ON THE SET OF POINTS OF CONVERGENCE OF A LACUNARY TRIGONOMETRIC SERIES AND THE EQUIDISTRIBUTION PROPERTIES OF RELATED SEQUENCES

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1. Introduction

We consider the convergence of the series

$$\sum_{k=1}^{\infty} \sin(n_k x + \mu_k), \tag{1}$$

where $\{\mu_k\}$ $(k = 1, 2, \ldots)$ is a sequence of constants satisfying $0 \leqslant \mu_k \leqslant 2\pi$ and $\{n_k\}$ $(k = 1, 2, \ldots)$ is an increasing sequence of integers satisfying

$$t_k = \frac{n_{k+1}}{n_k} \geqslant \rho > 1. \tag{2}$$

It is well known from the classical theory of trigonometric series that the series (1) cannot converge except possibly for values of x in a set of zero Lebesgue measure. Our object is to discover how 'thin' this set is for various types of sequence $\{n_k\}$. The first result of this kind is due to P. Turan who proved, in 1941, that the series (1) converges absolutely in a set of positive logarithmic capacity in the case

$$n_k = (k!)^2, \qquad \mu_k = 0 \qquad (k = 1, 2, \ldots). \tag{3}$$

It will follow from the results of the present paper that in the case (3) considered by Turan, the set of values of x for which (1) converges absolutely has dimension $\frac{1}{2}$, whereas (1) converges in a set of dimension 1.

The convergence, or absolute convergence, of the lacunary trigonometric series is intimately related to that of the series

$$\sum_{k=1}^{\infty} \{((n_k x)) - \alpha_k\}, \tag{4}$$

where $\{\alpha_k\}$ $(k = 1, 2, \ldots)$ is a sequence of real numbers satisfying $0 \leqslant \alpha_k \leqslant 1$, $\{n_k\}$ satisfies (2) and, for a positive real number β,

$$((\beta)) = \beta - [\beta]$$

denotes the non-integer part of β. We will consider the set of values of x which make (4) convergent or absolutely convergent concurrently with the corresponding sets for the series (1). Absolute convergence is considered in § 2, and convergence in § 3.

Our discussion of the convergence of the series (4) leads naturally to the problem of equidistribution of the sequence $\{((n_k x))\}$ $(k = 1, 2,...)$. By the result of Weyl (**3**), given any increasing sequence $\{n_k\}$ $(k = 1, 2,...)$ of positive integers, the set of values of x such that $((n_k x))$ $(k = 1, 2,...)$ is equidistributed in $(0, 1)$ has full measure in the Lebesgue sense. Our object in § 4 is to examine the exceptional set of values of x for which $((n_k x))$ $(k = 1, 2,...)$ is not equidistributed for different types of sequence $\{n_k\}$. Among other results obtained, we prove that if $\{n_k\}$ satisfies (2), then $((n_k x))$ $(k = 1, 2,...)$ is not equidistributed for values of x in a set of dimension 1.

For notation and definitions relating to the theory of Hausdorff measures, see, for example, (**2**). We need the following theorem which is a special case of a result due to Eggleston (**1**).

THEOREM A. *Suppose I_k $(k = 1, 2,...)$ is a linear set consisting of N_k closed intervals each of length δ_k. Let each interval of I_k contain $n_{k+1} \geqslant 2$ closed intervals of I_{k+1} so distributed that their minimum distance apart is $\rho_{k+1} > \delta_{k+1}$. Let*

$$P = \bigcap_{k=1}^{\infty} I_k.$$

Then, if

$$\liminf_{k\to\infty} N_{k+1}\,\rho_{k+1}\,\delta_k^{s-1} > 0,$$

the set P has positive Λ^s-measure.

2. Absolute convergence

We first consider the cardinal number of the set of absolute convergence of (1) or (4). For sets known to have the cardinal number of the continuum, we consider the dimension in the sense of Besicovitch. This classifies linear sets of zero Lebesgue measure by considering their measures with respect to the class of Hausdorff s-dimensional measures $0 < s \leqslant 1$.

The connexion between the absolute convergence of the series (1) and (4) is given by

LEMMA 1. *If the series $\sum \left|\left(\left(n_k \frac{x}{2\pi}\right)\right) - \frac{\mu_k}{2\pi}\right|$ converges, then the series $\sum \sin(n_k x - \mu_k)$ converges absolutely.*

The proof is trivial. The converse is not true, but we will see that the infinite cardinal or dimension of the sets of absolute convergence of (1) and (4) will always be the same. To save space we will state the theorems only for the series (1). Theorem nA will be the theorem for series (4) corresponding to the Theorem n for series (1).

We will see that the 'size' of the set of absolute convergence depends on the rate at which t_k increases. First we consider the case of t_k bounded.

In Theorem 16 of (**1**), Eggleston proves that, given y $(0 \leqslant y \leqslant 1)$, there cannot be more than a countable set of x for which $((n_k x)) \to y$ as $k \to \infty$. Trivial modifications of his proof give:

THEOREM 1. *If* $\{\mu_k\}$ $(k = 1, 2,\ldots)$ *satisfies* $0 \leqslant \mu_k \leqslant 2\pi$ *and* $\{n_k\}$ $(k = 1, 2,\ldots)$ *is an increasing sequence of integers such that* $1 < t_k \leqslant K < \infty$, *then there is at most a countable set of values x such that*

$$\sin(n_k x - \mu_k) \to 0 \quad as\ k \to \infty.$$

COROLLARY. *Under the conditions of Theorem* 1, *there is at most a countable set of values x such that*

$$\sum \sin(n_k x - \mu_k)$$

converges.

This theorem is best possible in the sense that the set of points where (1) converges absolutely can have power $\aleph_0$. For example, take $n_k = 2^k$, $\mu_k = 0$ $(k = 1, 2,\ldots)$, then (1) converges absolutely if $x = p\pi 2^{-q}$, for any positive integers p, q. However, the result can be sharpened in the sense that given $\{n_k\}$ with $\rho \leqslant t_k \leqslant K$, for 'almost all' sequences $\{\mu_k\}$ there will be no points x for which (1) converges. We content ourselves with proving

THEOREM 2. *Suppose the sequence of integers $\{n_k\}$ is such that*

$$1 < \rho \leqslant t_k \leqslant K < \infty.$$

Then there are at most enumerably many pairs (x, y) $(0 \leqslant x \leqslant 2\pi, 0 \leqslant y \leqslant 2\pi)$ *such that*

$$\sin(n_k x - y) \to 0 \quad as\ k \to \infty.$$

COROLLARY. *Under the conditions of Theorem* 2, *there is a linear set Q with cardinal number $\leqslant \aleph_0$ such that, when y is not in Q, there is no x with* $\sin(n_k x - y) \to 0$; *and therefore the series* $\sum \sin(n_k x - y)$ *does not converge for any x, unless $y \in Q$.*

Proof. Let ϵ satisfy

$$0 < \epsilon < \frac{1}{8}\frac{\rho - 1}{K^2}. \tag{5}$$

Let E_k be the subset of the closed square $0 \leqslant x \leqslant 2\pi$, $0 \leqslant y \leqslant 2\pi$ such that

$$|\sin(n_k x - y)| < \epsilon.$$

Since $\epsilon < \frac{1}{8}$, it follows that, for every (x, y) in E_k, there must exist an integer r such that

$$|n_k x - y - r\pi| < 2\epsilon.$$

Hence E_k is a subset of the set F_k consisting of the closed strips of the plane

$$n_k x + l\pi - 2\epsilon \leqslant y \leqslant n_k x + l\pi + 2\epsilon,$$

where l takes all integer values satisfying

$$-2(n_k + 1) \leqslant l \leqslant 2(n_k + 1).$$

Then $F_k \cap F_{k+1}$ consists of $(4n_k+5)(4n_{k+1}+5) = C_k$ closed parallelograms which are congruent, equally spaced, and similarly situated. Let the projection on the x-axis of one of these parallelograms have length δ. By elementary trigonometry it follows that, if $n_k > 10$,

$$\delta \leqslant \frac{16\epsilon}{n_{k+1}-n_k} = \frac{16\epsilon}{n_k(t_k-1)}.$$

Hence, by (2), we have $$\delta \leqslant \frac{16\epsilon}{n_k(\rho-1)}. \tag{6}$$

Now the parallel strips making up F_{k+2} are separated by a horizontal distance $d = \dfrac{\pi-4\epsilon}{n_{k+2}}$. Thus

$$d > \frac{2}{n_{k+2}} \geqslant \frac{2}{K^2}\frac{1}{n_k} > \delta, \quad \text{by (5) and (6).}$$

The gradient of the parallel strips of F_{k+2} is greater than that of either of the sides of the parallelograms of $F_k \cap F_{k+1}$. Hence not more than one strip of F_{k+2} can have a non-void intersection with a single parallelogram of $F_k \cap F_{k+1}$.

Suppose, if possible, (x_1, y_1), (x_2, y_2) are two points of $\bigcap_{i=k}^{\infty} F_i$ which are in a single parallelogram of $F_k \cap F_{k+1}$. Then for every positive integer r, the two points must be in a single parallelogram of $F_{k+r} \cap F_{k+r+1}$. But the projection of such a parallelogram on the x-axis has length which tends to zero as $r \to \infty$, by (6). Hence $x_1 = x_2$. Thus the projection $\bigcap_{i=k}^{\infty} F_i$ on the x-axis has at most C_k points. But $E_i \subset F_i$ $(i = 1, 2,\ldots)$, so the projection of $\bigcap_{i=k}^{\infty} E_i$ on the x-axis has at most C_k points.

Now if (x, y) is such that

$$\sin(n_k x - y) \to 0 \quad \text{as } k \to \infty, \tag{7}$$

then (x, y) is in E_i for all sufficiently large i; that is (x, y) is in $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i = E$. Since the projection of $\bigcap_{i=k}^{\infty} E_i$ on the x-axis is finite for each k, it follows that the projection of E on the x-axis is at most countable. Thus the set of x for which there is a y such that (x, y) satisfies (7) is at most countable. For each such x there are at most 3 values of y in $0 \leqslant y \leqslant 2\pi$ such that (7) is satisfied. Hence the set of pairs (x, y) satisfying (7) has cardinal number $\leqslant \aleph_0$.

We now consider the case of a sequence $\{n_k\}$ such that $t_k \to \infty$. In this case, there can be a set of power continuum for which (1) or (4) converges absolutely.

THEOREM 3. *If $\{n_k\}$ is such that t_k is an integer for large values of k, and $t_k \to \infty$ as $k \to \infty$, then the set of x such that $\sum \sin n_k x$ converges absolutely has power continuum.*

Proof. It is clearly sufficient to prove that $\sum ((n_k x))$ converges for x in a set of power continuum. Let k_1 be such that t_k is an integer for $k \geqslant k_1 - 1$. Define a sequence $\{k_i\}$ $(i = 1, 2,\ldots)$ inductively by letting k_i be the smallest integer such that $k_i > k_{i-1}$, and $t_k \geqslant 2^i$ for every $k \geqslant k_i - 1$. Let

$$x = \sum_{i=2}^{\infty} \frac{\eta_i}{n_{k_i}},$$

where $\eta_i = 0$ or 1 for each i. Let E be the set of values taken by x for all such sequences $\{\eta_i\}$ $(i = 2, 3,\ldots)$. Clearly E has the power of the continuum.

Suppose x is in E. Then if $k \geqslant k_2$,

$$((n_k x)) = \sum \frac{\eta_i}{n_{k_i}} n_k,$$

where the summation extends over those i for which $k_i > k$. Hence

$$\sum_{k=k_2}^{\infty} ((n_k x)) = \sum_{k=k_2}^{\infty} \left[\sum_{k_i > k} \frac{\eta_i}{n_{k_i}} n_k\right] = \sum_{i=3}^{\infty} \left[\sum_{k=k_2}^{k_i-1} \frac{\eta_i}{n_{k_i}} n_k\right].$$

Now when $k = k_i - 1$, $n_k/n_{k_i} \leqslant 2^{-i}$, and, for $k \geqslant k_2$, $n_{k-1}/n_k \leqslant \frac{1}{2}$. Hence

$$\sum_{k=k_2}^{k_i-1} \frac{\eta_i}{n_{k_i}} n_k \leqslant \frac{1}{2^i}[1+\tfrac{1}{2}+\ldots] = \frac{1}{2^{i-1}};$$

and so
$$\sum_{k=k_2}^{\infty} ((n_k x)) \leqslant \sum_{i=3}^{\infty} \frac{1}{2^{i-1}} = \frac{1}{2}.$$

Thus for all x in E, the series $\sum ((n_k x))$ converges.

Remark. The condition that t_k be an integer in the above theorem cannot be omitted. For, if $\{n_k\}$ is defined inductively by

$$n_1 = 1, \quad n_k = kn_{k-1}+1 \quad \text{for } k = 2, 3,\ldots,$$

then it can be shown that $\sum \sin(n_k x - y)$ converges absolutely for no pairs (x, y).

Theorem 3 shows that, when t_k is an integer, and $t_k \to \infty$, there are some sequences $\{\mu_k\}$ for which (1) converges absolutely in a set of power continuum. In fact if t_k increases smoothly in some sense and $\sum t_k^{-1}$ diverges, one can prove that for almost all y (in the Lebesgue sense) there is no value of x such that $\sum \sin(n_k x - y)$ converges absolutely. This means that if t_k increases slowly and smoothly the series (1) will not converge absolutely for any x unless $\{\mu_k\}$ is a special sequence. We do not make this idea precise because we have not obtained nice conditions which are best possible.

Instead we state a result which shows that the conclusion of Theorem 3 is more or less 'best possible'.

THEOREM 4. *Suppose* $n_k = k!$ $(k = 1, 2,...)$ *and* $0 < y < \pi$ *or* $\pi < y < 2\pi$. *Then the series* $\sum \sin(n_k x - y)$ *converges absolutely for no value of* x.

This is easily proved using the comparison test.

COROLLARY. *Suppose* $n_k = k!$ $(k = 1, 2,...)$, *then the series* $\sum |\cos n_k x|$ *diverges for every* x.

The above theorems show that if $\sum t_k^{-1}$ diverges, then the series (1) is unlikely to converge for a set of values of x of power continuum. The situation is completely different when t_k increases rapidly enough to make $\sum t_k^{-1}$ convergent. This is given by

THEOREM 5. *Suppose* $\{n_k\}$ *is such that* $\sum t_k^{-1}$ *converges; then for any* $\{\mu_k\}$ *the series* $\sum \sin(n_k x - \mu_k)$ *converges absolutely for values of* x *in a set of power continuum.*

Proof. In view of Lemma 1, it is sufficient to prove the absolute convergence of $\sum \{((n_k x)) - \alpha_k\}$ for continuum many values of x. Let k_0 be such that $t_k > 10$ for $k \geqslant k_0$. Let I_k be the set of closed intervals of x such that $0 \leqslant x \leqslant 1$, and

$$|((n_k x)) - \alpha_k| \leqslant \frac{4}{t_k}.$$

Then I_k contains at least $(n_k - 2)$ closed intervals of length l_k, whose centres are distance n_k^{-1} apart, where

$$\frac{4}{t_k n_k} = \frac{4}{n_{k+1}} \leqslant l_k \leqslant \frac{8}{n_{k+1}}.$$

Each interval of I_k contains at least 2 intervals of I_{k+1} of length l_{k+1}, whose centres are distance apart n_{k+1}^{-1}. Hence $\bigcap_{k=k_0}^{k_0+r} I_k$ contains at least $2^r(n_{k_0} - 2)$ disjoint closed intervals of length l_{k_0+r}. Thus $E = \bigcap_{k=k_0}^{\infty} I_k$ has a perfect subset, and therefore has power continuum. But, if x is in E, then

$$|((n_k x)) - \alpha_k| \leqslant \frac{4}{t_k} \quad \text{for } k \geqslant k_0,$$

and so $\sum |((n_k x)) - \alpha_k|$ converges.

We now study the dimension in the sense of Besicovitch of the sets of absolute convergence of (1) and (4). The dimension is interesting only in the case where the set has power continuum since enumerable sets necessarily have dimension 0. Various results can be obtained showing how the dimension depends on the rate at which $t_k \to \infty$. We prove only

Theorem 6. *Suppose $\lambda > 0$, $\mu > 0$, $\rho > 0$ are constants, and $\{n_k\}$ is an increasing sequence of integers such that $\lambda k^\rho \leqslant t_k \leqslant \mu k^\rho$ for each integer k, and $\{\mu_k\}$ is any sequence of constants $0 \leqslant \mu_k \leqslant 2\pi$. Then*

(i) *if $0 < \rho \leqslant 1$, the dimension of the set of x for which $\sum \sin(n_k x - \mu_k)$ converges absolutely is zero;*

(ii) *if $\rho > 1$, the dimension of the set of x for which $\sum \sin(n_k x - \mu_k)$ converges absolutely is $1 - 1/\rho$.*

To obtain Theorems 6 A (ii) and 6 (ii) it is sufficient to prove

(*a*) *the set where* (4) *converges absolutely has dimension at least $(1-1/\rho)$; and*

(*b*) *the set where* (1) *converges absolutely has dimension at most $(1-1/\rho)$.*

Proof of (*a*). Let ϵ satisfy $0 < \epsilon < \rho - 1$ and s satisfy

$$0 < s < 1 - \frac{1+\epsilon}{\rho}.$$

For any such s we will prove that the set of x for which (4) converges absolutely has positive Λ^s-measure. Then the result (*a*) will follow by taking values of ϵ which are arbitrarily small.

Put
$$\eta_k = \left[\frac{1}{2}\frac{t_k}{k^{1+\epsilon}}\right] \quad (k = 1, 2, \ldots), \tag{8}$$

and let k_0 be a fixed integer such that

$$\frac{t_k}{k^{1+\epsilon}} - 2 > \eta_k, \quad \text{for } k \geqslant k_0. \tag{9}$$

k_0 exists since $t_k > \lambda k^\rho$ and $\rho > 1+\epsilon$. Let P_k be the set of x such that $0 \leqslant x \leqslant 1$ and

$$|((n_k x)) - \alpha_k| \leqslant \frac{1}{k^{1+\epsilon}}. \tag{10}$$

Then P_k consists of a finite number of closed intervals, at least $(n_k - 1)$ of which are of the same length γ_k. If $\alpha_k = 0$ or 1, $\gamma_k = (n_k k^{1+\epsilon})^{-1}$, while if α_k is not near 0 or 1, $\gamma_k = 2(n_k k^{1+\epsilon})^{-1}$: in any case

$$\frac{1}{k^{1+\epsilon} n_k} \leqslant \gamma_k \leqslant \frac{2}{k^{1+\epsilon} n_k}.$$

The centres of the intervals of P_k are distant apart $1/n_k$. Now define a subset I_k of P_k $(k \geqslant k_0)$ as follows. Let

$$\delta_k = (n_k k^{1+\epsilon})^{-1} \leqslant \gamma_k. \tag{11}$$

Let I_{k_0} be a set of $(n_{k_0} - 1)$ closed intervals each of length δ_{k_0}, concentric with intervals of P_{k_0} of length γ_{k_0}. Each interval of I_{k_0} contains at least $(t_{k_0} k_0^{-1-\epsilon} - 2)$ intervals of P_{k_0+1} of length γ_{k_0+1}, by (10). Define I_{k_0+1} as a set consisting of closed intervals of length δ_{k_0+1} concentric with some of the complete intervals of $I_{k_0} \cap P_{k_0+1}$ so chosen that each interval of I_{k_0} contains precisely η_{k_0} intervals of I_{k_0+1}: this is possible by (9).

For $k > k_0$, suppose I_k has been defined and consists of closed intervals of length δ_k. Define I_{k+1} as a set consisting of closed intervals of length δ_{k+1} concentric with some of the intervals of length γ_{k+1} of $I_k \cap P_{k+1}$ so chosen that each interval of I_k contains precisely η_k intervals of I_{k+1}. Write $P = \bigcap_{k=k_0}^{\infty} I_k$. The conditions of Theorem A are satisfied with

$$N_{k+1} = (n_{k_0}-1)\eta_{k_0}\eta_{k_0+1}\dots\eta_k \geqslant 2^{-k} n_{k+1}\left[\frac{(k_0-1)!}{k!}\right]^{1+\epsilon}, \quad \text{by (8);} \tag{12}$$

$$\rho_{k+1} > \tfrac{1}{2} n_{k+1}^{-1} \tag{13}$$

for large k, by (10) since $k^{-1-\epsilon} < \frac{1}{4}$. Thus, by (11), (12), (13),

$$\begin{aligned} N_{k+1}\rho_{k+1}\delta_k^{s-1} &\geqslant (\tfrac{1}{2})^{k+1}(k!)^{-1-\epsilon}(n_k k^{1+\epsilon})^{1-s} \\ &\geqslant \tfrac{1}{2}(\tfrac{1}{2}\lambda)^k k^{(1+\epsilon)(1-s)}(k!)^{\{\rho(1-s)-1-\epsilon\}}, \end{aligned}$$

since $n_k \geqslant \lambda^k (k!)^\rho$. Since $\rho(1-s)-(1+\epsilon) > 0$, $N_{k+1}\rho_{k+1}\delta_k^{s-1} \to \infty$ as $k \to \infty$. By Theorem A, $\Lambda^s P > 0$. But P is a subset of the set of x such that

$$|((n_k x))-\alpha_k| \leqslant k^{-1-\epsilon}$$

for sufficiently large k. Hence this set has positive Λ^s-measure. But for x in this set, the series (4) converges absolutely. Hence the set of x for which (4) converges absolutely has positive Λ^s-measure. This completes the proof.

Proof of (*b*). Let E be the set of x such that $\sum |\sin(n_k x-\mu_k)|$ converges. Suppose $1 \geqslant s > 1-1/\rho$; then it is sufficient to prove that $\Lambda^s(E) = 0$ for all such s. Let ϵ satisfy

$$0 < \epsilon < 1-\rho(1-s). \tag{14}$$

For each x in E, let $k_1, k_2,\dots, k_q,\dots$ be the sequence of values of the integer k for which

$$|\sin(n_k x-\mu_k)| \geqslant \frac{1}{4k}. \tag{15}$$

Then since (1) converges absolutely for this x, the sequence $\{k_i\}$ $(i = 1, 2,\dots)$ must have zero density. This implies that there exists an integer N (depending on x in E) such that, when $n \geqslant N$, the number of integers $k \leqslant n$ which satisfy (15) is less than ϵn.

Let Q_k be the set of x such that $0 \leqslant x \leqslant 2\pi$, and

$$|\sin(n_k x-\mu_k)| \leqslant \frac{1}{4k}. \tag{16}$$

Then $Q_k \subset I_k$, where I_k consists of $(2n_k+3)$ closed intervals of length $1/kn_k$ and centres at the points $(\mu_k+l\pi)/n_k$ with l taking integer values between -2 and $2n_k$. Since the centres of the intervals of I_k are distance apart πn_k^{-1}, the number of intervals of I_{k+1} which have a non-void intersection with a single interval of I_k is at most

$$\frac{n_{k+1}}{k\pi n_k}+2 \leqslant \frac{1}{k}t_k.$$

Let $\bigcap_{k=N}^{m} I_k$ consist of $t_{N,m}$ closed intervals of length not greater than $1/mn_m$. Then

$$t_{N,m} \leqslant (2n_N+3)\frac{n_{N+1}}{Nn_N}\cdots\frac{n_m}{(m-1)n_{m-1}};$$

so that

$$t_{N,m} \leqslant 3n_m\frac{N!}{(m-1)!}. \tag{17}$$

Now suppose $\{k_i\}$ is an increasing sequence of integers such that

$$\left.\begin{aligned} N \leqslant k_1 < k_2 < \ldots < k_q < \ldots \\ q < \epsilon r \end{aligned}\right\}. \tag{18}$$

and, if $k_q \leqslant r$, then

The number of such sequences which differ in the range $N \leqslant k \leqslant m$ is certainly less than 2^m.

For each k_i ($i = 1, 2,\ldots$) the number of intervals in $\bigcap_{\substack{k\neq k_i\\ N\leqslant k\leqslant m}} I_k$ differs from the number in $\bigcap_{N\leqslant k\leqslant m} I_k$ by a factor $\leqslant 2\pi k_i$ since the intervals of I_{k_i} have length $1/k_i n_{k_i}$ and centres distance apart π/n_{k_i}. Hence for a fixed sequence satisfying (18), the number of intervals of length $1/mn_m$ needed to cover $\bigcap_{\substack{k\neq k_i\,(1\leqslant i\leqslant q)\\ N\leqslant k\leqslant m}} I_k$ is not greater than

$$\begin{aligned} w_{N\,m} &= t_{N,m}\prod_{i=1}^{q}(2\pi k_i) \\ &\leqslant 3(2\pi m)^q\frac{N!}{(m-1)!}n_m, \end{aligned}$$

by (17). Hence, by (18), we have

$$w_{N,m} \leqslant 3(2\pi m)^{\epsilon m}\frac{N!}{(m-1)!}n_m. \tag{19}$$

Thus, if E_N is the set of values of x such that x is in Q_k for $k \geqslant N$, except for a sequence $\{k_i\}$ satisfying (18) where (16) is not known to be satisfied, then E_N can be covered by $r_{N,m}$ intervals of length $1/mn_m = l_m$, where

$$r_{N,m} < 2^m w_{N,m} < 15^m m^{\epsilon m}\frac{N!}{(m-1)!}n_m,$$

by (19). Hence

$$r_{N,m}\,l_m^s \leqslant 15^m m^{\epsilon m}\frac{N!}{(m-1)!}n_m\left(\frac{1}{mn_m}\right)^s = 15^m m^{\epsilon m+1-s}n_m^{1-s}\frac{N!}{m!}.$$

But $n_m < \mu^m(m!)^\rho$, and hence

$$r_{N,m}\,l_m^s < \mu^{m(1-s)}15^m m^{\epsilon m+1}N!\,(m!)^{\rho(1-s)-1} < \nu^m m c m^{m[\epsilon+\rho(1-s)-1]},$$

where ν, c are suitable finite constants, since $m! > (\frac{1}{4}m)^m$. By (14), it follows that $\epsilon+\rho(1-s)-1 < 0$, and hence

$$r_{N,m}\, l_m^s \to 0 \quad \text{as } m \to \infty.$$

Thus $\Lambda^s(E_N) = 0$. But $E \subset \bigcup_{N=1}^{\infty} E_N$. Hence $\Lambda^s(E) = 0$ as required.

To prove Theorems 6 A (i) and 6 (i) it is sufficient to prove

(c) *When $0 < \rho \leqslant 1$, the set where* (1) *converges absolutely has dimension zero.*

This can be proved by making a few obvious modifications to the proof of (*b*).

Remark 1. If this theorem is applied to the case $n_k = k!$ of Theorem 4 we see that, if y is not a multiple of π, then $\sum \sin(n_k x-y)$ converges absolutely for no value of x, while if $y = 0$, π, or 2π, $\sum \sin n_k x$ converges absolutely in a set of power continuum but zero dimension.

Remark 2. In the hypothesis of Theorem 6, if one assumes only one-sided inequalities to be satisfied by t_k, then one obtains by the above method of proof upper or lower bounds for the dimension of the appropriate set of absolute convergence. In particular, if

$$n_1 = 1, \qquad n_2 = 2, \qquad t_k = k^k \quad (k = 2, 3,...),$$

the series (1) and (4) each converge absolutely on a set of dimension 1. The same methods can be used to prove the following

THEOREM 7. *If $\{\mu_k\}$ is any sequence of constants $0 \leqslant \mu_k \leqslant 2\pi$ and $h(z)$ is any measure function of class 1,† there exists an increasing sequence $\{n_k\}$ of integers such that the set of values of x for which $\sum \sin(n_k x-\mu_k)$ converges absolutely has infinite measure with respect to $h(z)$.*

3. Convergence

Clearly the series (1) or (4) may converge without converging absolutely, so that, in general the set of points x making (1) or (4) converge will be larger than the set making the series converge absolutely. However, if (1) is to converge, $\sin(n_k x-\mu_k)$ must tend to zero as $k \to \infty$. Thus, by Theorem 1, if t_k is bounded for all k, then the set of convergence of (1) is at most enumerable. We now see that if $t_k \to \infty$, however slowly, as $k \to \infty$, then the set of convergence has dimension 1. Thus the situation is much simpler than for absolute convergence.

THEOREM 8. *Suppose $\{\mu_k\}$ is any sequence of constants, $0 \leqslant \mu_k \leqslant 2\pi$, and $\{n_k\}$ is an increasing sequence of integers such that $t_k \to \infty$, then the set of values of x such that $\sum \sin(n_k x-\mu_k)$ converges has dimension 1.*

† See (2) for a definition of measure function of class 1.

Proof. Let E be the set of x such that the series (1) converges. Suppose s is fixed and $0 < s < 1$. It is sufficient to prove that, for any such s, $\Lambda^s(E) > 0$. We prove this by defining a perfect set P and an integer j such that, when x is in P,

$$\sin(n_k x - \mu_k) \to 0 \quad \text{as } k \to \infty; \tag{20}$$

and, for $k \geqslant j$,

$$\sin(n_{k+1} x - \mu_{k+1}) \text{ and } \sum_{r=j}^{k} \sin(n_r x - \mu_r) \text{ have opposite signs.} \tag{21}$$

Clearly P is a subset of E, and so it is sufficient to prove that

$$\Lambda^s(P) > 0. \tag{22}$$

Now $n_k^{-1/k} \to 0$ as $k \to \infty$, since $t_k \to \infty$. For $k = 1, 2, \ldots$, let

$$g(k) = \max\left\{40\frac{n_k}{n_{k+1}},\ 8n_k^{-(1-s)/k}\right\}, \tag{23}$$

$$h(k) = \sup_{i \geqslant k} g(i). \tag{24}$$

Then $g(k) \to 0$, $h(k) \to 0$ as $k \to \infty$, $h(k)$ decreases as k increases, and $h(k) \geqslant g(k)$ $(k = 1, 2, \ldots)$. Put

$$\zeta_k = \tfrac{1}{8} t_k h(k). \tag{25}$$

Then

$$\zeta_k \leqslant \frac{h(k) n_{k+1}}{\pi n_k} - 4, \tag{26}$$

by (23) and (24).

Choose an integer j so large that

$$n_j \geqslant 100 \quad \text{and} \quad h(k) \leqslant \tfrac{1}{10} \quad \text{for } k \geqslant j. \tag{27}$$

By (23) and (24),

$$n_k^{(1-s)/k} \geqslant \{\tfrac{1}{8} g(k)\}^{-1} \geqslant \{\tfrac{1}{8} h(k)\}^{-1},$$

and therefore

$$n_k^{1-s} \geqslant \{\tfrac{1}{8} h(k)\}^{-k}.$$

Since $h(k)$ decreases, there is a constant C such that

$$n_{j+r}^{1-s} \geqslant C\{\tfrac{1}{8}h(j)\tfrac{1}{8}h(j+1)\ldots\tfrac{1}{8}h(j+r)\}^{-1} \quad (r = 0, 1, 2, \ldots). \tag{28}$$

Let R_k be the set of closed intervals given by

$$x = \frac{1}{n_k}\{\mu_k + l\pi + (-1)^l \xi\} \quad (0 \leqslant \xi \leqslant h(k)),$$

with l taking all integer values. Let S_k be the corresponding set given by

$$x = \frac{1}{n_k}\{\mu_k + l\pi + (-1)^l \xi\} \quad (-h(k) \leqslant \xi \leqslant 0).$$

Then

$$\sin(n_k x - \mu_k) \begin{cases} \geqslant 0 & \text{for } x \text{ in } R_k, \\ \leqslant 0 & \text{for } x \text{ in } S_k. \end{cases} \tag{29}$$

Each of the sets R_k, S_k consists of closed intervals of length $h(k) n_k^{-1}$ separated by distances of either πn_k^{-1} or $\{\pi-2h(k)\}n_k^{-1}$. By (27), there are at least $(2n_j-1)$ complete intervals of R_j in $0 \leqslant x \leqslant 2\pi$. Choose precisely n_j of these, and call this set I_0. In each interval of I_0 there are at least

$$\left\{h(j)\frac{n_{j+1}}{\pi n_j}-2\right\}$$

complete intervals of S_{j+1}. By (25), we can choose exactly ζ_j such intervals in each interval of I_0: call this set I_1. Then for any x in $I_0 \cap I_1$, (21) is true for $k = j$. We proceed inductively. Suppose I_r $(r \geqslant 1)$ has been defined and consists of closed intervals of length

$$\delta_r = h(j+r)n_{j+r}^{-1}. \tag{30}$$

Let (l, m) be a typical interval of I_r. There exists a point γ $(l \leqslant \gamma \leqslant m)$ such that

$$\left.\begin{aligned} &\sum_{k=j}^{j+r}\sin(n_k x-\mu_k) < 0 \quad \text{when } l < x < \gamma \\ \text{but}\quad &\sum_{k=j}^{j+r}\sin(n_k x-\mu_k) > 0 \quad \text{when } \gamma < x < m \end{aligned}\right\}. \tag{31}$$

In (l, γ) there are at least $\{(\gamma-l)(n_{j+r+1}/\pi)-2\}$ complete intervals of R_{j+r+1}, and in (γ, m) there are at least $\{(m-\gamma)(n_{j+r+1}/\pi)-2\}$ complete intervals of S_{j+r+1}. Hence, by (26) and (30), since $\delta_r = m-l$, we can choose precisely ζ_{j+r} intervals of length δ_{r+1} such that some of these are complete intervals of $(l,\gamma) \cap R_{j+r+1}$ and others are complete intervals of $(\gamma, m) \cap S_{j+r+1}$. Call the set obtained by treating each interval of I_r in this way I_{r+1}. Then, by (31), for x in I_{r+1} (21) is satisfied with $k = j+r$. Thus the set $P = \bigcap_{r=0}^{\infty} I_r$ satisfies the conditions (20) and (21). We now apply Theorem A to this set:

$$N_{r+1} = n_j\,\zeta_j\,\zeta_{j+1}\ldots\zeta_{j+r}; \qquad \rho_{r+1} \geqslant \frac{\pi-2h(j+r+1)}{n_{j+r+1}} \geqslant n_{j+r+1}^{-1};$$

and δ_r is given by (29). Hence

$$\begin{aligned} N_{r+1}\,\rho_{r+1}\,\delta_r^{s-1} &\geqslant \frac{n_j}{n_{j+r+1}}\zeta_j\ldots\zeta_{j+r}\,n_{j+r}^{1-s}\{h(j+r)\}^{-(1-s)} \\ &\geqslant \frac{n_j}{n_{j+r+1}}t_j\,t_{j+1}\ldots t_{j+r}\,C\{h(j+r)\}^{-(1-s)}, \end{aligned}$$

by (28) and (25). Thus

$$N_{r+1}\,\rho_{r+1}\,\delta_r^{s-1} \geqslant C\{h(j+r)\}^{-(1-s)} \geqslant C, \quad \text{by (27)};$$

and P satisfies (22), as required.

Remark 1. Theorem 8 A is not true for completely arbitrary $\{\alpha_k\}$: for example it is not true if $\alpha_k \equiv 0$, since in this case the series (4) converges only if it converges absolutely. However, if $0 < \delta \leqslant \alpha_k \leqslant 1-\delta$ $(k = 1, 2,\ldots)$ and $t_k \to \infty$, it can be proved that (4) converges for x in a set of dimension 1.

Remark 2. By making some modifications to the argument of Theorem 8, it can be shown that, with the same hypothesis, and any real number K, the set of values of x such that

$$\sum_{k=1}^{\infty} \sin(n_k x - \mu_k) = K$$

has dimension 1.

4. Equidistribution of $((n_k x))$

We say that the sequence $z_1, z_2, \ldots, z_r, \ldots$ is equidistributed in $(0, 1)$ if, for every l, m satisfying $0 \leqslant l < m \leqslant 1$, the density of integers r for which $l \leqslant z_r \leqslant m$ is exactly $(m-l)$: that is, if

$$\epsilon_r = \begin{cases} 1 & \text{when } l \leqslant z_r \leqslant m, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\lim_{t \to \infty} \left[\frac{1}{t} \sum_{r=1}^{t} \epsilon_r\right] = m - l.$$

THEOREM 9. *The sequence $z_1, z_2, \ldots$ of real numbers is equidistributed in $(0, 1)$ if and only if*

$$\lim_{t \to \infty} \frac{1}{t} \sum_{r=1}^{t} \exp(q z_r 2\pi i) = 0$$

for every positive integer q.

This result is due to Weyl (**3**). By the method of Weyl one can prove easily

THEOREM 10. *If $\{n_k\}$ $(k = 1, 2, \ldots)$ is an increasing sequence of integers, then the set of values of x such that $((n_k x))$ $(k = 1, 2, \ldots)$ is not equidistributed in $(0, 1)$ has zero Lebesgue measure.*

The 'size' of the exceptional set of x for which $((n_k x))$ is not equidistributed depends on the sequence $\{n_k\}$. For example, it is known that if n_k is given by a polynomial in k with integer coefficients, then the set of x for which $((n_k x))$ is not equidistributed is enumerable. In this case $t_k \to 1$ as $k \to \infty$. However, we first see that $t_k \to 1$ is not a sufficient condition to ensure that the exceptional set has power $\aleph_0$.

THEOREM 11. *There exists a finite constant C, and an increasing sequence of integers $\{n_k\}$ such that*

$$n_{k+1} - n_k < C \quad (k = 1, 2, \ldots)$$

and the set of x such that $((n_k x))$ is not equidistributed is not enumerable.

Proof. We define a sequence $\{n_k\}$ for which there is a G_δ-set E, such that E is dense in an interval, and for x in E, $((n_k x))$ is not equidistributed. By the Baire category theorem, a G_δ-set which is dense in an interval cannot have power $\leqslant \aleph_0$ so that it is clearly sufficient for the truth of the theorem to define such a sequence.

Suppose $\{\lambda_i\}$ $(i = 1, 2,...)$ contains all the rationals ρ satisfying $\frac{1}{8} \leqslant \rho \leqslant \frac{1}{6}$, and each rational occurs in the sequence infinitely often. Let

$$k_s = 5^s \quad (s = 0, 1, 2,...). \tag{32}$$

Put $n_1 = 1$. Suppose for some positive integer r, n_k has been defined for $k \leqslant k_{r-1}$. We define n_k by induction in the range

$$k_{r-1} < k \leqslant k_r \quad (r = 1, 2,...)$$

as follows. Suppose n_{k-1} has been defined. Let n_k be the smallest integer greater than n_{k-1} for which

$$\cos(n_k \lambda_r 2\pi) > \tfrac{1}{2}. \tag{33}$$

Since $\frac{1}{6} \geqslant \lambda_r \geqslant \frac{1}{8}$, it is clear that

$$n_k - n_{k-1} < \frac{3\pi}{\lambda_r} < 24\pi$$

so that
$$n_{k+1} - n_k < 100 \quad (k = 1, 2,...). \tag{34}$$

By (33),
$$\sum_{k=k_{r-1}+1}^{k_r} \cos(n_k \lambda_r 2\pi) > \tfrac{1}{2}(k_r - k_{r-1}) = 2k_{r-1}, \quad \text{by (32).}$$

Hence
$$\frac{1}{k_r} \sum_{k=1}^{k_r} \cos(n_k \lambda_r 2\pi) > \frac{k_{r-1}}{k_r} = \frac{1}{5}.$$

Let I_r be an open interval containing λ_r such that if x is in I_r, then

$$\frac{1}{k_r} \sum_{k=1}^{k_r} \cos(n_k x 2\pi) > \tfrac{1}{5}. \tag{35}$$

Let
$$E = \bigcap_{q=1}^{\infty} \bigcup_{r=q}^{\infty} I_r.$$

Then E contains all points x which are in infinitely many I_r. Thus E contains every rational in $\frac{1}{8} \leqslant \rho \leqslant \frac{1}{6}$, and is therefore everywhere dense in the interval $(\frac{1}{8}, \frac{1}{6})$. Further, E is a G_δ-set.

If $x \in E$, then given N, x is in I_r for some $r > N$. By (35), there is an integer $t = k_r > N$ such that

$$\frac{1}{t} \sum_{k=1}^{t} \cos\{((n_k x))2\pi\} = \frac{1}{t} \sum_{k=1}^{t} \cos(n_k x 2\pi) > \tfrac{1}{5}.$$

Hence, for any x in E,

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \cos\{((n_k x))2\pi\} \geqslant \tfrac{1}{5},$$

and therefore, by Theorem 9, $((n_k x))$ is not equidistributed in $(0, 1)$. By (34), the sequence $\{n_k\}$ satisfies the required conditions with $C = 100$.

It now becomes interesting to ask what is the dimension of the exceptional set of non-equidistribution for a sequence satisfying the conditions of Theorem 11.

THEOREM 12. *Suppose C is a constant, and $\{n_k\}$ an increasing sequence of integers such that*

$$n_{k+1}-n_k < C \quad (k = 1, 2,..),$$

then the set of points x for which $((n_k x))$ is not equidistributed has dimension zero.

Proof. Under the given conditions, there is a constant λ such that

$$n_k < \lambda k \quad (k = 1, 2,...). \tag{36}$$

For $s = 1, 2,..., t = 1, 2,...,$ put

$$f_{s,t}(x) = \sum_{k=1}^{t} \cos(n_k x\, 2\pi s). \tag{37}$$

For any rational $\mu > 0$, let $F_{s,t,\mu}$ be the set of x such that

$$|f_{s,t}(x)| > \mu t. \tag{38}$$

Put $a_m = [\exp(m/\log m)]$ $(m = 3, 4,...)$: then $\{a_m\}$ $(m = 3, 4,...)$ is an increasing sequence of integers such that

$$\frac{a_{m+1}}{a_m} \to 1 \quad \text{as } m \to \infty.$$

Then if (38) is satisfied for infinitely many integers t (fixed s, μ), it must be satisfied for infinitely many integers of the sequence $\{a_m\}$, that is

$$E_{s,\mu} = \bigcap_{l=3}^{\infty} \bigcup_{m=l}^{\infty} F_{s,a_m,\mu} \tag{39}$$

contains the set of x such that

$$\limsup_{t\to\infty} \left|\frac{1}{t} f_{s,t}(x)\right| > \mu.$$

Given α satisfying $1 > \alpha > 0$, we prove that

$$\Lambda^{\alpha} E_{s,\mu} = 0. \tag{40}$$

This, together with the corresponding result for polynomials of sines instead of cosines, implies the truth of the theorem by taking the union of $E_{s,\mu}$ for $s = 1, 2,...,$ and all positive rationals μ, and applying Theorem 9.

Now
$$\int_0^{2\pi} |f_{s,t}(x)|^2\, dx = \pi t,$$

so
$$|F_{s,t,\mu}| < \frac{\pi}{\mu^2}\frac{1}{t}, \tag{41}$$

by (38). Now

$$\left|\frac{d}{dx} f_{s,t}(x)\right| < 2\pi \sum_1^t n_k$$

$$< 2\pi\lambda \sum_{k=1}^t k, \quad \text{by (36)},$$

$$< \pi\lambda t^2.$$

Hence, if $x_0 \in F_{s,t,\mu}$, there is an interval containing x_0 of length at least $\frac{\mu}{\pi\lambda}\frac{1}{t}$ which is a subset of $F_{s,t,\frac{1}{2}\mu}$. The total length of $F_{s,t,\frac{1}{2}\mu}$ is less than $\frac{4\pi}{\mu^2}\frac{1}{t}$, by (41), so that $F_{s,t,\mu}$ can be enclosed in a finite set of not more than $K_{s,\mu}$ intervals of total length $\frac{4\pi}{\mu^2}\frac{1}{t}$ where

$$K_{s,\mu} = \left[\frac{4\pi}{\mu^2}\frac{\pi\lambda}{\mu}\right]+1.$$

If $l_1, l_2,\ldots, l_p$ ($p \leqslant K_{s,\mu}$) are the lengths of a set of intervals covering $F_{s,t,\mu}$, but contained in $F_{s,t,\frac{1}{2}\mu}$, we have

$$\sum_{i=1}^p l_i \leqslant |F_{s,t,\frac{1}{2}\mu}| < \frac{4\pi}{\mu^2}\frac{1}{t},$$

and therefore

$$\sum_{i=1}^p l_i^\alpha \leqslant K_{s,\mu}\left(\frac{4\pi}{\mu^2}\frac{1}{t}\frac{1}{K_{s,\mu}}\right)^\alpha,$$

since the real function z^α is convex. It follows that there is a constant $L_{s,\mu}$, and a covering by a finite set of intervals of lengths $l_1, l_2,\ldots, l_p$ of the set $F_{s,t,\mu}$ such that

$$\sum_{i=1}^p l_i^\alpha < L_{s,\mu} t^{-\alpha}.$$

By (39) the set $E_{s,\mu} \subset \bigcup_{m=l}^\infty F_{s,a_m,\mu}$ ($l = 3, 4,\ldots$) and therefore $E_{s,\mu}$ can be covered by a sequence of intervals of lengths $l_1, l_2,\ldots$ such that

$$\sum l_i^\alpha < L_{s,\mu} \sum_{m=q}^\infty a_m^{-\alpha}.$$

Since the series on the right-hand side of this expression converges (40) is proved. This completes the proof of the theorem.

Remark. The method of proof used is good enough to strengthen the result in Theorem 12. Thus the exceptional set of x for which $((n_k x))$ is not equidistributed has zero measure with respect to the measure function $[\log(1/z)]^{-1-\epsilon}$ for every $\epsilon > 0$. We have been unable to decide whether or not this set must have zero capacity—this would follow if the measure with respect to $[\log(1/z)]^{-1}$ is finite.

By the same method of proof used in Theorem 12 one can prove

THEOREM 13. *Suppose $C \geqslant 0$, $\rho \geqslant 1$ are constants and $\{n_k\}$ is an increasing sequence of integers such that*

$$n_k < Ck^{\rho} \quad (k = 1, 2, \ldots),$$

then the set of points x for which $((n_k x))$ is not equidistributed has dimension not greater than $(1-1/\rho)$.

By constructing a special sequence $\{n_k\}$ one can prove that the bound $(1-1/\rho)$ of this theorem can be attained.

So far, in the present section, we have been considering sequences $\{n_k\}$ which do not increase too quickly. They are certainly not lacunary, for under the hypotheses of Theorem 12 or 13

$$\liminf_{k\to\infty} t_k = 1.$$

The case $t \to \infty$ is easily decided. For in § 3 we proved that the set E of values of x such that $\sum \{((n_k x))-\alpha\}$ converges, $0 < \alpha < 1$, has dimension 1. For x in this set E,

$$((n_k x)) \to \alpha,$$

and therefore $((n_k x))$ is certainly not equidistributed. We now show that the condition (2) that the sequence $\{n_k\}$ be lacunary is sufficient to imply that the exceptional set of x for which $((n_k x))$ is not equidistributed has dimension 1. Zygmund, in (**4**), proved that the set of x for which

$$\limsup_{t\to\infty} \left|\frac{1}{t}\sum \cos(n_k x\, 2\pi)\right| > 0$$

is everywhere dense in $(0, 1)$, but his method does not seem to give the dimension.

THEOREM 14. *If $\{n_k\}$ is an increasing sequence of integers such that $t_k \geqslant \rho > 1$, then the set E of values of x such that $((n_k x))$ is not equidistributed in $(0, 1)$ has dimension 1.*

Proof. It is sufficient to show that E has positive Λ^s-measure for any s satisfying $0 < s < 1$. Let $\beta > 1/(1-s)$. Choose C so that $\frac{1}{10} > C > 0$, and

$$C^{1-\beta(1-s)} > 3. \tag{42}$$

Let

$$v = \left[\frac{\beta \log(1/C)}{\log \rho}\right] + 1, \tag{43}$$

where ρ satisfies (2). Then for any positive integer k,

$$\frac{n_{k+v}}{n_k} \geqslant C^{-\beta} > 10. \tag{44}$$

Let Q_r be the set of x such that $0 \leqslant x \leqslant 1$, and

$$0 \leqslant ((n_{rv} x)) \leqslant C \quad (r = 1, 2, \ldots).$$

Then Q_r contains at least n_{rv} closed intervals of length

$$\delta_r = \frac{C}{n_{rv}}, \tag{45}$$

whose centres are distance n_{rv}^{-1} apart. Put

$$\gamma_r = \left[C\frac{n_{(r+1)v}}{n_{rv}} - 2\right] \quad (r = 1, 2, \ldots).$$

Then, by (44),
$$\gamma_r \geqslant \tfrac{1}{2}C\frac{n_{(r+1)v}}{n_{rv}} \geqslant \tfrac{1}{2}C^{1-\beta}, \tag{46}$$

and each interval of Q_r contains at least γ_r complete intervals of Q_{r+1}.

Let I_1 consist of n_v complete intervals of Q_1. Suppose I_r has been defined, $r \geqslant 1$, so that it consists of some of the intervals of Q_r of length δ_r. In each interval of I_r choose γ_r complete intervals of Q_{r+1}, and call this set I_{r+1}. Now let us apply Theorem A to $P = \bigcap_{r=1}^{\infty} I_r$. δ_r is given by (45);

$$\rho_{r+1} > \tfrac{1}{2}n_{(r+1)v}^{-1}; \qquad N_{r+1} = n_v\gamma_1\gamma_2\ldots\gamma_r \geqslant \frac{C^{r+1}}{2^r}n_{(r+1)v}, \quad \text{by (46)}.$$

Hence
$$N_{r+1}\,\rho_{r+1}\,\delta_r^{s-1} \geqslant \left(\frac{C}{2}\right)^r C^{s-1}(n_{rv})^{1-s}.$$

By (44), $n_{rv} \geqslant n_v C^{-\beta(r-1)} \geqslant C^{-\beta(r-1)}$, and so

$$N_{r+1}\,\rho_{r+1}\,\delta_r^{s-1} \geqslant \left\{\frac{C^{1-\beta(1-s)}}{2}\right\}^r \frac{C^{1+\beta-\beta s}}{2} \to \infty$$

as $r \to \infty$ by (42).

Thus $\Lambda^s(P) > 0$. Now, if x is in P, the lower density of integers q such that $0 \leqslant ((n_q x)) \leqslant C$ is at least $1/v$. By (43), if C is small enough,

$$\frac{1}{v} \geqslant 10C.$$

Thus, for x in P, $((n_k, x))$ has too many members in the interval $(0, C)$, and is therefore not equidistributed in $(0, 1)$. Hence $P \subset E$, and $\Lambda^s(E) > 0$, as required.

REFERENCES

1. H. G. Eggleston, 'Sets of fractional dimensions which occur in some problems of number theory', *Proc. London Math. Soc.* 54 (1951–2) 42–93.
2. S. J. Taylor, 'On cartesian product sets', *J. London Math. Soc.* 27 (1952) 295–304.
3. H. Weyl, 'Über die Gleichwertteilung von Zahlen mod Eins', *Math. Ann.* 77 (1916) 313–54.
4. A. Zygmund, 'Quelques théorèmes sur les séries trigonométriques et celles de puissances', *Studia Math.* 3 (1931) 77–91.

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