A PROBLEM ON ORDERED SETS

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Introduction.

1. Let S be an ordered set, of power $|S|$ and order type $\bar{S} = \phi$. We denote by ϕ^* the *converse* of ϕ, *i.e.* the order type obtained from ϕ by replacing every order relation $x < y$ by the corresponding relation $y < x$, and by ω_n the least ordinal number of power $\aleph_n$. It is easy to see that, if $|S| = \aleph_0$, then S contains a subset S' such that either $\bar{S'} = \omega_0$ or $\bar{S}' = \omega_0^*$. For cardinals $\aleph_n > \aleph_0$ the corresponding property, with ω_0 replaced by ω_n, no longer holds. Thus, the linear continuum C, ordered by magnitude, satisfies $|C| = \aleph_m \geqslant \aleph_1$ but contains no subset of any of the types ω_m, ω_m^*. If, however, we assume the continuum hypothesis $2^{\aleph_0} = \aleph_1$, then $m = 1$, and the following statement is true. *Given any ordinal $\alpha < \omega_1$, there are subsets C_1 and C_2 of C, of order types α and α^* respectively.*

The question arises whether not only C but every ordered set S of cardinal $\aleph_1$ contains either (i) a subset of type ω_1, or (ii) a subset of type ω_1^*, or (iii) two subsets of types α and α^* respectively, corresponding to every ordinal $\alpha < \omega_1$. We shall show, assuming the continuum hypothesis‡ and making free use of the axiom of choice, that this is, in fact, true. More generally, we shall obtain, as principal result of this note, a simple characterization of those cardinals $\aleph_n$ which possess the following

Property P. If S is an ordered set, $|S| = \aleph_n$, and α is an ordinal number, $\alpha < \omega_n$, then either (i) there is $S' \subset S$ such that $\bar{S}' = \omega_n$, or (ii) there is $S'' \subset S$ such that $\bar{S}'' = \omega_n^*$, or (iii) there are subsets S_1 and S_2 of S such that $\bar{S}_1 = \alpha$; $\bar{S}_2 = \alpha^*$.

We denote, for any cardinal number a, by a^- the immediate predecessor of a provided that such a predecessor exists, and we put $a^- = a$ in all other cases, *i.e.* when a is a limit number. We recall that a is called *singular* if a can be represented in the form $a = \sum_{\rho \in R} a_\rho$, where $|R| < a$; $a_\rho < a$, and *regular* if no such representation exists.

We shall prove the following

Theorem. *Suppose that the generalized continuum hypothesis $2^{\aleph_\nu} = \aleph_{\nu+1}$ holds for every ν. Then a cardinal number $\aleph_n$ possesses the property P if, and only if, $\aleph_n^-$ is regular.*

In fact, the continuum hypothesis is not required for the proof that P does not hold when $\aleph_n^-$ is singular, and for the proof of the converse proposition it is only required for $\nu < n$.

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‡ See addendum.

Since $\aleph_{\omega_0}$ $(=\aleph_0+\aleph_1+\ldots)$ is singular, the theorem asserts, for instance, that $\aleph_{\omega_0+2}$ possesses the property P, and that neither $\aleph_{\omega_0+1}$ nor $\aleph_{\omega_0}$ possesses that property.

J. C. Shepherdson (**1**) has investigated the structure of ordered sets which contain only well-ordered subsets of given types. His methods and results do not appear to have any direct connection with the problem considered in the present paper. He has, however, informed us that he has since proved that $\aleph_1$ has the property P, and W. Sierpiński has obtained the more general result that every ordered set of cardinal $\aleph_1$ contains a subset of one of the types ω_1, $\omega_1{}^*$, η, where η is the order type of the set of rational numbers ordered by magnitude.

Notation and definitions.

2. Small Greek letters denote order types, *i.e.* ordinal numbers as well as order types of sets whose order is not a well-order. Small Latin letters are used to denote either ordinal numbers or cardinal numbers as well as elements of abstract sets. Instead of order type, ordinal number, and cardinal number we say *type*, *ordinal*, and *cardinal* respectively. We need not distinguish between a finite ordinal and the corresponding finite cardinal. The relation $S' \subset S$ denotes set inclusion in the wide sense.

If the set S is ordered by the order relation $x<y$ then the type of S is denoted by $\bar{S}_{<}$, and, if there is no risk of confusion, by $\bar{S}$. If a second ordering of the same set has to be introduced the new order relation will be denoted by $<<$ and the new order type by $\bar{S}_{<<}$. If $\bar{S}=\alpha$, then we define the *cardinal* $|\alpha|$ of α by putting $|\alpha|=|S|$. The relation $\beta\leqslant\alpha$ means that, if $\bar{S}=\alpha$, then there is $S'\subset S$ such that $\bar{S}'=\beta$. The relation $\beta\nleqslant\alpha$ means that the relation $\beta\leqslant\alpha$ is false. It is worth noting that a set T of types is not ordered, in the strict sense of the word, by our relation "$\leqslant$", but that only a quasi-ordering† is defined in T, which means that the relation "$\leqslant$" between types is transitive but that there may be two distinct types α and β satisfying both $\beta\leqslant\alpha$ and $\alpha\leqslant\beta$. If, however, α is an ordinal, then $\beta\leqslant\alpha$ implies that β is an ordinal, and in this case the two relations $\beta\leqslant\alpha$; $\alpha\leqslant\beta$ only hold if $\alpha=\beta$.

If $\bar{S}_{<}=\phi$, and if the relation "$<<$" is the converse of the relation "$<$", so that $x<<y$ is equivalent to $y<x$, then $\bar{S}_{<<}$ is the *converse* of ϕ and is denoted by ϕ^*. Clearly‡, $\beta^*\leqslant\alpha^*$ holds if, and only if, $\beta\leqslant\alpha$.

The property P is equivalent to the following

Property P'. If $|\phi|=\aleph_n$; $\alpha<\omega_n$, then either $\omega_n{}^*\leqslant\phi$ or $\alpha\leqslant\phi$.

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† (**2**), p. 4.

‡ This is contrary to the convention used in (**1**), p. 292, where, in the case o fordinals α and β, the relation $\beta^*\leqslant\alpha^*$ is taken to be equivalent to $\alpha\leqslant\beta$.

For, first of all, suppose that $\aleph_n$ has property P. Let $|\phi| = \aleph_n$, $\alpha < \omega_n$, and consider an ordered set S such that $\bar{S} = \phi$. Then (i) of property P implies $\omega_n \leqslant \phi$ and hence $\alpha \leqslant \phi$, (ii) of property P implies $\omega_n{}^* \leqslant \phi$, while (iii) of property P implies $\alpha \leqslant \phi$. Therefore P implies P'.

On the other hand, suppose that P' holds for some $\aleph_n$. Let $|S| = \aleph_n$; $\bar{S}_< = \phi$; $\bar{S}_{<<} = \phi^*$. Then, by applying the definition of P' to both the types, ϕ and ϕ^*, we find subsets S' and S'' of S such that

$$\textit{either} \quad \bar{S}'_< = \omega_n{}^* \textit{ or } \quad \bar{S}'_< = \alpha \tag{1}$$

and

$$\textit{either } \bar{S}''_{<<} = \omega_n{}^* \textit{ or } \bar{S}''_{<<} = \alpha. \tag{2}$$

Now, (1) implies that

$$\textit{either } \omega_n{}^* \leqslant \phi \textit{ or } \alpha \leqslant \phi, \tag{3}$$

and (2) implies that

$$\textit{either } \omega_n{}^* \leqslant \phi^* \textit{ or } \alpha \leqslant \phi^*. \tag{4}$$

By combining, in the four possible ways, one alternative of (3) with one of (4), we find that, in any case,

$$\textit{either } \text{(i) } \omega_n \leqslant \phi \textit{ or } \text{(ii) } \omega_n{}^* \leqslant \phi \textit{ or}\dagger \text{ (iii) } \alpha, \alpha^* \leqslant \phi,$$

so that $\aleph_n$ satisfies P.

3. The description of our arguments is greatly simplified by the introduction of the *decomposition relation*

$$a \rightarrow (b_1, b_2)^2 \tag{5}$$

between cardinals a, b_1, b_2 which will now be defined. For any set S we denote by $\Omega_2(S)$ the set of all sets $S' \subset S$ such that $|S'| = 2$. Then we say that (5) holds if, and only if, the following statement is true. Whenever

$$|S| = a; \quad \Omega_2(S) = K_1 + K_2,$$

then there is $S' \subset S$ and $\lambda \varepsilon \{1, 2\}$ such that

$$|S'| = b_\lambda; \quad \Omega_2(S') \subset K_\lambda.$$

The relation (5) is fundamental in many investigations in set theory. The authors hope to deal in another paper with its numerous interesting properties and generalizations. In the present note it only serves as a convenient abbreviation. Clearly, (5) is equivalent to

$$a \rightarrow (b_2, b_1)^2. \tag{6}$$

† The relation α, $\alpha^* \leqslant \phi$, and similarly in other cases, means that both, $\alpha \leqslant \phi$ and $\alpha^* \leqslant \phi$, hold.

Proof of the negative part of the Theorem.

4. We begin by proving two lemmas.

LEMMA 1. *If $k,\ n \geqslant 0$, and $\alpha_\nu < \omega_n$ for $\nu < \omega_k$, then*†

$$\omega_n{}^*,\ \omega_{k+1} \nleqslant \Sigma \nu < \omega_k \,\Box\, \alpha_\nu{}^*.$$

Proof. Put $S = \Sigma_{\lambda < \alpha_\nu}^{\nu < \omega_k} \Box \{(\nu,\ \lambda)\}$. We order S by putting

$$(\nu,\ \lambda) < (\nu',\ \lambda')$$

if, and only if, either (i) $\nu < \nu'$ or (ii) $\nu = \nu'$; $\lambda > \lambda'$. Then

$$\bar{S} = \Sigma\, \nu < \omega_k \,\Box\, \alpha_\nu{}^* = \phi, \quad \text{say.}$$

If $S' \subset S$; $\bar{S}' = \omega_{k+1}$ then, for every $\nu_0 < \omega_k$,

$$|\Sigma (\nu_0,\ \lambda) \,\varepsilon\, S' \,\Box\, \{\lambda\}| < \aleph_0,$$

and hence $$|S'| \leqslant \Sigma\, \nu_0 < \omega_k \,\Box\, \aleph_0 = \aleph_k \aleph_0 = \aleph_k,$$

which is the desired contradiction.

On the other hand, if $S'' \subset S$; $\bar{S}'' = \omega_n{}^*$ then

$$|\Sigma (\nu,\ \lambda) \,\varepsilon\, S'' \,\Box\, \{\nu\}| < \aleph_0,$$

and therefore, for some finite number of ordinals ν_k,

$$|S''| \leqslant \Sigma\, |\alpha_{\nu_k}| < \aleph_n$$

which, again, is a contradiction. This proves Lemma 1.

5. Let $a = \aleph_m$ and $b = \aleph_l$ be infinite cardinals. We denote by F_{ab} the set of all functions $f(\lambda)$, defined for $\lambda < \omega_l$, whose functional values are ordinals $f(\lambda) < \omega_m$. We order F_{ab} alphabetically, *i.e.* we put $f_1 < f_2$ if, and only if, there is $\lambda_0 < \omega_l$ such that $f_1(\lambda) = f_2(\lambda)$ for $\lambda < \lambda_0$; $f_1(\lambda_0) < f_2(\lambda_0)$.

LEMMA 2. *Let $a = \aleph_m$, and let $b = \aleph_l$ be the least cardinal such that $a^b > a$. Then $\omega_{l+1}^*,\ \omega_{m+1} \nleqslant \bar{F}_{ab}$.*

Proof. The letter λ denotes ordinals, $\lambda < \omega_l$, and $F = F_{ab}$.

(i) Let $F' \subset F$; $\bar{F}' = \omega_{m+1}$. We have to deduce a contradiction. Let $f \,\varepsilon\, F'$. We define a function $\tau(f) \,\varepsilon\, F$ as follows. In F', f has an immediate successor $\sigma(f) = g$, say. Then there is $\lambda_0 = \lambda_0(f)$ such that $f(\lambda) = g(\lambda)$ for $\lambda < \lambda_0$; $f(\lambda_0) < g(\lambda_0)$. Now we put $\tau(f) = h$, where $h(\lambda) = g(\lambda)$

† For typographical convenience we write $\Sigma \nu < \omega_k \Box \alpha_\nu{}^*$ instead of $\sum\limits_{\nu < \omega_k} \alpha_\nu{}^*$, and similarly in other cases, where the sign $\Box$ is used to separate the summation conditions from the terms to be summed.

for $\lambda \leqslant \lambda_0$, and $h(\lambda) = 0$ for $\lambda > \lambda_0$. Then†, if $\{f_1, f_2\}_< \subset F'$,

$$\tau(f_1) \leqslant \sigma(f_1) \leqslant f_2 < \tau(f_2).$$

We conclude that $|F'| \leqslant |F''|$, where F'' is the set of all functions $j(\lambda)$ such that, for some $\bar{\lambda} = \bar{\lambda}(j) < \omega_l$, $j(\lambda) < \omega_m$ for $\lambda \leqslant \bar{\lambda}$, and $j(\lambda) = 0$ for $\lambda > \bar{\lambda}$. Hence, using the minimum property of b,

$$a < |F'| \leqslant |F''| = \Sigma\lambda < \omega_l \square a^{|\lambda|} = \Sigma\lambda < \omega_l \square a = ab = a,$$

which is the desired contradiction. The last equation follows from the fact that $a^a > a$, so that $b \leqslant a$.

(ii) Let $F' \subset F$; $\bar{F}' = \omega_{l+1}^*$. Then there is $f_\mu \,\varepsilon\, F'$, for $\mu < \omega_{l+1}$, such that $f_\mu > f_\nu$ for $\mu < \nu < \omega_{l+1}$. The letter μ always denotes numbers such that $\mu < \omega_{l+1}$. We define, inductively, numbers $\bar{\mu}(\lambda)$ as follows. Let $\lambda_0 < \omega_l$, and suppose that $\bar{\mu}(\lambda)$ has already been defined for $\lambda < \lambda_0$, and that, for $\lambda < \lambda_0$; $\mu \geqslant \bar{\mu}(\lambda)$, we have

$$f_\mu(\lambda) = f_{\bar{\mu}(\lambda)}(\lambda). \tag{7}$$

Then
$$\Sigma\lambda < \lambda_0 \square |\bar{\mu}(\lambda)| \leqslant |\lambda_0| b \leqslant b,$$

and hence there is μ_0 such that $\bar{\mu}(\lambda) \leqslant \mu_0$ for $\lambda < \lambda_0$. Then

$$f_\mu(\lambda) = f_{\mu_0}(\lambda) \text{ for } \lambda < \lambda_0;\ \mu \geqslant \mu_0,$$

and $f_{\mu_1}(\lambda_0) \geqslant f_{\mu_2}(\lambda_0)$, if $\mu_0 \leqslant \mu_1 \leqslant \mu_2$. Hence, by the definition of well-ordering, there is $\bar{\mu}(\lambda_0) \geqslant \mu_0$ such that (7) holds for $\lambda = \lambda_0$; $\mu \geqslant \bar{\mu}(\lambda_0)$. This completes the inductive definition of $\bar{\mu}(\lambda)$ such that (7) holds for $\lambda < \omega_l$; $\mu \geqslant \bar{\mu}(\lambda)$. Then

$$\Sigma\lambda < \omega_l \square |\bar{\mu}(\lambda)| \leqslant bb = b,$$

and hence there is μ_3 such that $\bar{\mu}(\lambda) \leqslant \mu_3$ for all $\lambda < \omega_l$. Then, for all $\lambda < \omega_l$, $f_{\mu_3}(\lambda) = f_{\mu_3+1}(\lambda)$, so that $f_{\mu_3} = f_{\mu_3+1}$, which is a contradiction. This proves the lemma.

6. We can now prove the negative part of our theorem. Let us assume that $\aleph_n^-$ is singular.

Case 1. Let $\aleph_n^- = \aleph_n$. Then $\aleph_n = \Sigma\nu < \omega_k \square \aleph_{m_\nu}$, where $k < n$; $m_\nu < n$. Then, by Lemma 1, $\omega_n{}^*$, $\alpha \nleqslant \phi$, where

$$\alpha = \omega_{k+1} < \omega_n;\ \ \phi = \Sigma\nu < \omega_k \ulcorner \omega_{m_\nu}^*;\ \ |\phi| = \aleph_n.$$

Hence $\aleph_n$ does not possess the property P'.

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† The symbol $\{f_1, f_2\}_<$ denotes the set $\{f_1, f_2\}$ and, at the same time, expresses the fact that $f_1 < f_2$. A similar notation is used in other cases when sets and relations between the elements of these sets are to be exhibited.

Case 2. Let $\aleph_n^- = \aleph_m < \aleph_n$. Then $n = m+1$, and $a = \aleph_m$ is singular. There is a representation $a = \Sigma\, \nu \,\varepsilon\, N \,[]\, a_\nu$ such that $|N|$, $a_\nu < a$. Then, by König's Theorem, $a < \Pi\, \nu \,\varepsilon\, N \,[]\, a = a^{|N|}$, and if $b = \aleph_l$ is the least cardinal satisfying $a^b > a$, then $b \leqslant |N| < a$; $l < m$. Now, by Lemma 2, we have ω_{l+1}^*, $\omega_{m+1} \not\leqslant \bar{F}_{ab}$, and hence $\omega_n{}^*$, $\alpha \not\leqslant \phi$, where $\phi = (\bar{F}_{ab})^*$,

$$|\phi| = a^b > a = \aleph_m; \quad |\phi| \geqslant \aleph_n; \quad \alpha = \omega_{l+1} < \omega_n.$$

Again, it follows that $\aleph_n$ does not possess the property P'.

The case of a limit number $\aleph_n$.

7. Throughout the rest of this paper we assume the generalized continuum hypothesis, *i.e.* the equation

$$\aleph_{\nu+1} = 2^{\aleph_\nu},$$

for all ν.

LEMMA 3. *Let a be a regular limit cardinal, and $b < a$. Then* $a \rightarrow (b, a)^2$.

Proof. Let $|S| = a$; $\Omega_2(S) = K_1 + K_2$. Suppose that,

$$\textit{if } S' \subset S;\ \Omega_2(S') \subset K_2 \textit{ then } |S'| < a. \tag{8}$$

Our aim is to find a set $S'' \subset S$ such that

$$\Omega_2(S'') \subset K_1; \quad |S''| = b. \tag{9}$$

Corresponding to every set $T \subset S$ we choose a set $B(T)$ such that (i) $B(T) \subset T$, (ii) $\Omega_2\big(B(T)\big) \subset K_2$, (iii) for fixed T the set $B(T)$ is maximal among all sets $U \subset T$ such that $\Omega_2(U) \subset K_2$. The existence of such a set $B(T)$ follows from Zorn's Lemma. We choose a fixed ordinal $\rho^{(0)}$ such that $|\rho^{(0)}| > |S|$, and we agree that the letters λ, μ, ν always denote ordinals less than $\rho^{(0)}$.

We well-order S. For all ν and all $x \,\varepsilon\, S$, we define $f_\nu(x) \,\varepsilon\, S$ as follows. Let x be fixed, and suppose that, for some ν_0, the elements $f_\nu(x)$ have already been defined for $\nu < \nu_0$, and that

$$\{f_\nu(x),\, x\} \,\varepsilon\, K_1$$

for all those $\nu < \nu_0$ for which $f_\nu(x) \neq x$. We shall now define $f_{\nu_0}(x)$.

Case 1. Suppose that $f_\nu(x) \neq x$ for $\nu < \nu_0$. Then we denote by S_0 the set of all $y \,\varepsilon\, S$ such that

$$\{f_\nu(x),\, y\} \,\varepsilon\, K_1 \text{ for } \nu < \nu_0.$$

Thus $x \,\varepsilon\, S_0$.

There are now two possibilities. Firstly, if $x \,\varepsilon\, B(S_0)$, we put $f_{\nu_0}(x) = x$. Secondly, if $x \notin B(S_0)$, then, by part (iii) of the definition of $B(T)$, there is a first element z of $B(S_0)$ such that $\{z, x\} \,\varepsilon\, K_1$. Then we put $f_{\nu_0}(x) = z$.

Case 2. Suppose that there is a number $\nu < \nu_0$ such that $f_\nu(x) = x$. Then we put $f_{\nu_0}(x) = x$. This completes the definition, by transfinite construction, of elements $f_\nu(x)$, for $\nu < \rho^{(0)}$ and $x \varepsilon S$, such that

$$\{f_\mu(x),\ f_\nu(x)\} \in K_1 \tag{10}$$

if $\mu < \nu < \rho^{(0)}$; $f_\mu(x) \neq x$.

Since $|\rho^{(0)}| > |S|$, it is impossible that, for some fixed x, all $f_\nu(x)$ are different from each other, and hence it is impossible that, for some fixed x, $f_\nu(x) \neq x$ for all ν. Hence there is $\bar{\nu}(x)$ such that $f_\nu(x) \neq x$ for $\nu < \bar{\nu}(x)$, and $f_\nu(x) = x$ for $\nu \geqslant \bar{\nu}(x)$. We put $M_\nu = \Sigma x \varepsilon S \,\square\, \{f_\nu(x)\}$. We shall show that, if $|\nu| < a$,

$$|M_\nu| < a. \tag{11}$$

First of all, $M_0 \subset B(S)$, and hence, by (8), $|M_0| \leqslant |B(S)| < a$. Thus (11) holds for $\nu = 0$. Now let $0 < |\nu_3| < a$, and suppose that (11) holds for $\nu < \nu_3$. Corresponding to every $x \varepsilon S$ there belongs a system of elements $y_\nu = f_\nu(x)$ $(\nu < \nu_3)$. The number n of distinct systems y_ν arising in this way satisfies

$$n \leqslant \Pi\, \nu < \nu_3 \,\square\, |M_\nu|.$$

Now, since a is a regular cardinal,

$$\Sigma\, \nu < \nu_3 \,\square\, |M_\nu| = d < a.$$

A. Tarski† proved that, for every regular limit cardinal a, and $a_1, a_2 < a$, we have $a_1^{a_2} < a$. By applying this result to $a_1 = d$, $a_2 = |\nu_3|$, we obtain

$$n \leqslant d^{|\nu_3|} < a. \tag{12}$$

It follows from (8) and the definition of $f_{\nu_3}(x)$ that, given any system of elements $\bar{y}_\nu \varepsilon S$ $(\nu < \nu_3)$, the cardinal of the set of all elements $f_{\nu_3}(x)$ corresponding to x's such that $f_\nu(x) = \bar{y}_\nu$ for all $\nu < \nu_3$, is less than a, *i.e.* that

$$|\Sigma f_\nu(x) = \bar{y}_\nu \text{ for all } \nu < \nu_3 \,\square\, \{f_{\nu_3}(x)\}| < a.$$

By (12), the number of distinct systems $\bar{y}_\nu$ $(\nu < \nu_3)$ which need be considered is also less than a. Hence $M_{\nu_3} < aa = a$. This shows that (11) holds for all ν such that $|\nu| < a$. Then, in view of the regularity of a,

$$|\Sigma_{x \varepsilon S}^{|\nu| < b} \,\square\, \{f_\nu(x)\}| \leqslant \Sigma\, |\nu| < b \,\square\, |M_\nu| < a = |S|,$$

and therefore there exists $\bar{x}_0 \varepsilon S$ such that

$$\bar{x}_0 \neq f_\nu(x) \text{ for } |\nu| < b;\ x \varepsilon S.$$

In particular, $\bar{x}_0 \neq f_\nu(\bar{x}_0)$ for $|\nu| < b$. Put

$$S'' = \Sigma\, |\nu| < b \,\square\, \{f_\nu(\bar{x}_0)\}.$$

Then (9) follows from (10).

† (3), Satz 9.

8. We shall now prove the positive part of the theorem in the case when $\aleph_n$ is a regular limit number, so that $\aleph_n^- = \aleph_n$. This number will be fixed throughout the proof. Our aim is to establish, for $\alpha < \omega_n$, the following

PROPOSITION Q_α. *If $|\phi| \geqslant \aleph_n$, then either $\omega_n^* \leqslant \phi$ or $\alpha \leqslant \phi$.*

For if this is shown then $\aleph_n$ possesses the property P' defined in § 2, and this was seen to be equivalent to $\aleph_n$ possessing the property P defined in the introduction.

Let $\beta < \omega_n$, and assume that Q_α is true for $\alpha < \beta$ but that Q_β is false. We shall deduce a contradiction. Clearly, $\beta \geqslant \omega_0$.

We suppose then that there is an ordered set S_0 such that $|S_0| \geqslant \aleph_n$,

$$\omega_n^*,\ \beta \not\leqslant \bar{S}_0. \tag{13}$$

Then $n > 0$. Let S be the set of all sections L of S_0, *i.e.* of all subsets L of S_0 such that $x < y \,\varepsilon\, L$ implies $x \,\varepsilon\, L$. We order S by inclusion, *i.e.* we put $L < L'$ if, and only if, $L \subset \neq L'$. Then $|S| \geqslant |S_0| \geqslant \aleph_n$. For every non-empty subset S' of S we denote by

$$\underline{\mathrm{bd}}\, L \,\varepsilon\, S' \,\square\, L, \quad \overline{\mathrm{bd}}\, L \,\varepsilon\, S' \,\square\, L \tag{14}$$

the intersection and the union respectively of all sets $L \,\varepsilon\, S'$. Then the two sets (14) are elements of S. We have

$$\omega_n^*,\ \beta \not\leqslant \bar{S}. \tag{15}$$

For $\omega_n^* \leqslant \bar{S}$ would imply the existence of a system $L_\nu \,\varepsilon\, S$ $(\nu < \omega_n)$ such that $L_\nu \subset \neq L_\mu$ for $\mu < \nu < \omega_n$. Then we could choose $a_\nu \,\varepsilon\, L_\nu - L_{\nu+1}$ and find that†

$$\omega_n^* = Tp(\Sigma\, \nu < \omega_n \,\square\, \{a_\nu\}) \leqslant \bar{S}_0,$$

which contradicts (13). Similarly, $\beta \leqslant \bar{S}$ would imply the existence of $L_\nu' \,\varepsilon\, S$ $(\nu < \beta)$ such that $L_\mu' \subset \neq L_\nu'$ for $\mu < \nu < \beta$. Then, choosing $a_\nu' \,\varepsilon\, L_{\nu+1}' - L_\nu'$, we find

$$\beta = Tp(\Sigma\, \nu < \beta \,\square\, \{a_\nu'\}) \leqslant \bar{S}_0,$$

again a contradiction against (13). This proves (15).

For the rest of this proof the letters x, y, z denote typical elements of S. If $x \leqslant y$ then we put $x \equiv y$ if, and only if,

$$|\Sigma\, x \leqslant z \leqslant y \,\square\, \{z\}| < \aleph_n,$$

and we define the relation $y \equiv x$ to be equivalent to $x \equiv y$. Then "$\equiv$" is an equivalence relation. Let x_ρ $(\rho \,\varepsilon\, R)$ be a system of representatives of

† Occasionally we write $Tp(U)$ instead of $\bar{U}$.

the corresponding equivalence classes, so that, given any x, there is exactly one $\rho \varepsilon R$ such that $x \equiv x_\rho$. Put

$$y_\rho = \overline{\mathrm{bd}}\, x \equiv x_\rho \,\square\, x; \quad y_\rho' = \underline{\mathrm{bd}}\, x \equiv x_\rho \,\square\, x.$$

Then
$$y_\rho \equiv y_\rho' \equiv x_\rho. \tag{16}$$

For, first of all, let us assume that, for some ρ, $y_\rho \not\equiv x_\rho$. Then $x_\rho < y_\rho$. We define, inductively, elements z_ν as follows. Put $z_0 = x_\rho$. If $\nu < \omega_n$, and if z_μ has already been defined for $\mu < \nu$, such that

$$x_\rho \leqslant z_\mu \equiv x_\rho \text{ for } \mu < \nu,$$

then we conclude from $|\nu| < \aleph_n$; $x_\rho \equiv z_\mu$, and the regularity of $\aleph_n$, that

$$\left| \sum_{\mu < \nu} \Sigma x_\rho \leqslant z \leqslant z_\mu \,\square\, \{z\} \right| < \aleph_n = |\Sigma x_\rho < z < y_\rho \,\square\, \{z\}|,$$

and hence that there exists z_ν such that $x_\rho < z_\nu < y_\rho$,

$$z_\mu < z_\nu \text{ for } \mu < \nu.$$

Then, by definition of y_ρ, there is $x' \equiv x_\rho$ such that $x_\rho \leqslant z_\nu \leqslant x'$. Then, by definition of "$\equiv$", we have $z_\nu \equiv x_\rho$. This completes the construction of z_ν for all $\nu < \omega_n$. We have, however,

$$\omega_n = Tp(\Sigma\, \nu < \omega_n \,\square\, \{z_\nu\}) \leqslant \bar{S}$$

which, in view of $\beta < \omega_n$, contradicts (15).

For reasons of symmetry, the assumption $y_\rho' \not\equiv x_\rho$ would lead to $\omega_n{}^* \leqslant \bar{S}$, which, again, contradicts (15). Hence (16) holds.

Let the letters ρ and σ denote typical elements of R. Clearly, $|R| \leqslant |S| = \aleph_n$. Also,

$$|\Sigma x \equiv x_\rho \,\square\, \{x\}| = |\Sigma y_\rho' \leqslant z \leqslant y_\rho \,\square\, \{z\}| < \aleph_n \quad (\rho \,\varepsilon\, R).$$

If, now, $|R| < \aleph_n$, then we obtain, since $\aleph_n$ is regular,

$$|S| = \left| \sum_\rho \Sigma y_\rho' \leqslant z \leqslant y_\rho \,\square\, \{z\} \right| < \aleph_n,$$

which is false. Hence $|R| = \aleph_n$. We well-order R, by means of a relation "$<<$", and put†

$$\Omega_2(R) = K_1 + K_2, \tag{17}$$

where K_1 is the set of all $\{\rho, \sigma\}_<$ such that $x_\rho << x_\sigma$, and K_2 is the set of all $\{\rho, \sigma\}_<$ such that $x_\rho >> x_\sigma$. Since $\aleph_n$ is a limit cardinal $> \aleph_0$ and $\beta < \omega_n$, there is $m < n$ such that $\beta < \omega_m$. Now, by applying Lemma 3, with $a = \aleph_n$; $b = \aleph_m$, we find a set $R' \subset R$ such that either

$$|R'| = \aleph_m; \quad \Omega_2(R') \subset K_1 \tag{18}$$

or
$$|R'| = \aleph_n; \quad \Omega_2(R') \subset K_2. \tag{19}$$

† This idea of defining a decomposition of $\Omega_2(R)$ by means of two order relations on R was first used by Sierpiński.

If (19) holds then, in view of the definition of K_2,

$$\omega_n^* \leqslant Tp(\Sigma\rho \,\varepsilon\, R' \,\square\, \{x_\rho\}) \leqslant \bar{S}$$

which contradicts (15). Hence (18) holds. The ordering of S induces an ordering of the set of all x_ρ, for $\rho \,\varepsilon\, R'$, and we may put

$$\Sigma\, \rho \,\varepsilon\, R' \,\square\, \{x_\rho\} = \Sigma\, \nu < \nu_0 \,\square\, \{x_\nu'\},$$

where $\nu_0 \geqslant \omega_m$, and $x_\mu' < x_\nu'$ whenever $\mu < \nu < \nu_0$. Let

$$S_\nu = \Sigma\, x_\nu' < z < x_{\nu+1}' \,\square\, \{z\} \quad (\nu < \omega_m).$$

Then, by definition of "$\equiv$", $|S_\nu| \geqslant \aleph_n$.

We can write

$$\Sigma\, 0 < \alpha < \beta \,\square\, \{\alpha\} = \Sigma\, \nu < \omega_m \,\square\, \{\alpha_\nu\},$$

where the α_ν are not necessarily mutually distinct. Since Q_α holds for $\alpha < \beta$, there is $S_\nu' \subset S_\nu$ such that

$$\bar{S}_\nu' = \alpha_\nu,$$

$$\beta \leqslant \Sigma\, \nu < \omega_m \,\square\, \alpha_\nu = Tp(\Sigma\, \nu < \omega_m \,\square\, S_\nu') \leqslant \bar{S},$$

which contradicts (15). This completes the proof of the theorem in the case when $\aleph_n$ is a limit number.

$\aleph_n$ not a limit number.

9. For any cardinal a, we denote by a^+ the next larger cardinal.

LEMMA 4. *If $a \geqslant \aleph_0$, and b is the least cardinal such that $a^b > a$, then $a^+ \to (b, a^+)^2$.*

Proof. The proof is similar to that of Lemma 3. Let

$$|S| = a^+; \quad \Omega_2(S) = K_1 + K_2,$$

and suppose that

$$\left.\begin{array}{ll} \textit{whenever} & S' \subset S;\ \Omega_2(S') \subset K_2, \\ \textit{then} & |S'| < a^+. \end{array}\right\} \tag{20}$$

Our aim is to find a set $S'' \subset S$ such that

$$\Omega_2(S'') \subset K_1; \quad |S''| = b. \tag{21}$$

We well-order S, and we choose $B(T)$, $\rho^{(0)}$ and define $f_\nu(x)$ and M_ν exactly as in the proof of Lemma 3. Then, for $|\nu| < b$,

$$|M_\nu| \leqslant a. \tag{22}$$

For, $M_0 \subset B(S)$, and hence, by (20), $|M_0| \leqslant a$. Let $0 < |\nu_3| < b$, and suppose that (22) holds for $\nu < \nu_3$. Then the number n of distinct systems of elements $f_\nu(x)$ $(\nu < \nu_3)$ satisfies, in view of $|\nu_3| < b$ and the minimum property of b,

$$n \leqslant \Pi\, \nu < \nu_3 \square |M_\nu| \leqslant a^{|\nu_3|} \leqslant a. \tag{23}$$

It follows from (20) and the definition of $f_{\nu_3}(x)$ that, given any $\bar{y}_\nu \,\varepsilon\, S$ $(\nu < \nu_3)$, the cardinal of the set of all elements $f_{\nu_3}(x)$ such that $f_\nu(x) = \bar{y}_\nu$ for all $\nu < \nu_3$, is at most equal to a. Thus, for fixed $\bar{y}_\nu$,

$$|\Sigma f_\nu(x) = \bar{y}_\nu \ (\nu < \nu_3) \square \{f_{\nu_3}(x)\}| \leqslant a.$$

Hence, in view of (23),

$$|M_{\nu_3}| \leqslant aa = a.$$

This establishes, by induction, the inequality (22) for all ν such that $|\nu| < b$.

Using $b \leqslant a$, which follows from $a^a > a$ and the definition of b, we conclude that

$$|\Sigma_{x \,\varepsilon\, S}^{|\nu| < b} \square \{f_\nu(x)\}| \leqslant \Sigma\, |\nu| < b \square |M_\nu| \leqslant ba \leqslant a < |S|,$$

and hence deduce the existence of $\bar{x}_0 \,\varepsilon\, S$ such that $\bar{x}_0 \neq f_\nu(x)$ for $|\nu| < b$; $x \,\varepsilon\, S$. Then the set

$$S'' = \Sigma\, |\nu| < b \square \{f_\nu(\bar{x}_0)\}$$

satisfies (21), and Lemma 4 is proved.

10. We shall now prove the positive part of the theorem in the case when $\aleph_n^- = \aleph_m$ is regular, $n = m+1$. We note that $\aleph_n$ is regular. The proof is identical with that of § 8 up to the definition of the decomposition (17).

It follows from results of A. Tarski† that, if a is regular and $a_1 < a$, then $a^{a_1} \leqslant a$. Hence, applying this result to $a = \aleph_m$, we find that the cardinal b of Lemma 4 is equal to a, so that, by Lemma 4,

$$|R| = \aleph_n \rightarrow (\aleph_m, \aleph_n)^2.$$

By applying the information contained in this relation to the decomposition (17) we find that there is a set $R' \subset R$ such that

$$\text{either} \quad |R'| = \aleph_m; \quad \Omega_2(R') \subset K_1 \tag{24}$$

$$\text{or} \quad |R'| = \aleph_n; \quad \Omega_2(R') \subset K_2. \tag{25}$$

But (25) implies that, by definition of K_2,

$$\omega_n{}^* \leqslant Tp(\Sigma\, \rho \,\varepsilon\, R' \square \{x_\rho\}) \leqslant \bar{S},$$

which contradicts (15). Hence (24) holds. The rest of the argument is again identical with that used in § 8, from the point onwards when (18) had been established. This completes the proof of the theorem.

† (3), Satz 9 and Satz 13, for limit numbers a. For non-limit numbers the statement is trivial.

Concluding remarks.

11. We shall now prove the following results which show that Lemma 4 is best possible.

If $a \geqslant \aleph_0$, *and* b *is the least cardinal such that* $a^b > a$, *then*†

$$a^+ \to (b, a^+)^2, \tag{26}$$

$$a^+ \not\to (b^+, a^+)^2, \tag{27}$$

$$a^b \not\to (b^+, a^+)^2. \tag{28}$$

One of us‡ has proved, assuming, as we do in the present note, the generalized continuum hypothesis, that

$$d^{++} \to (d^+, d^{++})^2 \tag{29}$$

for $d \geqslant \aleph_0$. This result is equivalent to the special case of (26) when a is not a limit number. For if $a = d^+ \geqslant \aleph_1$, then $a^d = (2^d)^d = a$, so that§ the cardinal b in (26) is equal to a, and (29) is the same as (26).

In order to prove (26)-(28), we note that (28) implies (27), and that (26) is Lemma 4. There remains the proof of (28). We define m, l and $F_{ab} = F$ as in Lemma 2, and we denote by $x < y$ the order relation in F defined in § 5. Let $x << y$ be a well-ordering of F, and put

$$\Omega_2(F) = K_1 + K_2,$$

where K_1 is the set of all sets $\{x, y\}_< = \{x, y\}_{>>} \subset F$, and K_2 is the set of all sets $\{x, y\}_< = \{x, y\}_{<<} \subset F$. We have $|F| = a^b$. If (28) were false then we could find $F' \subset F$ such that

$$\text{either} \quad |F'| = b^+; \quad \Omega_2(F') \subset K_1 \tag{30}$$

$$\text{or} \quad |F'| = a^+; \quad \Omega_2(F') \subset K_2. \tag{31}$$

But (30) would imply that

$$\omega^*_{l+1} \leqslant (\bar{F}')^*_{<<} = (\bar{F}')_< \leqslant (\bar{F})_<,$$

and (31) that

$$\omega_{m+1} \leqslant (\bar{F}')_{<<} = (\bar{F}')_< \leqslant (\bar{F})_<$$

which, in either case, contradicts Lemma 2.

12. It is easy to obtain from the argument leading to Lemma 2 some more information about the order type $\bar{F} = (\bar{F})_\leqslant$. Let S be any ordered set of type ϕ. We define the *Lusin index* $\Lambda(\phi)$ of ϕ as the least cardinal n

† We denote by (27) the negation of the relation $a^+ \to (b^+, a^+)^2$, and similarly in the case of (28).

‡ (4), Theorem II.

§ More generally, for any $a \geqslant \aleph_0$, b is the least cardinal such that a is representable as a sum of b cardinals less than a (Tarski).

which has the property that it is impossible to find n mutually non-overlapping open intervals in S, which means that, whenever R is a set, $|R| = n$, and $\{x_\rho, y_\rho\}_< \subset S$ for $\rho \varepsilon R$, then there is $\{\rho, \sigma\}_{\neq} \subset R$ such that $x_\rho \leqslant x_\sigma < y_\rho$. Then we have the following result.

Let $a \geqslant \aleph_0$, and let b be the least cardinal such that $a^b > a$. Then, if $\bar{F}_{ab}$ is the order type defined in § 5, $\Lambda(\bar{F}_{ab}) = a^+$.

The proof may be left to the reader.

Added in proof. L. Gillman has since proved that the generalized continuum hypothesis H_i " $2^{\aleph_\nu} = \aleph_{\nu+1}$ for all ν " is necessary for the truth of the assertion of our theorem, so that H is equivalent to the statement: $\aleph_n$ has the property P if, and only if, $\aleph_n^-$ is regular.

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