10) and since the intervals $[A_i m, A_i m(1 + k)]$ and $[A_{i+1}m, A_{i+1}m(1 + k)]$ overlap we have for $A_1 m \leqq x \leqq a_1 A_1 m$

$$\frac{F(x)}{(x+1)^\rho} > c + \min \delta_{\alpha,\beta} = c + \delta, \tag{11}$$

for sufficiently large m, where δ is fixed and depends only on c and C. Consider now the integers x satisfying $a_1 A_1 m \leqq x \leqq a_1^2 A_1 m$, by (4), (6) and (11) we obtain as in (8) and (9)

$$\frac{F(x)}{(x+1)^\rho} > (c + \delta)\frac{1}{a_1^\rho} + c \sum_{a_i > a_1} \frac{1}{a_i^\rho} - o(1) = c + \delta\left(1 - \sum_{a_i > a_1} \frac{1}{a_i^\rho}\right) - o(1).$$

$\left(\text{i.e. } \frac{x}{a_1} \text{ lies in } [A_1 m, A_1 m(1 + k)]\right)$. Similarly for the integers satisfying $a_1^2 A_1 m \leqq x \leqq a_1^3 A_1 m$ we have

$$\frac{F(x)}{(x+1)^\rho} > \left[c + \delta\left(1 - \sum_{a_i > a_1} \frac{1}{a_i^\rho}\right)\right] \sum_{a_i \leqq a_1^2} \frac{1}{a_i^\rho} + c \sum_{a_i > a_1^2} \frac{1}{a_i^\rho}$$

$$- o(1) > c + \delta\left(1 - \sum_{a_i > a_1} \frac{1}{a_i^\rho}\right)\left(1 - \sum_{a_i > a_1^2} \frac{1}{a_i^\rho}\right) - o(1).$$

Finally we obtain for $a_1^{k-1}A_1m \leqq x \leqq a_1^kA_1m$ (k fixed, m sufficiently large)

$$(12) \qquad \frac{F(x)}{(x+1)^\rho} > c + \delta \prod_{r=1}^{k} \left(1 - \sum_{a_i > a_1^r} \frac{1}{a_i^\rho}\right) - o(1).$$

Denote

$$\prod_{r=1}^{\infty} \left(1 - \sum_{a_i > a_1^r} \frac{1}{a_i^\rho}\right) = \eta.$$

The product converges since $\sum \frac{\log a_i}{a_i^\rho}$ converges. From (12) we have for $A_1m \leqq x \leqq a_1^kA_1m$

$$(13) \qquad \frac{F(x)}{(x+1)^\rho} > c + \frac{\delta\eta}{2}.$$

Now choose k so great that

$$(14) \qquad \prod_{r>k} \sum_{a_i \leqq a_1^r} \frac{1}{a_i^\rho} > \frac{c + \frac{1}{4}\delta\eta}{c + \frac{1}{2}\delta\eta}.$$

Then from (13) and (4) we have for $A_1a_1^km \leqq x \leqq A_1a_1^{k+1}m$

$$F(x) > \sum_{a_i \leqq a_1^{k+1}} F\left[\frac{x}{a_i}\right] > \left(c + \frac{\delta\eta}{2}\right) \sum_{a_i \leqq a_1^{k+1}} \frac{(x+1)^\rho}{a_i^\rho}.$$

Similarly for any r, in the interval $A_1a_1^rm \leqq x \leqq A_1a_1^{r+1}m$ we have by (14)

$$\frac{F(x)}{(x+1)^\rho} > \left(c + \frac{\delta\eta}{2}\right) \prod_{l>k} \sum_{a_i < a_1^l} \frac{(x+1)}{\rho} > \frac{c + \delta\eta}{4}.$$

Thus $\lim \frac{F(x)}{(x+1)^\rho} > c$. This contradicts (6) and completes the proof of our theorem.

It is easy to see that in our theorem, we can replace the assumption that $\sum \frac{1}{a_i^{1+\epsilon}}$ converges by the following slightly more general one: There exists a $k > 0$ such that $\sum \frac{1}{a_i^k}$ converges, and $\sum \frac{\log a_i}{a_i^k}$ converges too.

Let $a_k = k + 1$. By using Lemma 2 we can prove that constants c_1 and c_2 exist, $0 < c_2 < c_1 < 1$, such that for infinitely many n

$$f(n) > \frac{n^\rho}{e^{(\log n)^{c_1}}}$$

and that for all $n > n_0$

$$f(n) < \frac{n^{\rho}}{e^{(\log n)^{c_2}}}.^{4}$$

As I shall show in another paper the methods used here yield some asymptotic formulas in the theory of partitions.

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[4] E. Hille proved that $f(n) > n^{\rho-\epsilon}$ for infinitely many n (ibid).