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TRIPLE POINTS OF BROWNIAN PATHS IN 3-SPACE

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In (3), some of us proved that Brownian paths in n-space have double points with probability 1 if $n = 2$ or 3; but, for $n \geqslant 4$, there are no double points with probability 1. The question naturally arises as to whether or not Brownian paths in n-space ($n = 2$ or 3) have triple points. The case of paths in the plane is settled by (4), where it is shown that, with probability 1, Brownian paths in the plane have points of multiplicity k ($k = 2, 3, 4, \ldots$). The purpose of the present paper is to settle the remaining case, $n = 3$. We prove that, with probability 1, Brownian paths in 3-dimensional space have no triple points. The general idea behind our proof is to show that there are not too many double points. That is, we show that the set of double points has sigma-finite linear measure, and therefore zero capacity, with probability 1.

1. *Definitions and preliminary results.* Let $(\Omega, \mathscr{E}, \mu)$ be a probability space, i.e. $\Omega = \{\omega\}$ is a set of elements ω, $\mathscr{E} = \{E\}$ is a Borel field of subsets of Ω called events, and μ is a countably additive measure defined on $\mathscr{E}$ and satisfying $\mu(\Omega) = 1$. $\mu(E)$ is called the probability of the event E.

A *one-dimensional Brownian motion* (see (1), (2), (7)) is a real-valued function $x(t, \omega)$ of the two variables t and ω, defined for all non-negative real numbers t, $0 \leqslant t < \infty$, and for all $\omega \in \Omega$, which has the following properties:

(*a*) $x(0, \omega) \equiv 0$;

(*b*) for any real numbers s, t with $0 \leqslant s < t < \infty$, the increment $\{x(t, \omega) - x(s, \omega)\}$ is $\mathscr{E}$-measurable in ω and has a normal distribution with mean 0 and variance $t - s$, that is, if

$$E(x, s, t, \alpha) = \{\omega \colon x(t, \omega) - x(s, \omega) < \alpha\}, \tag{1}$$

where $\{\omega \colon \ldots\}$ denotes the set of ω having the properties following the colon, then $E(x, s, t, \alpha)$ is measurable and

$$\mu\{E(x, s, t, \alpha)\} = [2\pi(t-s)]^{-\frac{1}{2}} \int_{-\infty}^{\alpha} \exp[-u^2/2(t-s)]\, du \tag{2}$$

for every real number α;

(*c*) for any real numbers s_i, t_i ($i = 1, 2, \ldots, m$) with

$$0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \ldots \leqslant s_m < t_m < \infty,$$

the increments $\{x(t_i, \omega) - x(s_i, \omega)\}$, $i = 1, 2, \ldots, m$, are independent in the sense of probability theory, i.e.

$$\mu\left\{\bigcap_{i=1}^{m} E(x, s_i, t_i, \alpha_i)\right\} = \prod_{i=1}^{m} \mu\{E(x, s_i, t_i, \alpha_i)\} \tag{3}$$

for any real α_i, $i = 1, 2, \ldots, m$.

A *three-dimensional Brownian motion* is an ordered triple of three mutually independent one-dimensional Brownian motions $x(t,\omega)$, $y(t,\omega)$, $z(t,\omega)$ with the property that

$$\begin{aligned}\mu\{E(x,s,t,\alpha) \cap E(y,s',t',\alpha') \cap E(z,s'',t'',\alpha'')\} \\ = \mu\{E(x,s,t,\alpha)\}\,\mu\{E(y,s',t',\alpha')\}\,\mu\{E(z,s'',t'',\alpha'')\} \quad (4)\end{aligned}$$

for any real numbers $s, t, \alpha, s', t', \alpha', s'', t'', \alpha''$, with $0 \leqslant s < t$, $0 \leqslant s' < t'$, $0 \leqslant s'' < t''$.

If we consider $\mathbf{r}(t,\omega) = [x(t,\omega), y(t,\omega), z(t,\omega)]$ as a point in Euclidean 3-space, then for each fixed ω, $\mathbf{r}(t,\omega)$ may be considered as a function of t, defined for $0 \leqslant t < \infty$, and assuming as values, points (or vectors) in 3-space.

It is easy to see that this definition of Brownian motion in 3-space is independent of the choice of the rectangular coordinate system; i.e. the motion is isotropic, it is invariant *vis-à-vis* rotations of the coordinate system.

It is further assumed that the Borel field $\mathscr{E}$ is sufficiently large to contain the subset C of Ω consisting of all ω for which $x(t,\omega)$ is continuous as a function of t $(0 \leqslant t < \infty)$, and $\mu(C) = 1$. This means that Brownian motion is a separable stochastic process in the sense of Doob (1).

For any point $\mathbf{r}'$ in 3-space, for any $\omega \in \Omega$ and any real numbers a, b with $0 \leqslant a < b < \infty$, let us put

$$L(a,b;\mathbf{r}';\omega) = \{\mathbf{r}' + \mathbf{r}(t,\omega)\colon a \leqslant t \leqslant b\}, \quad (5)$$

$$L(a,\infty;\mathbf{r}';\omega) = \{\mathbf{r}' + \mathbf{r}(t,\omega)\colon a \leqslant t < \infty\}, \quad (6)$$

$$L(\mathbf{r}';\omega) = \{\mathbf{r}' + \mathbf{r}(t,\omega)\colon 0 \leqslant t < \infty\}, \quad (7)$$

where the $+$ sign in the above formula (as well as $+$ and $-$ in similar context in the sequel) refers to vector addition. Furthermore, when $\mathbf{r}' = 0$, i.e. coincides with the origin, we use the abbreviations

$$L(a,b;\omega) = L(a,b;\mathbf{0};\omega), \quad L(\omega) = L(\mathbf{0};\omega). \quad (8)$$

$L(a,b;\mathbf{r}';\omega)$ is called the (a,b) path of the Brownian motion starting from $\mathbf{r}'$, and $L(\mathbf{r}';\omega)$ is called the *path* of the Brownian motion starting from $\mathbf{r}'$.

For almost all ω, $L(a,b;\mathbf{r}';\omega)$ is the continuous image of the finite closed interval $\{t\colon a \leqslant t \leqslant b\}$ and is therefore a compact subset of the 3-space.

A point $\mathbf{r}_0$ in 3-space is called a triple point of $L(\omega)$ if there exist three real numbers t_1, t_2, t_3 with $0 \leqslant t_1 < t_2 < t_3 < \infty$ for which $\mathbf{r}(t_1,\omega) = \mathbf{r}(t_2,\omega) = \mathbf{r}(t_3,\omega) = \mathbf{r}_0$.

Let $|\mathbf{r}_1 - \mathbf{r}_2|$ denote the Euclidean distance from $\mathbf{r}_1$ to $\mathbf{r}_2$, and $|\mathbf{r}|$ the distance from $\mathbf{r}$ to the origin.

We need the concept of *capacity* (see, for example, Frostman (6)). Let F be a compact subset of 3-space. Let $\mathscr{M}(F)$ be the family of all countably additive measures $m(B)$ defined for all Borel subsets B of F with $m(F) = 1$. Put

$$\lambda(F) = \inf \iint \frac{m(d\mathbf{r}_1)\,m(d\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|},$$

where the double integral is extended over $F \times F$, and inf is taken over all measures $m \in \mathscr{M}(F)$. Now $\lambda(F) = \infty$ if and only if the double integral is ∞ for all $m \in \mathscr{M}(F)$. The (Newtonian) capacity of F is defined by

$$C(F) = \begin{cases} [\lambda(F)]^{-1} & \text{if } \lambda(F) < \infty, \\ 0 & \text{if } \lambda(F) = \infty. \end{cases}$$

We need the result that Brownian paths $L(\mathbf{r}', \omega)$ starting from a point $\mathbf{r}'$ not in F will have no points in common with F, with probability 1, if $C(F) = 0$. This is

LEMMA 1·1. *Let F be a compact set in R^3 with $C(F) = 0$. Then, for every $\mathbf{r}$ not in F, the path $L(\mathbf{r}; \omega)$ and F are disjoint with probability* 1.

This result is due to Kakutani (7).

We also need to define *linear measure*. If A is any set in R^3, let $\mathscr{U}(A, \delta)$ be a sequence of convex sets E_i $(i = 1, 2, \ldots)$ such that $d(E_i)$, the diameter of E_i, satisfies $d(E_i) < \delta$, and $A \subset \bigcup_{i=1}^{\infty} E_i$. Put

$$\Lambda_\delta(A) = \inf \sum_{i=1}^{\infty} d(E_i), \tag{9}$$

the infimum being over all such coverings $\mathscr{U}(A, \delta)$, and

$$\Lambda^*(A) = \lim_{\delta \to 0} \Lambda_\delta(A). \tag{10}$$

The set function $\Lambda^*(\cdot)$ is an outer measure in the sense of Carathéodory. If A is measurable with respect to this outer measure, then the value of $\Lambda^*(A)$ is called the linear measure of A, and denoted by $\Lambda(A)$.

For sets A in R^3, there is a connexion between $\Lambda(A)$ and $C(A)$ given by

LEMMA 1·2. *If E is a bounded closed set in Euclidean space of 3 dimensions such that $\Lambda(E)$ is finite, then $C(E) = 0$.*

This lemma was proved for plane sets (where logarithmic measure and logarithmic capacity are concerned) by Erdős and Gillis (5). The present result is due to Kametani (8) who used the methods of Ugaheri (9).

We also need a result which shows that very large movements of $\mathbf{r}(t, \omega)$ in a small interval $a \leqslant t \leqslant a + \delta$ are unlikely.

LEMMA 1·3. *For any $\eta > 0$, $\delta > 0$, $t_1 = t_0 + \delta$ and ω in the space Ω,*

$$\mu\{\omega: \sup_{t_0 \leqslant t \leqslant t_1} |x(t, \omega) - x(t_0, \omega)| > \eta\} < \frac{2}{\eta}\left(\frac{2\delta}{\pi}\right)^{\frac{1}{2}} \exp[-\eta^2/2\delta].$$

This result has been known for a long time. It follows immediately from Theorem 2·1 of (1).

LEMMA 1·4. *For any $\eta > 0$, $\delta > 0$, $t_1 = t_0 + \delta$ and ω in the space Ω,*

$$\mu\{\omega: \sup_{t_0 \leqslant t \leqslant t_1} |\mathbf{r}(t, \omega) - \mathbf{r}(t_0, \omega)| > \eta\} < 10 \frac{\delta^{\frac{1}{2}}}{\eta} \exp[-\eta^2/8\delta].$$

Proof. If $\sup_{t_0 \leqslant t \leqslant t_1} |\mathbf{r}(t, \omega) - \mathbf{r}(t_0, \omega)| > \eta$, then for at least one of the coordinates, say x,

$$\sup_{t_0 \leqslant t \leqslant t_1} |x(t, \omega) - x(t_0, \omega)| > \tfrac{1}{2}\eta.$$

Now, by Lemma 1·3,

$$\mu\{\omega: \sup_{t_0 \leqslant t \leqslant t_1} |x(t, \omega) - x(t_0, \omega)| > \tfrac{1}{2}\eta\} < \frac{4}{\eta}\left(\frac{2\delta}{\pi}\right)^{\frac{1}{2}} \exp[-\eta^2/8\delta].$$

Hence

$$\mu\{\omega\colon \sup_{t_0\leqslant t\leqslant t_1} |\mathbf{r}(t,\omega)-\mathbf{r}(t_0,\omega)|>\eta\} \leqslant 3\mu\{\omega\colon \sup_{t_0\leqslant t\leqslant t_1} |x(t,\omega)-x(t_0,\omega)|>\tfrac{1}{2}\eta\}$$
$$< 12\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\delta^{\frac{1}{2}}}{\eta}\exp[-\eta^2/8\delta]$$
$$< 10\frac{\delta^{\frac{1}{2}}}{\eta}\exp[-\eta^2/8\delta].$$

2. *Proof of the main result.* We first obtain a covering for the subset of the double points of $L(\omega)$ given by
$$L(0,1;\omega)\cap L(2,\infty;\omega).$$
The set of double points is difficult to obtain by approximation methods, so we cover a larger set of 'near returns'. For a given $\mathbf{r}$ in $L(0,1;\omega)$ we say there is a near δ-return in $L(2,\infty;\omega)$ if $\inf_{t\geqslant 2}|\mathbf{r}-\mathbf{r}(t,\omega)|<\delta$. If we cover the subset of $L(0,1,\omega)$ for which there is a near δ-return in $L(2,\infty;\omega)$, then we have certainly covered the set
$$L(0,1;\omega)\cap L(2,\infty;\omega).$$
Our first step is to obtain the probability of near return.

LEMMA 2·1. *If* $\epsilon>0$,
$$\mu\{\omega\colon \inf_{t\geqslant 1}|\mathbf{r}(t,\omega)|<\epsilon\}\leqslant\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\epsilon.$$

This is a special case of Lemma 2 of (**2**).

Remark. The estimate for $\mu\{\omega\colon \inf_{t\geqslant 1}|\mathbf{r}(t,\omega)|<\epsilon\}$ given by Lemma 2·1 is a good one, the error being $o(\epsilon)$ as $\epsilon\to 0$.

To obtain a covering of $L(0,1;\omega)\cap L(2,\infty;\omega)$ we divide $L(0,1;\omega)$ into k^2 parts and consider the likelihood of a near $1/k$-return for some point of one of the parts $L(t_{i-1},t_i;\omega)$. If this event occurs, we take a sphere centre $\mathbf{r}(t_i,\omega)$ large enough to enclose all of $L(t_{i-1},t_i;\omega)$.

Let k be a positive integer. Let $t_i=i/k^2$, $(i=0,1,2,\ldots,k^2)$ define values of t, $0\leqslant t_i\leqslant 1$. Then we say there is a near $1/k$-return in $L(t_{i-1},t_i;\omega)$ if there are values of t, τ such that $t_{i-1}\leqslant t\leqslant t_i$, $\tau\geqslant 2$, and $|\mathbf{r}(\tau,\omega)-\mathbf{r}(t,\omega)|<1/k$. Call this event $F_{i,k}$. Then $F_{i,k}$ can occur only if either $\mathbf{r}(t,\omega)$ returns very close to $\mathbf{r}(t_i,\omega)$ for values of $t\geqslant 2$, or the path returns fairly close and in addition $L(t_{i-1},t_i;\omega)$ has a large oscillation. Thus if $F_{i,k}$ occurs, then one of the events $E_{i,k,s}$ $(s=0,1,\ldots)$ must occur where $E_{i,k,0}$ is the set of ω such that
$$\sup_{t_{i-1}\leqslant t\leqslant t_i}|\mathbf{r}(t,\omega)-\mathbf{r}(t_i,\omega)|\leqslant\frac{8}{k}, \tag{11}$$
and
$$\inf_{t\geqslant 2}|\mathbf{r}(t,\omega)-\mathbf{r}(t_i,\omega)|<\frac{9}{k}. \tag{12}$$
$E_{i,k,s}$ $(s=1,2,\ldots)$ is the set of ω such that
$$\frac{2^{s+2}}{k}<\sup_{t_{i-1}\leqslant t\leqslant t_i}|\mathbf{r}(t,\omega)-\mathbf{r}(t_i,\omega)|\leqslant\frac{2^{s+3}}{k}, \tag{13}$$
and
$$\inf_{t\geqslant 2}|\mathbf{r}(t,\omega)-\mathbf{r}(t_i,\omega)|<\frac{2^{s+3}+1}{k}. \tag{14}$$

Define a random variable $l_{i,k}(\omega)$ $(i = 1, 2, \ldots, k^2)$, by

$$l_{i,k}(\omega) = \begin{cases} \dfrac{2^{s+3}}{k} & \text{if} \quad \omega \in E_{i,k,s} \quad (s = 0, 1, 2, \ldots), \\ 0 & \text{if} \quad \omega \text{ is not in } F_{i,k}. \end{cases}$$

For $i = 1, 2, \ldots, k^2$, define a sphere $S_{i,k}$ with centre $\mathbf{r}(t_i, k)$ and radius $l_{i,k}$. Then by construction, $\bigcup_{i=1}^{k^2} S_{i,k}$ covers $L(0, 1; \omega) \cap L(2, \infty; \omega)$. Let

$$\sum_{i=1}^{k^2} d(S_{i,k}) = 2 \sum_{i=1}^{k^2} l_{i,k}(\omega) = l_k(\omega). \tag{15}$$

We now prove that $l_k(\omega)$ has finite expectation. Now

$$\mu\{E_{i,k,0}\} \leqslant \mu\left\{\omega\colon \inf_{t \geqslant t_i+1} |\mathbf{r}(t,\omega) - \mathbf{r}(t_i,\omega)| < \frac{9}{k}\right\}$$

$$= \mu\left\{\omega\colon \inf_{t \geqslant 1} |\mathbf{r}(t,\omega)| < \frac{9}{k}\right\}, \quad \text{by (2)}.$$

Hence, by Lemma 2·1,

$$\mu\{E_{i,k,0}\} \leqslant \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{9}{k}. \tag{16}$$

By (3), the conditions (13) and (14) are independent. Hence, for $s = 1, 2, \ldots$,

$$\mu\{E_{i,k,s}\}$$

$$= \mu\left\{\omega\colon \frac{2^{s+2}}{k} < \sup_{t_{i-1} \leqslant t \leqslant t_i} |\mathbf{r}(t,\omega) - \mathbf{r}(t_i,\omega)| \leqslant \frac{2^{s+3}}{k}\right\} \mu\left\{\omega\colon \inf_{t \geqslant 2} |\mathbf{r}(t,\omega) - \mathbf{r}(t_i,\omega)| < \frac{2^{s+3}+1}{k}\right\}$$

$$< \mu\left\{\omega\colon \sup_{t_{i-1} \leqslant t \leqslant t_i} |\mathbf{r}(t,\omega) - \mathbf{r}(t_i,\omega)| > \frac{2^{s+2}}{k}\right\} \mu\left\{\omega\colon \inf_{t \geqslant t_i+1} |\mathbf{r}(t,\omega) - \mathbf{r}(t_i,\omega)| < \frac{2^{s+3}+1}{k}\right\}$$

$$\leqslant 10\,.\,2^{-s-2} \exp(-2^{2s+1}) \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2^{s+3}+1}{k},$$

by Lemmas (1·4), (2·1). Thus we have, for $s = 1, 2, \ldots$,

$$\mu\{E_{i,k,s}\} < \frac{10}{k} \exp(-4^s). \tag{17}$$

Thus

$$\int_\Omega l_k(\omega)\, d\omega = 2 \sum_{i=1}^{k^2} \int_\Omega l_{i,k}(\omega)\, d\omega$$

$$= 2 \sum_{i=1}^{k^2} \sum_{s=0}^{\infty} \frac{2^{s+3}}{k} \mu\{E_{i,k,s}\}$$

$$< 144 + 2 \sum_{s=1}^{\infty} 10\,.\,2^{s+3} \exp(-4^s),$$

by (16) and (17). Hence

$$\int_\Omega l_k(\omega)\, d\omega < M \quad (k = 1, 2, \ldots), \tag{18}$$

where M is some finite constant. This allows us to prove

LEMMA 2·2. *For almost all ω of Ω,*

$$\Lambda\{L(0,1;\omega) \cap L(2,\infty;\omega)\} < \infty.$$

Proof. Let $T_{k,j} = \{\omega : l_k(\omega) \leqslant j \,.\, M\}$, $j = 1, 2, \ldots$. It follows from (18) that

$$\mu\{T_{k,j}\} \geqslant 1 - 1/j \quad (k = 1, 2, \ldots). \tag{19}$$

Let
$$T_j = \bigcap_{k_0=1}^{\infty} \left\{ \bigcup_{k=k_0}^{\infty} T_{k,j} \right\};$$

then T_j is the set of ω which are in $T_{k,j}$ for infinitely many positive integers k. By (19), it follows that

$$\mu(T_j) \geqslant 1 - 1/j. \tag{20}$$

Hence $\mu\left\{\bigcup_{j=1}^{\infty} T_j\right\} = 1$, and almost all ω of Ω are in T_{j_0} for some integer j_0. Given a fixed ω_0, let j_0 be such that $\omega_0 \in T_{j_0}$. Then

$$\sum_{i=1}^{k^2} d(S_{i,k}) = l_k(\omega) \leqslant j_0 M, \tag{21}$$

for all integers k in a certain subsequence

$$k_1, \; k_2, \; \ldots \tag{22}$$

of the positive integers.

Since $\mathbf{r}(t, \omega)$ is continuous with probability 1, it is uniformly continuous $0 \leqslant t \leqslant 1$, and hence

$$\max_{1 \leqslant i \leqslant k^2} d(S_{i,k}) \to 0 \quad \text{as} \quad k \to \infty.$$

For any $\delta > 0$, choose K such that

$$\max_{1 \leqslant i \leqslant k^2} d(S_{i,k}) < \delta \quad \text{for} \quad k \geqslant K.$$

Then, if k_r is any integer of the subsequence (22) such that $k_r \geqslant K$, then $\bigcup_{i=1}^{k^2} S_{i,k_r}$ is a covering of $L(0,1;\omega) \cap L(2,\infty;\omega)$ by spheres of diameter $< \delta$ and, by (21), (9),

$$\Lambda_\delta\{L(0,1;\omega) \cap L(2,\infty;\omega)\} \leqslant j_0 M.$$

Since this is true for every $\delta > 0$, we have, by (10),

$$\Lambda\{L(0,1;\omega) \cap L(2,\infty;\omega)\} \leqslant j_0 M.$$

Thus the lemma is proved.

We can now prove

THEOREM. *Almost all Brownian paths $L(\omega)$ in 3-space have no triple points.*

Proof. By obvious modifications of the proof of Lemma 2·2, it follows that if $0 \leqslant a < b < c < d$ then with probability 1,

$$\Lambda\{L(a,b;\omega) \cap L(c,d;\omega)\} < \infty.$$

By Lemma 1·2, it follows that, with probability 1,

$$C\{L(a,b;\omega) \cap L(c,d;\omega)\} = 0.$$

Hence, if $e > d$, by Lemma 1·1, there is probability 1 that

$$L(a,b;\omega) \cap L(c,d;\omega) \cap L(e,\infty;\omega) = \emptyset. \tag{23}$$

Let a, c, b, d, e take all rational values satisfying

$$0 \leqslant a < b < c < d < e < \infty.$$

Then with probability 1, (23) is satisfied for each of these sets of values. Hence, with probability 1, there cannot exist t, t', t'' with $0 \leqslant t < t' < t'' < \infty$ and

$$\mathbf{r}(t,\omega) = \mathbf{r}(t',\omega) = \mathbf{r}(t'',\omega).$$

This completes the proof.

3. *Further problems.* We proved that in 3-space the Λ-measure of sets of the form $L(a,b;\omega) \cap L(c,d;\omega)$ is finite; and therefore with probability 1, the Λ-measure of the set of double points of $L(\omega)$ is sigma-finite. The question arises as to whether this result is best possible: i.e..what is the α-dimensional measure of the set of double points of $L(\omega)$ $(0 < \alpha \leqslant 1)$? Our conjecture is that with probability 1, the set of double points has zero Λ-measure, but that the Λ^α-measure $(0 < \alpha < 1)$ is infinite, i.e. the set of double points has dimension 1.

Similar questions can be asked in the case of Brownian motion in the plane. The methods of the present paper are good enough to show that the Λ-measure of the set of double points is positive with probability 1. However, this result is far from best possible. Our conjecture here is that the set of k-multiple points $(k = 2, 3, \ldots)$, which exists with probability 1, by (4), actually has dimension 2, i.e. there is probability 1 that, for $0 < \alpha < 2$ the Λ^α-measure of the set of k-multiple points $(k = 2, 3, \ldots)$ is infinite.

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