ACTA ARITHMETICA
V (1959)

Remarks on number theory II

Some problems on the σ function

by

P. ERDÖS (Budapest)

H. J. Kanold and I (see [1] and [4]) observed that if a and b, where $a \neq b$, are squarefree integers then $\sigma(a)/a \neq \sigma(b)/b$. The proof is very simple. Assume $\sigma(a)/a = \sigma(b)/b$; we can clearly assume $(a, b) = 1$. Let p be the greatest prime factor of ab, say $p|a$, $p \nmid b$. But then $a\sigma(b) = b\sigma(a)$ is clearly impossible, since the left side is a multiple of p and the right side is not.

On the other hand the equation

$$\frac{\sigma(a)}{a} = \frac{\sigma(b)}{b} \tag{1}$$

clearly has infinitely many solutions, e. g. if $(n, 42) = 1$,

$$\frac{\sigma(6n)}{6n} = \frac{\sigma(28n)}{28n} = 2\frac{\sigma(n)}{n}.$$

A solution of (1) is called *primitive* if

$$\frac{\sigma(a)}{a} = \frac{\sigma(b)}{b} \text{ but for every } d|a,\ d|b,$$
$$\left(d, \frac{a}{b}\right) = \left(d, \frac{b}{d}\right) = 1, \quad \sigma\left(\frac{a}{d}\right) \neq \sigma\left(\frac{b}{d}\right), \tag{2}$$

in other words a and b are called *primitive solutions of* (1) if no prime p divides a and b to the same exponent. Clearly every solution a_1, b_1 of (1) can be written in the form $a_1 = au_1$, $b_1 = bu$ where a and b are primitive solutions and $(u, ab) = 1$.

It is very probable that if $\{a_1, b_1\}, \{a_2, b_2\}$ are primitive solutions then $a_2 = ka_1$, $b_2 = kb_1$ is impossible.

It seems very likely that (1) has infinitely many primitive solutions, but I cannot prove this. Perhaps even the equation

$$\frac{\sigma(a)}{a} = \frac{\sigma(b)}{b}, \quad (a, b) = 1, \tag{3}$$

has infinitely many solutions. (3) clearly implies that $\sigma(a) \equiv 0 (\bmod a)$, $\sigma(b) \equiv 0 (\bmod b)$, i. e. that a and b are multiply perfect. In fact, no solution of (3) is known, since no odd multiply perfect number is known.

In the present paper I shall prove that *the number of distinct numbers of the form*

$$\frac{\sigma(n)}{n}, \quad 1 \leqslant n \leqslant x,$$

equals $c_1 x + o(x)$ *where* $6/\pi^2 < c_1 < 1$.

Further I shall outline the proof of the following result:

The number of solutions of (1) *satisfying* $a < b \leqslant x$ *equals* $c_2 x + o(x)$ *for some constant* $0 < c_2 < \infty$.

The analogous questions for $\varphi(n)$ are all trivial, since it is easy to see that $\varphi(a)/a = \varphi(b)/b$ holds if and only if a and b have the same prime factors. To see this observe that if a and b are both composed of the primes $p_1, p_2, \ldots, p_k$ then

$$\frac{\varphi(a)}{a} = \frac{\varphi(b)}{b} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

If a and b do not have the same prime factors we can write $a = a_1 d d_1$, $b = b_1 d d_2$ where $(a_1, b_1) = 1$ and not both $a_1 = 1$, $b_1 = 1$ and all prime factors of d_1 and d_2 divide d. Then $\varphi(a)/a = \varphi(b)/b$ would clearly imply $\varphi(a_1)/a_1 = \varphi(b_1)/b_1$, and this is clearly impossible.

I would finally like to call attention to three simple problems which as far as I know are still unsolved (see [6], p. 193 and 198).

Is it true that the equation $\sigma(n) = \varphi(m)$ has infinitely many solutions? The answer certainly must be yes.

Let $1 \leqslant c \leqslant \infty$. Does there exist an infinite sequence of integers n_k, m_k, where $n_k \neq m_k$, for which $\sigma(n_k) = \sigma(m_k)$ and $m_k/n_k \to c$? It is not difficult to see that for $c = 1$ the answer is positive, but I cannot decide the general question, in particular $c = \infty$ is open. The analogous question for the function φ can easily be answered affirmatively.

Is it true that the number $g(x)$ of solutions of

$$\sigma(a) = \sigma(b), \quad (a, b) = 1 \tag{4}$$

satisfies $g(x)/x \to \infty$?

THEOREM 1. *The number of distinct numbers of the form*

$$\frac{\sigma(n)}{n}, \quad 1 \leqslant n \leqslant x$$

equals $c_1 x + o(x)$ (compare [5]).

Write $n = A_n B_n$ where A_n is the squarefree part and B_n the quadratic part of n, i. e.

$$A_n = \prod_{p|n, p^2 \nmid n} p, \quad B_n = \frac{n}{A_n}, \quad (A_n, B_n) = 1.$$

Now we prove the following

LEMMA. *Let v_1 and v_2 be two integers whose all prime factors occur with an exponent greater than* 1, (*i. e. whose squarefree part is* 1). *Then there exists at most one pair of squarefree integers u_1 and u_2 satisfying*

$$(5) \qquad \frac{\sigma(u_1 v_1)}{u_1 v_1} = \frac{\sigma(u_2 v_2)}{u_2 v_2}, \quad (u_1, v_1) = (u_2, v_2) = (u_1, u_2) = 1.$$

Suppose that there is a second pair u_1', u_2' satisfying (5). Then we should have

$$(6) \quad \frac{\sigma(u_1)}{u_1} \frac{u_2}{\sigma(u_2)} = \frac{\sigma(u_1')}{u_1'} \frac{u_2'}{\sigma(u_2')} = \frac{r}{s}, \quad (u_1, u_2) = (u_1', u_2') = (r, s) = 1.$$

Now we show that (6) has no solutions (except if $u_1 = u_1', u_2 = u_2'$ or $u_1 = u_2', u_2 = u_1'$), and this contradiction will complete the proof of the Lemma. Assume that u_1, u_2, u_1', u_2' is a solution of (6) for which the product $u_1 u_2 u_1' u_2'$ is minimal (it clearly must be greater than 1 since not all the u's can be 1). Let $p > 1$ be the greatest prime factor of $u_1 u_2 u_1' u_2'$; assume say $p|u_1, p \nmid u_2$. Clearly $p|s$ (since $\sigma(u_1) \not\equiv 0 \pmod p$ as u_1 is squarefree). But then by (6) $u_1' \sigma(u_2') \equiv 0 \pmod p$ or $u_1' \equiv 0 \pmod p$, $u_2' \not\equiv 0 \pmod p$. But then $u_1/p, u_2, u_1'/p, u_2'$ also satisfy (6), which contradicts the minimality of the product $u_1 u_2 u_1' u_2'$.

In the same way we can prove that for squarefree integers u_i, u_j' the equation

$$\prod_{i=1}^{r} \frac{\sigma(u_i)}{u_i} = \prod_{j=1}^{s} \frac{\sigma(u_j')}{u_j'}$$

is impossible except if $\prod_{i=1}^{r} u_i = \prod_{j=1}^{s} u_j'$.

Now let $1 = v_1 < v_2 < \ldots$ be the sequence of the integers whose all prime factors occur with an exponent greater than 1. Clearly

$$\sum_{i=1}^{\infty} \frac{1}{v_i} = \prod_{p} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \ldots\right) < \infty, \tag{7}$$

and it is easy to see by a simple sieve process that the density of integers n whose quadratic part is v_i equals

$$\frac{1}{v_i} \prod_{p \mid v_i} \left(1 - \frac{1}{p}\right) \prod_{p \nmid v_i} \left(1 - \frac{1}{p^2}\right). \tag{8}$$

It clearly follows from (7) and (8) that

$$\sum_{i=1}^{\infty} \frac{1}{v_i} \prod_{p \mid v_i} \left(1 - \frac{1}{p}\right) \prod_{p \nmid v_i} \left(1 - \frac{1}{p^2}\right) = 1. \tag{9}$$

Now denote by $a_1^{(i)} < a_2^{(i)} < \ldots$ the integers whose quadratic part is v_i. Clearly

$$\frac{\sigma(a_k^{(i)})}{a_k^{(i)}} = \frac{\sigma(v_i)}{v_i} \cdot \frac{\sigma(u_k)}{u_k}, \quad \text{where } u_k \text{ is squarefree and } (u_k, v_i) = 1.$$

Thus the numbers $\sigma(a_k^{(i)})/a_k^{(i)}$ are all different. Next we show that the number of numbers $\sigma(a_k^{(i)})/a_k^{(i)}$, $v_i \leqslant a_k^{(i)} \leqslant x$, which differ from all the numbers of the form $\sigma(a_k^{(j)})/a_k^{(j)}$, $1 \leqslant j < i$, $a_k^{(j)} \leqslant x$ (i. e. which differ from all the numbers of the form $\sigma(n)/n$ whose quadratic part is less than v_i) equals

$$\frac{a_i x}{v_i} \prod_{p \mid v_i} \left(1 - \frac{1}{p}\right) \prod_{p \nmid v_i} \left(1 - \frac{1}{p^2}\right) + o(x), \quad 0 < a_i \leqslant 1. \tag{10}$$

To prove (10) observe that

$$\sigma(a_k^{(i)})/a_k^{(i)} = \sigma(a_{k'}^{(j)})/a_{k'}^{(j)}, \quad 1 \leqslant j < i, \tag{11}$$

holds if and only if there is a primitive solution n_l, m_l of (1) so that

$$a_k^{(i)} = t n_l, \quad a_{k'}^{(j)} = t m_l, \quad (t, n_l m_l) = 1 \tag{12}$$

Clearly the quadratic part of n_l and m_l must be less than or equal to v_i; thus by our Lemma there is only a finite number of possible choices for n_l and m_l (in fact the number of choices is at most $i-1$). Thus (11) does not hold if $a_k^{(i)}$ is not of the form (12). (10) now follows by a simple sieve process.

Theorem 1 clearly follows from (7) and (10).

THEOREM 2. *The number of solutions of the equation*

$$\sigma(a)/a = \sigma(b)/b, \quad a < b \leqslant x \tag{13}$$

equals $c_2 x + o(x)$.

We will only sketch the proof of Theorem 2. Denote by $\{a_i, b_i\}$, $a_i < b_i$, the set of all the primitive solutions of (1). Since every solution of (13) is a multiple of a primitive solution, Theorem 2 will follow by a simple sieve process if we succeed in proving that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty. \tag{14}$$

Let v_k and v_l ($v_k < v_l$) be any two integers whose squarefree part is 1. From our Lemma it follows that there is at most one primitive solution of (1) $\{a_i, b_i\}$ for which $B_{a_i} = v_k$, $B_{b_i} = v_l$.

Thus clearly

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \sum_{j=1}^{\infty} \frac{j}{v_j}. \tag{15}$$

Unfortunately $\sum_{j=1}^{\infty} j/v_j = \infty$, since it is well known that (see [2]) $v_j = cj^2 + O(j)$. Thus to prove (14) we need somewhat more complicated arguments, and from now on we will omit most of the details since they are somewhat cumbersome, but not really difficult and similar to arguments used in previous papers of mine [2].

To prove the convergence of (14) we first split the pairs (v_k, v_l) which give rise to primitive solutions $\{a_i, b_i\}$ into two classes. In the first class are the pairs satisfying $v_k < v_l/(\log v_l)^4$. For these pairs we have as in (15)

$$\sum{}' \frac{1}{b_i} < \sum_{l} f(v_l)/v_l \tag{16}$$

where the accent in the summation indicates that the summation is extended only over those pairs $\{a_i, b_i\}$ which correspond to pairs (v_k, v_l) of the first class, and $f(v_l)$ denotes the number of the v's not exceeding $v_l/(\log v_l)_4$. From $v_j = cj^2 + O(j)$ we evidently have

$$f(v_l) < c_3 l/(\log l)^2. \tag{17}$$

(16) and (17) clearly implies that $\sum{}' 1/b_i < \infty$.

Henceforth we can restrict ourselves to the pairs (v_k, v_l) satisfying

$$v_l/(\log v_l)^4 < v_k < v_l. \tag{18}$$

Now put

$$a_i = uv, \quad b_i = u'v', \quad (u, u') = 1 \tag{19}$$

where (v, v') is a pair satisfying (18). We split the pairs satisfying (18) again into two classes. In the first class are the pairs for which

$$\max(u, u') > (\log v)^5. \tag{20}$$

It easily follows from (15), (18) and (20) that

$$\sum{}'' \frac{1}{b_i} < \sum_{j=1}^{\infty} \frac{j}{v_j^2(\log v_j^2)} < \infty \tag{21}$$

where $\sum''$ denotes that the summation is extended over the pairs (v, v') satisfying (20).

Thus finally we can assume that (20) does not hold. But then if (v_k, v_l) give rise to the primitive pair (a_i, b_i) we must have

$$\frac{\sigma(v_k)}{v_k} = \frac{u_k}{\sigma(u_k)} \cdot \frac{\sigma(u_l)}{u_l} \cdot \frac{\sigma(v_l)}{v_l}, \quad (a_i = u_k v_k,\ b_i = u_l v_l). \tag{22}$$

Since (20) does not hold, there are at most $(\log v_l)^{10}$ choices for

$$\frac{u_k}{\sigma(u_k)} \cdot \frac{\sigma(u_l)}{u_l},$$

or there are at most $(\log v_l)^{10}$ possible choices for $\sigma(v_k)/v_k$. I can prove the following

THEOREM 3. *Let $1 \leqslant a < \infty$. Then the number of solutions of $\sigma(n)/n = a$, $1 \leqslant n \leqslant x$ is less than $c_4 x^{1/2 - c_5}$, where c_4 and c_5 are independent of a.*

We do not give the proof of Theorem 3 since it is similar to one used in a previous paper [1] and also uses the remark that for squarefree n the numbers $\sigma(n)/n$ are all different. It is very likely that Theorem 3 is very far from being best possible and I would guess that the number of solutions of $\sigma(n)/n = a$, $1 \leqslant n \leqslant x$ is $o(x^{\varepsilon})$. Possibly one can prove this by using the method of Hornfeck and Wirsing [3].

From Theorem 3 it follows that the number of solutions of (22) is less than

$$c_4 v_l^{1/2 - c_5} (\log v_l)^{10} < v_l^{1/2 - c_6} < c_7 l^{1 - 2c_6} \tag{23}$$

for sufficiently large l.

From (23) it follows that (as in (15))

$$(24) \qquad \sum\nolimits''' \frac{1}{b_j} < \sum_{j=1}^{\infty} \frac{c_7 j^{1-2c_6}}{v_j} < \sum_{j=1}^{\infty} \frac{c_8}{j^{1+2c_6}} < \infty,$$

where in $\sum'''$ the summation is extended over those $\{a_i, b_i\}$ which give rise to the pair (v_k, v_l), which does not satisfy (20). (16), (17), (21), and (24) prove (14) and thus the proof of Theorem 2 is complete.

References

[1] P. Erdös, *On primitive abundant numbers*, J. Lond. Math. Soc. 10 (1935), p. 49-58.

[2] — and G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Litt. Sci. Szeged 7 (1934), p. 95-102.

[3] B. Hornfeck und E. Wirsing, *Über die Häufigkeit vollkommener Zahlen*, Math. Ann. 133 (1957), p. 431-438.

[4] H.-J. Kanold, *Über die Verteilung der vollkommenen Zahlen und allgemeinerer Zahlenmengen*, Math. Ann. 132 (1957), p. 442-450.

[5] — *Über zahlentheoretische Funktionen*, J. Reine Angew. Math. 195 (1955), p. 180-191.

[6] A. Schinzel et W. Sierpiński, *Sur certains hypothèses concernant les nombres premiers*, Acta Arithmetica 4 (1958), p. 185-208.

Reçu par la Rédaction le 28. 8. 1958