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ON THE INTEGERS HAVING EXACTLY K PRIME FACTORS

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Denote by $\pi_k(n)$ the number of integers not exceeding n having exactly k prime factors (multiple factors are counted only once). Hardy and Ramanujan[1] proved that

$$(1) \quad \pi_k(n) < c(n/\log n)\cdot(\log\log n + c)^{k-1}/(k-1)! < c(n/(\log\log n)^{\frac{1}{2}}).$$

(c denotes constants not necessarily the same.) Put $x = [\log\log n]$ Hardy[2] conjectured that

$$(2) \qquad \pi_k(n) > c(n/(\log\log n)^{\frac{1}{2}}) = c(n/\log n)\cdot x^{k-1}/(k-1)!.$$

Pillai and I proved this conjecture (independently).[2] In fact we both proved that for $x - cx^{\frac{1}{2}} \leqq k \leqq x + cx^{\frac{1}{2}}$ (the interval $(x - cx^{\frac{1}{2}}, x + cx^{\frac{1}{2}})$ will be denoted by I).

$$(3) \qquad \pi_k(n) > c(n/(\log\log n)^{\frac{1}{2}}),$$

and Pillai proved that for $k < cx$

$$\pi_k(n) > c(n/\log n)\cdot x^{k-1}/(k-1)!.$$

In the present paper we shall prove that for k in I

$$(4) \qquad \pi_k(n) = (1 + o(1))\cdot(n/\log n)\cdot x^{k-1}/(k-1)!.$$

I believe that a formula like (4) holds for $k < cx$, but the proof presents difficulties which I have not yet been able to overcome.

In the proof of (4) we will have to use the prime number theorem. It will be relatively easy to prove (3) (Lemma 3), and the prime number theorem will not be required for the proof of (3).

Throughout this paper k and r will denote integers in I. $a_i^{(k)}, i = 1, 2 \cdots$ denotes the integers $\leqq n$ having exactly k prime factors (multiple factors are counted only once). $\sum (1/a_i^{(k)})$ will indicate that the summation is extended over i.

LEMMA 1.

$$\sum_{k \text{ in I}} \sum \frac{1}{a_i^{(k)}} > \tfrac{1}{2} \log n.$$

[1] Collected papers of S. Ramanujan, p. 262–275.
[2] "Ramanujan" by G. H. Hardy, p. 56.

Hardy and Ramanujan[3] prove that for large c the number of integers $\leqq n$, for which the number of prime factors is in I, is $> (1 - \epsilon)n$. Let $n^{\delta} \leqq m \leqq n$, clearly $\log\log m = \log\log n + 0\,(1)$. Thus it easily follows that the number of integers $\leqq m$ for which the number of prime factors is in $(x - cx^{\frac{1}{2}}, x + cx^{\frac{1}{2}})$ is $> (1 - \epsilon)m$. Thus

$$\sum_{k \text{ in I}} \sum \frac{1}{a_i^{(k)}} > (1 - \epsilon) \sum_{t=n^{\delta}}^{n} \frac{1}{t} > \tfrac{1}{2} \log n,$$

which proves the lemma.

LEMMA 2.

$$\sum \frac{1}{a_i^{(k)}} > \frac{c}{x^{\frac{1}{2}}} \log n.$$

It follows from Lemma 1 that for some r

$$\sum \frac{1}{a_i^{(r)}} > \frac{c}{x^{\frac{1}{2}}} \log n. \tag{5}$$

Further

$$(k + 1) \sum \frac{1}{a_i^{(k+1)}} < \sum \frac{1}{a_i^{(k)}} \cdot \sum_{p^{\alpha} \leqq n} \frac{1}{p^{\alpha}} < (x + c) \sum \frac{1}{a_i^{(k)}}.$$

Hence

$$\sum \frac{1}{a_i^{(k+1)}} < (x + c)/(k + 1) \sum \frac{1}{a_i^{(k)}} < (1 + c/x^{\frac{1}{2}}) \sum \frac{1}{a_i^{(k)}}. \tag{6}$$

We obtain from (5) and (6) by a simple computation that for every $k \leqq r$

$$\sum \frac{1}{a_i^{(k)}} > \frac{c \log n}{x^{\frac{1}{2}}}.$$

Now

$$(r + 1) \sum \frac{1}{a_i^{(r+1)}} = \sum \frac{1}{a_i^{(r)}} \sum{}' \frac{1}{p^{\alpha}}.$$

The prime indicates that the summation is extended over the primes p with $p^{\alpha} \leqq n/a_i^{(r)}$, $p \nmid a_i^{(r)}$. Hence

$$(r + 1) \sum 1/a_i^{(r+1)} > \sum 1/a_i^{(r)} (\log\log\,(n/a_i^{(r)}) - c \log x) \tag{7}$$

since, as is well known,

$$\sum_{p \mid a_i^{(r)}} \frac{1}{p^{\alpha}} < c \log x.$$

Now

$$\sum \left(\frac{1}{a_i^{(r)}}\right) \log\log \frac{n}{a_i^{(r)}} > \sum{}'' \frac{1}{a_i^{(r)}} (x - 2 \log x) \tag{8}$$

[3] See [1].

where the two primes indicate that the summation is extended over the

$$a_i^{(r)} < n^{1-1/x^2}$$

(hence $n/a_i^{(r)} > n^{1/x^2}$). Also

$$\sum'' \frac{1}{a_i^{(r)}} = \sum \frac{1}{a_i^{(r)}} + O\left(\frac{1}{x^2}\log n\right). \tag{9}$$

Hence from (7), (8) and (9)

$$(r+1)\sum \frac{1}{a_i^{(r+1)}} > (x - c\log x)\sum \frac{1}{a_i^{(r)}} - 0\left(\frac{1}{x}\log n\right).$$

Thus from (5)

$$\sum \frac{1}{a_i^{(r+1)}} > \frac{x - c\log x}{r+1}\sum \frac{1}{a_i^{(r)}} - 0\left(\frac{1}{x^2}\log n\right) > \left(1 - \frac{c}{x^{\frac{1}{2}}}\right)\sum \frac{1}{a_i^{(r)}}. \tag{10}$$

We obtain from (5) and (10) by a simple computation that for $k \geqq r$

$$\sum \frac{1}{a_i^{(k)}} > \frac{c}{x^{\frac{1}{2}}}\log n,$$

which completes the proof of the lemma.

LEMMA 3.

$$\pi_k(n) > c\,\frac{n}{x^{\frac{1}{2}}}.$$

We evidently have for $y < n^{\frac{1}{2}}$

$$\pi_1'\left(\frac{n}{y}\right) - \pi_1\left(\frac{n}{2y}\right) > \frac{cn}{y\log n},$$

where $\pi_1'(n/y)$ denotes the number of primes and powers of primes $\leqq n/y$ with $p \nmid y$, ($\pi_1(n/2y)$ denotes the number of primes and their powers $\leqq (n/2y)$. Thus from Lemma 2

$$\sum'\left(\pi_1'\left(\frac{n}{a_i^{(k-1)}}\right) - \pi_1\left(\frac{n}{2a_i^{(k-1)}}\right)\right) > c\,\frac{n}{x^{\frac{1}{2}}}, \tag{11}$$

where in $\sum'$, $a_i^{(k-1)} < \frac{1}{2}n^{\frac{1}{2}}$. But it is easy to see that the sum (11) is not greater than $\pi_k(n)$ (i.e. every $a_i^{(k)} > n/2$ can be written in at most one way in the form $a_i^{(k-1)}\cdot p$, where $a_i^{(k-1)} < \frac{1}{2}n^{\frac{1}{2}}$, $n/a_i^{(k-1)} > p > n/2a_i^{(k-1)}$, i.e. $p > n^{\frac{1}{2}}$, $p \nmid a_i^{(k-1)}$), (and the $a_i^{(k)} \leqq n/2$ do not occur at all) which proves the lemma.

LEMMA 4.

$$\sum \frac{1}{a_i^{(k)}} = \left(1 + 0\left(\frac{1}{x^{\frac{1}{2}}}\right)\right)\sum \frac{1}{a_i^{(k+1)}}.$$

This follows from (6) and (10).

LEMMA 5.

$$\sum \frac{1}{a_i^{(k)}} = (1 + o(1)) \frac{x^k}{k!}$$

Suppose that Lemma 5 is false. We can assume without loss of generality that, for infinitely many n, there exists a k and a fixed $\epsilon > 0$ such that

$$\sum \frac{1}{a_i^{(k)}} > (1 + \epsilon) \frac{x^k}{k!}.$$

(It will be clear from the proof that if $\epsilon < 0$, the argument remains unchanged.) It follows from Lemma 4 by a simple calculation that there exists a $c = c(\epsilon)$ such that for every $k \leqq r \leqq k + cx^{\frac{1}{2}}$

$$\sum \frac{1}{a_i^{(r)}} > \left(1 + \frac{3\epsilon}{4}\right) \frac{x^k}{k!} > \left(1 + \frac{\epsilon}{2}\right) \frac{x^r}{r!}.$$

Thus

$$\sum_{r=k}^{k+cx^{\frac{1}{2}}} \sum \frac{1}{a_i^{(r)}} > \left(1 + \frac{\epsilon}{2}\right) \sum_{r=k}^{k+cx^{\frac{1}{2}}} \frac{x^r}{r!}.$$

Put

$$\sum_{r=k}^{k+cx^{\frac{1}{2}}} \frac{x^r}{r!} = c_1 \log n \left(\log n = e^x = \sum_{t=1}^{\infty} \frac{x^{t-1}}{(t-1)!}\right).$$

Thus we have

$$\sum_{r=k}^{k+cx^{\frac{1}{2}}} \sum \frac{1}{a_i^{(r)}} > \left(1 + \frac{\epsilon}{2}\right) c_1 \log n. \tag{12}$$

On the other hand, it follows from a result of Kac and myself[4] that for every $m > n^{\delta}$ ($\delta > 0$ an arbitrary number)

$$\sum_{r=k}^{k+cx^{\frac{1}{2}}} \pi_r(m) = (1 + o(1)) c_1 m.$$

Thus a simple calculation shows that

$$\sum_{r=k}^{k+cx^{\frac{1}{2}}} \sum \frac{1}{a_i^{(r)}} = \sum_{u=n^{\delta}}^{n} \frac{(1 + o(1))c_1}{u} + 0\left(\sum_{v<n^{\delta}} \frac{1}{v}\right) = (1 + o(1)) c_1 \log n$$

which contradicts (12), and completes the proof of the lemma.

Now we introduce some notations. $y = y(n)$, $y_1 = y_1(n) \cdots$ denote functions of n tending to infinity together with n in a manner which will be specified later

$$f_y(m) = \prod_{p^{\alpha} \| m}' p^{\alpha}$$

where $p^{\alpha} \| m$ means that $p^{\alpha} \mid m$, $p^{\alpha+1} \nmid m$ and the dash indicates that the product

[4] Amer. Journal of Math., Vol. 62 (1940) p. 738–742.

is extended over the $p \leqq n^{1/y}$. By $V_y(m)$ we shall denote the number of prime factors $p > n^{1/y}$ of m (multiple factors counted only once). Now we prove

LEMMA 6. *The number of integers* $a_i^{(k)} \leqq n$ *which do not satisfy*

$$(13) \qquad (1-\epsilon)\log y \leqq V_y(a_i^{(k)}) \leqq (1+\epsilon)\log y$$

is $o(n/x^{\frac{1}{2}})$. (i.e. *is* $o(\pi_k(n))$ *by Lemma* 3).

We denote the $a_i^{(k)}$ not satisfying (13) by $b_i^{(k)}$. To prove Lemma 6 we need several lemmas.

LEMMA 7. *The number* N_1 *of integers* $a_i^{(k)} \leqq n$ *for which*

$$f_y(a_i^{(k)}) > n^{y_1/y}$$

is $o(n/x^{\frac{1}{2}})$.

We evidently have

$$(14) \qquad \prod_i f_y(a_i^{(k)}) \leqq \prod_{p,\alpha}{}' (p^\alpha)^{\pi_{k-1}(n/p^\alpha)} = \prod{}'' \cdot \prod{}'''$$

where in $\prod'$, $p \leqq n^{1/y}$ in $\prod''$, $p^\alpha \leqq n^{\frac{2}{3}}$ and in $\prod'''$ $p^\alpha > n^{\frac{2}{3}}$.
Thus from (1)

$$\prod{}'' < \prod_{p,\alpha}{}'' (p^\alpha)^{c(n/p^\alpha x^{\frac{1}{2}})} < \exp\,(cn \log n/yx^{\frac{1}{2}})$$

since by a well known result

$$\sum_{p<z,\alpha} \log\,(p^\alpha)/p^\alpha < c \log z.$$

For $y = o(\log n)$

$$\prod{}''' < n^{n^{1/3+1/y}\cdot\log n} < \exp\,(cn \log n/yx^{\frac{1}{2}})$$

(the number of integers p^α, with

$$p < n^{1/y},\, n^{\frac{2}{3}} < p^\alpha \leqq n$$

is $\leqq n^{1/y} \log n$). Thus finally

$$\prod_i f_y(a_i^{(k)}) < \exp\,(cn \log n/yx^{\frac{1}{2}}).$$

Hence

$$\exp\,[(N_1 y_1/y) \log n] < \exp\,(cn \log n/y\, x^{\frac{1}{2}})$$

since the contribution of each of the $a_i^{(k)}$ with

$$f_y(a_i^k) > n^{y_1/y}$$

to the product (14) is $> n^{y_1/y}$. Thus

$$N_1 < cn/y_1\, x^{\frac{1}{2}} = o(n/x^{\frac{1}{2}}).$$

LEMMA 8. *Let* $u \leqq n^{y_2/y}$ *where* $y_2/y \to 0$. *Then the number of integers* $m \leqq n$ *with* $f_y(m) = u$ *does not exceed* $cny/u \log n$.

Lemma 8 follows immediately from Brun's method.[5]

LEMMA 9. *Let $\delta = \delta(\epsilon)$ be sufficiently small. Then*

$$\sum_i 1/b_i^{(k)} < c \log n/y^{\delta}x^{\frac{1}{2}}$$

(the $b_i^{(k)}$ are the $a_i^{(k)}$ not satisfying (13).

Clearly

$$\sum_i 1/b_i^{(\kappa)} < \sum_v ((\textstyle\sum' 1/p^{\alpha})^v/v!)((\sum'' 1/p^{\alpha})^{k-v}/(k-v)!)$$

where in $\sum'$, p runs in the interval $n^{1/y} \leqq p \leqq n$ and in $\sum''$, $p \leqq n^{1/y}$ and v satisfies

$$0 \leqq v \leqq (1-\epsilon)\log y \text{ or } (1+\epsilon)\log y \leqq v.$$

We obtain by a simple calculation (using Stirling's formula) that

$$\frac{(\sum'' 1/p^{\alpha})^{k-v}}{(k-v)!} < c \log n/yx^{\frac{1}{2}} (\text{since } \textstyle\sum'' 1/p^{\alpha} = \text{loglog } n - \log y + o(1),$$

if y does not tend to infinity too fast ($y < (\log n)^{1-\epsilon}$). Thus

$$\sum_i 1/b_i^{(k)} < (c \log n/yx^{\frac{1}{2}}) \sum_v ((\textstyle\sum' 1/p^{\alpha})^v/v!).$$

Now it is easy to see by a simple computation that

$$\sum_v ((\textstyle\sum' 1/p^{\alpha})v/v!) < cy^{1-\delta}, \ \delta = \delta(\epsilon) \ (\text{since } \textstyle\sum' 1/p^{\alpha} = \log y + o(1)).$$

Thus finally

$$\sum_i 1/b_i^{(k)} < c \log n/y^{\delta}x^{\frac{1}{2}}$$

which proves Lemma 9.

Now we can prove Lemma 6. We split the integers not satisfying (13) into two classes. In class I. are the integers with

$$f_{y/y_3}(b_i^{(k)}) > n^{y_3y_4/y}$$

where $y_3y_4/y \to 0$ and $y/y_3 = o(\log y)$. In the second class are the other $b_i^{(k)}$.

By Lemma 7 (replacing y by y/y_3, y_1 by y_4), the number of integers of the first class is $o(n/x^{\frac{1}{2}})$. Let $b_i^{(k)}$ be any integer of the second class. Put

$$B = f_{y/y_3}(b_i^{(k)}),$$

and consider the set of all B's. These integers satisfy the following conditions: 1) $B \leqq n^{y_3y_4/y}$; 2) $k - y/y_3 \leqq v(B) \leqq k(v(B)$ denotes the number of prime factors of B, multiple factors counted only once); $v_y(B)$ does not lie in the interval $[(1-\epsilon)\log y, (1+\epsilon)\log y - y/y_3]$. 1) is clearly satisfied, 2) and 3) hold since the number of prime factors $> n^{y_3/y}$ of any integer $\leqq n$ is $< y/y_3$.

[5] Ibid., lemma 2, p. 739.

Since $y/y_3 = o(\log y)$ 3) means that $v_y(B)$ does not lie in $[(1 - \epsilon/2) \log y, (1 + \epsilon/2) \log y]$ and since by 2) $v(B)$ can assume only $y/y_3 = o(\log y)$ values we obtain from Lemma 9 that

$$\sum 1/B < (cy/y_3) \frac{\log n}{y^\delta x^{\frac{1}{2}}} < \frac{\log n}{y^{\delta/2} x^{\frac{1}{2}}}. \tag{15}$$

From Lemma 8 (with $y = y/y_3$, $y_2 = y_4$) and (15) we obtain that the number of integers $m \leqq n$ for which $f_{y/y_3}(m)$ is one of the B's does not exceed

$$\frac{cny}{y_3 \log n} \sum 1/B < c \frac{n \log y}{y^{\delta/2} x^{\frac{1}{2}}} = o(n/x^{\frac{1}{2}}).$$

Thus the number of integers of the second class is also $o(n/x^{\frac{1}{2}})$, which completes the proof of the lemma.

LEMMA 10.

$$\sum' 1/a_i^{(k)} = (1 + o(1))x^k/k!.$$

The dash indicates that the summation is extended over the $a_i^{(k)}$ satisfying (13); in other words, the $b_i^{(k)}$ are omitted.

Lemma 10 follows immediately from Lemmas 5 and 9 since

$$x^k/k! > ((c \log n(/x^{\frac{1}{2}}))).$$

Now we can prove

THEOREM I.

$$\pi_k(n) = (1 + o(1)) \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}.$$

Consider

$$M = \sum' (\pi_1'(n/a_i^{(k-1)}) - \pi_1 (n^{1/y})) \tag{16}$$

where $\pi_1'(n/t)$ denotes the number of primes and powers of primes $p^\alpha \leqq n/t$ with $p \nmid t$ in $\sum'$, $a_i^{(k-1)} \leqq n^{1-1/y}$ and $a_i^{(k-1)}$ satisfies (13).

We have by the prime number theorem, if y tends to infinity sufficiently slowly (this is the only place where the prime number theorem is used)

$$\begin{aligned}\sum' \pi_1'\left(\frac{n}{a_i^{(k-1)}}\right) &= (1 + o(1)) \sum' \frac{n}{a_i^{(\kappa-1)} \log \frac{n}{a_i^{(\kappa-1)}}} \\ &= \sum_1' + \sum_2' + \cdots \sum_{y-1}'\end{aligned} \tag{17}$$

where in $\sum_j'$, $a_i^{(k-1)}$ satisfies (13) and

$$n^{j-1/y} \leqq a_i^{(k-1)} \leqq n^{j/y}.$$

We have for $y - j \to \infty$

$$\sum_j' = (1 + o(1)) \frac{yn}{(y-j)\log n} \sum_i' \frac{1}{a_i^{(k-1)}}. \tag{18}$$

Further

$$\sum_j' 1/a_i^{(k-1)} = \sum_u - \sum_v$$

where in $\sum_u$, $a_i^{(k-1)} \leqq n^{j/y}$ and $a_i^{(k-1)}$ satisfies (13), in $\sum_v$, $a_i^{(k-1)} \leqq n^{j-1/y}$, and $a_i^{(k-1)}$ satisfies (13). We have from Lemmas 5 and 10 if y tends to infinity sufficiently slowly (replace x by loglog $n^{j/y} = x + \log j/y$) and note $k = (1 + o(1)x)$

$$\sum_u = (1 + o(1)) \frac{(x + \log j/y)^{k-1}}{(k-1)!} = (1 + o(1)) \frac{j}{y} \frac{x^{k-1}}{(k-1)!}$$

and similarly

$$\sum_v = (1 + o(1)) \frac{j-1}{y} \frac{x^{k-1}}{(k-1)!}.$$

Thus if y tends to infinity sufficiently slowly

$$\sum_j' 1/a_i^{(k-1)} = (1 + o(1)) \frac{x^{k-1}}{y(k-1)!}. \tag{19}$$

Further if $y - j = 0(1)$ and y tends to infinity sufficiently slowly

$$\sum_{y-1}' + \sum_{y-2}' + \dots \sum_{y-c} < \frac{yn}{\log n} \sum'' \frac{1}{a_i^{(k-1)}} < c \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!} \tag{20}$$

by Lemma 5, in $\sum''$, $n^{y-c/y} \leqq a_i^{(k-1)} \leqq n^{1-1/y}$.
Thus from (17), (18), (19) and (20)

$$\begin{aligned} \sum' \pi_1(n/a_i^{(k-1)}(&= (1 + o(1)) \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!} \sum_{j<y} \frac{1}{y-j} \\ &+ 0\left(\frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}\right) = (1 + o(1)) \frac{n \log y}{\log n} \frac{x^{k-1}}{(k-1)!}. \end{aligned} \tag{21}$$

Also

$$\sum' \pi_1(n^{1/y}) < c \frac{ny}{\log n} = o\left(\frac{n}{x}\right)$$

if $y = o((\log n)/x)$. Thus finally

$$M = (1 + o(1)) \frac{n \log y}{\log n} \frac{x^{k-1}}{(k-1)!}. \tag{22}$$

Clearly M equals the number of integers not exceeding n of the form

$$a_i^{(k-1)} \cdot p^\alpha, \text{ with } n^{1/y} < p, \text{ p} \nmid a_i^{(k-1)}, a_i^{(k-1)} \leqq n^{1-1/y}, \tag{23}$$

and $a_i^{(k-1)}$ satisfies (13). Every integer u of (23) has exactly k prime factors and satisfies (13), thus $u = a_i^{(k)}$. Further every $a_i^{(k)}$ satisfying (13) can be written in exactly $V_y(a_i^{(k)})$ ways in the form (23). Thus by (13) we obtain that the number of $a_i^{(k)} \leqq n$ satisfying (13) is

$$(1 + o(1)) \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}. \tag{24}$$

Thus finally from (24) and Lemma 6 we obtain

$$\pi_k(n) = (1 + o(1)) \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}$$

which proves Theorem 1.

By the same argument we can prove the following results:

THEOREM II. *Denote by $\pi_k'(n)$ the number of integers $\leqq n$ having exactly k prime factors multiple factors counted multiply. Then*

$$\pi_k'(n) = (1 + o(1)) \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}.$$

THEOREM III. *Denote by $\pi_k''(n)$ the number of square-free integers $\leqq n$ having exactly k prime factors. Then*

$$\pi_k''(n) = (1 + o(1)) \frac{6}{\pi^2} \frac{n}{\log n} \frac{x^{k-1}}{(k-1)!}.$$

It would be interesting to investigate that for what values of l(l depending on n) is $\lim \pi_l(n)/\pi_l'(n) = 1$ and $\lim \pi_l(n)/\pi_l''(n) = \pi^2/6$.

We obtain from Theorem 1 (confirming a conjecture of Hardy)[2] that

$$\pi_x(n) = (1 + o(1)) \frac{n}{(2\pi x)^{\frac{1}{2}}}. \tag{25}$$

From (25) and a theorem of Behrend[6] we deduce

THEOREM IV. *Let $a_1 < a_2 \cdots < a_u \leqq n$ be a sequence of integers no one of which divides the other, then for n sufficiently large*

$$\sum \frac{1}{a_i} < \left(\frac{1}{\sqrt{2\pi}} + \epsilon\right) \frac{\log n}{x^{\frac{1}{2}}} \tag{26}$$

and for a suitable sequence a_i

$$\sum \frac{1}{a_i} > \left(\frac{1}{\sqrt{2\pi}} - \epsilon\right) \frac{\log n}{x^{\frac{1}{2}}}. \tag{27}$$

Theorem IV means that

$$\overline{\lim} \sum \frac{1}{a_i} \frac{x^{\frac{1}{2}}}{\log n} = \frac{1}{\sqrt{2\pi}}.$$

[6] London Math. Soc. Journal, Vol. X (1935) p. 42–44.

We are going to give only the outline of the proof.

Behrend proved that

$$\sum \frac{1}{a_i} < c\,\frac{\log n}{x^{\frac{1}{2}}}$$

and it is not difficult to show that we can take $1/(\sqrt{2\pi}) + \epsilon$ for Behrend's c, (we omit the details). This proves (26). Further if we consider the integers having exactly x prime factors, we obtain (24) from (25) by a simple computation, this proves Theorem IV.

We can raise several problems:

1). For which k is $\pi_k(n)$ maximal? It immediately follows from (1) and (4) that $k = x + 0(x^{\frac{1}{2}})$, the same holds for the k which maximizes $\pi_k'(n)$ and $\pi_k''(n)$. It seems likely that $k = x + 0(1)$.

2). Does there exist a k_0 such that for $k_1 \leqq k_2 < k_0 \leqq k_3 < k_4$

$$\pi_{k_1}(n) < \pi_{k_2}(n); \qquad \pi_{k_3}(n) > \pi_{k_4}(n).^{7}$$

So far, I could not make any progress with 2). But we will solve an analogous question for

$$A_l = \sum_i \frac{1}{a_i^{(l)}}\,.$$

First of all it follows from (6) that for $x + c < l_1 < l_2$

$$A_{l_2} < A_{l_1}\,. \tag{28}$$

Thus it suffices to consider the values $l \leqq x + c$. First we show that for every $l \leqq x + c$

$$A_l > cx^l/l!. \tag{29}$$

Suppose that (28) is not true. Then for some $l \leqq x + c$

$$A_l < \epsilon x^l/l!$$

But then from (6) for all $l_1 > l$

$$A_{l_1} < \epsilon(x^l/l!)\,\frac{(x + c)^{l_1 - l}}{(l + 1)\cdots l_1} < \epsilon\,\frac{(x + c)^{l_1}}{l_1!}$$

but this clearly contradicts Lemma 2 for $l_1 < x + x^{\frac{1}{2}}$ say (if ϵ is sufficiently small), thus (29) is proved.

We have as in the proof of Lemma 2

$$(l + 1)\sum_i 1/a_i^{(l+1)} > \sum_i{}'' \frac{1}{a_i^{(l)}} \log\log \frac{n}{a_i^{(l)}} - c\log x \sum_i 1/a_i^{(l)} \tag{30}$$

[7] An analogous conjecture had been made by Auluck, Chowla and Gupta for $p_k(n)$ the number of partition of n into k summands. (Indian Journal of Math. 1942.)

where the two dashes indicate that the summation is extended over the $a_i^{(l)} < n^{1-1/x}$. Let $n^{1-1/x} < m < n$. Then we have from (1)

$$\sum_{m < a_i^{(l)} < 2m} 1/a_i^{(l)} < c \frac{x^l}{l! \log n}.$$

Thus we obtain from (29) and Lemma 5 by a simple calculation

$$\text{(31)} \quad \sum_i{}'' 1/a_i^{(l)} = \sum 1/a_i^{(l)} - \sum{}''' 1/a_i^{(l)} > \sum 1/a_i^{(l)} - c\, x^l/l! \frac{\frac{\log n}{x}}{\log n} > (1 - c/x) \sum 1/a_i^{(l)}$$

where in $\sum'''$

$$n^{1-1/x} \leqq a_i^{(l)} \leqq n.$$

Thus from (30) and (31)

$$\sum 1/a_i^{(l+1)} > (1 - c/x) \frac{x - \log x}{l+1} \sum 1/a_i^{(l)} - \frac{c \log x}{l+1} \sum 1/a_i^{(l)} > \frac{x - c\log x}{l+1} \sum 1/a_i^{(l)}.$$

Hence

$$\text{(32)} \qquad A_{l+1} > A_l$$

for $l < x - c \log x$. Thus we only have to consider the interval $x - c \log x < l \leqq x + c$. The method which we will now use applies to all l satisfying $x - cx^{\frac{1}{2}} < l \leqq x + c$.

We have as in the proof of Lemma 2.

$$\text{(33)} \quad (l+1) \sum 1/a_i^{(l+1)} = \sum 1/a_i^{(l)} \sum{}' \frac{1}{p^\alpha} = \sum \frac{1}{a_i^{(l)}} \sum{}'' \frac{1}{p^\alpha} - \sum \frac{1}{a_i^{(l)}} \sum{}''' \frac{1}{p^\alpha} = \sum{}_1 - \sum{}_2$$

the prime indicates that $p^\alpha \leqq n/a_i^{(l)}$ and $p^\alpha \nmid a_i^{(l)}$, the two primes that $p^\alpha \leqq n/a_i^{(l)}$, the three primes that $p^\alpha \leqq n/a_i^{(l)}$ and $p^\alpha \mid a_i^{(l)}$. We evidently have

$$\text{(34)} \qquad \sum{}_2 = \sum \frac{1}{p^{2\alpha}} \left(\sum{}_3 \frac{1}{a_i^{(l-1)}} + \sum{}_4 \frac{1}{a_i^{(l)}} \right).$$

In $\sum_3$

$$a_i^{(l-1)} \leqq \frac{n}{p^{2\alpha}}, \qquad a_i^{(l-1)} \not\equiv 0 \pmod p,$$

and in $\sum_4$

$$a_i^{(l)} \leqq \frac{n}{p^{2\alpha}}, \qquad a_i^{(l)} \equiv 0 \pmod p.$$

From Lemma 5, we obtain by a simple calculation that for $p \leqq n^{\epsilon}$

$$\sum\nolimits_3 \frac{1}{a_i^{(l-1)}} = (1 + o(1)) \frac{x^{l-1}}{(l-1)!}\left(1 - \frac{1}{p}\right) \tag{35}$$

and

$$\sum\nolimits_4 \frac{1}{a_i^{(l)}} = (1 + o(1)) \frac{x^l}{pl!}. \tag{36}$$

We have

$$\begin{aligned} &\sum_{p > n^{\epsilon}} \frac{1}{p^{2\alpha}}\left(\sum\nolimits_3 \frac{1}{a_i^{(l-1)}} + \sum\nolimits_4 \frac{1}{a_i^{(l)}}\right) \\ &\qquad \leqq \sum_{p > n^{\epsilon}} \frac{1}{p^{2\alpha}} \sum_{t \leqq (n/p^{2\alpha})} \frac{1}{t} < c \log n \sum_{p > n^{\epsilon}} \frac{1}{p^{2\alpha}} = o(1). \end{aligned} \tag{37}$$

Thus, finally from (34), (35), (36), and (37)

$$\sum\nolimits_2 = (1 + o(1)) \frac{x^l}{l!} \sum_p \frac{1}{p^{2\alpha}} = A_l \sum_p \frac{1}{p^{2\alpha}} + o(A_l). \tag{38}$$

Further

$$\sum\nolimits_1 = \sum \frac{1}{a_i^{(l)}} \sum_{p^{\alpha} \leqq n} \frac{1}{p^{\alpha}} - \sum \frac{1}{a_i^{(l)}} \sum\nolimits'''' \frac{1}{p^{\alpha}},$$

in $\sum''''$

$$\frac{n}{a_i^{(l)}} < p^{\alpha} \leqq n.$$

Thus it is easy to see that

$$\begin{aligned} \sum\nolimits_1 = A_l &\left(\log\log n + 0{\cdot}261 \cdots + \sum_{\alpha > 1} \frac{1}{p^{\alpha}}\right) \\ &- \sum_i \frac{1}{a_i^{(l)}}\left(\log\log n - \log\log \frac{n}{a_i^{(l)}}\right) + o(A_l). \end{aligned} \tag{39}$$

Now we obtain by a simple calculation

$$\begin{aligned} \sum = \sum \frac{1}{a_i^{(l)}}\left(\log\log n - \log\log \frac{n}{a_i^{(l)}}\right) &= \sum_{t=1}^{y-1} \sum\nolimits_t \frac{1}{a_i^{(l)}}\left(-\log \frac{t}{y}\right) \\ &+ \sum\nolimits_y \frac{1}{a_i^{(l)}}\left(\log\log n - \log\log \frac{n}{a_i^{(l)}}\right) + o(A_l), \end{aligned} \tag{40}$$

where in $\sum_t$

$$n^{t-1/y} \leqq a_i^{(l)} \leqq n^{t/y}.$$

We have by (1) for $n^{1-1/y} < m \leqq n$

$$\sum_{m \leqq a_i^{(l)} \leqq 2m} \frac{1}{a_i^{(l)}} < c \frac{x^l}{l! \log n}.$$

Thus by a simple calculation

$$\sum_{y} \frac{1}{a_i^{(l)}}\left(\log\log n - \log\log \frac{n}{a_i^{(l)}}\right)$$
$$< c\,\frac{x^l}{l!}\,\frac{1}{\log n}\sum_{k=1}^{(1/y)\log n} (\operatorname{loglog} n - \log k) = o\left(\frac{x^l}{l!}\right) = o(A_l).$$

Hence from (19) by a simple computation

(40a) $$\sum = A_l + o(A_l).$$

Thus from (39) and (40)

(41) $$\sum\nolimits_1 = A_l\left(\operatorname{loglog} n + 0\cdot261 \cdots + \sum_{\alpha>1} \frac{1}{p^\alpha} - 1\right) + o(A_l).$$

Hence from (33), (38) and (41)

(42) $$A_{l+1} = A_l\,\frac{\log\log n + 0\cdot261 + \sum_{p,\alpha} \frac{1}{p^{2\alpha+1}} - 1 + o(1)}{l+1}.$$

Put

$$C = 0\cdot261 + \sum_{p,\alpha}\frac{1}{p^{2\alpha+1}} - 1, \qquad -1 < C < 0.$$

Thus if n is sufficiently large

(43) $$A_{l+1} > A_l \text{ for } \operatorname{loglog} n + C - \epsilon > l + 1$$

and

(44) $$A_{l+1} < A_l \text{ for } \operatorname{loglog} n + C + \epsilon < l + 1.$$

To complete our proof, we shall show that

(45) $$A_{l+1} = A_l\,, \qquad l = \operatorname{loglog} n + 0(1)$$

is impossible for large n. Assume that

$$\sum \frac{1}{a_i^{(l)}} = \sum \frac{1}{a_i^{(l+1)}}\,.$$

Denote by P the greatest prime $\leqq n/2.3. \cdots p_{l-1}$. It follows from a theorem of Chebichev that (the theorem in question states that there always is a prime between t and $2t$)

(46) $$\frac{n}{2.2.3.\cdots p_{l-1}} < P < \frac{n}{2.3.\cdots p_{l-1}}.$$

Thus clearly none of the $a_i^{(l+1)}$ are multiples of P. Write

$$A_l = A_l' + A_l''$$

where A_l' is the sum of the reciprocals of the $a_i^{(l)}$ with $a_i^{(l)} \equiv 0 \pmod{P}$ and A_l'' is the sum of the other $1/a_i^{(l)}$. We must have

$$A_l' = A_{l+1} - A_l'' .$$

The smallest common denominator of the fractions on the left side is not a multiple of P. On the right side all the denominators are multiples of P. Write

$$A_l' = \frac{1}{P} \sum_j \frac{1}{x_j}.$$

It follows from (46) that all the x_j are square-free integers composed of primes $\leqq 2p_{l-1}$. Thus the number of x_j is $\leqq 2^{2p_{l-1}}$. Their common denominator is not greater than

$$\prod_{p \leqq 2p_{l-1}} p < 4^{2p_{l-1}}.$$

Thus from (45), if we write

$$A_l' = \frac{u}{Pv}, \qquad (u, v) = 1$$

$u < 8^{2p_{l-1}} < (\log n)^c < P$. Thus (45) is impossible, since P appears in the denominator of one side and not of the other. Thus we can state

THEOREM V. *Denote by A_{l_0} the greatest A. Then for sufficiently large n*

$$l_0 = [\log\log n + C], \qquad C = 0\cdot261 \cdots + \sum_{p,\alpha} \frac{1}{p^{2\alpha+1}} - 1, \qquad -1 < C < 0$$

except if $\log\log n + C = J + \epsilon$, *($j$ integer, ϵ small) in which case l_0 can be J or $J - 1$. Also if $l_1 < l_2 < l_0 < l_3 < l_4$*

$$A_{l_1} < A_{l_2} < A_{l_0} \textit{ and } A_{l_4} < A_{l_3} < A_{l_0}.$$

Denote

$$A_l' = \sum_i \frac{1}{b_i^{(l)}}, \qquad A_l'' = \sum \frac{1}{c_i^{(l)}}$$

where $b_i^{(l)}$ denotes the integers $\leqq n$ having exactly l prime factors, multiple factors counted multiply, and $c_i^{(l)}$ denotes the square-free integers $\leqq n$ having exactly l prime factors. An analogous theorem holds for A_l' and A_l' of course with different values of l_0.

ADDED IN PROOF: Recently I learned that about the same time as I, Sathe also obtained these results, Sathe's results have not yet been published.

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