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THE HAUSDORFF MEASURE OF THE INTERSECTION OF SETS OF POSITIVE LEBESGUE MEASURE

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Erdős, Kestelman and Rogers [**1**] showed that, if $A_1, A_2, \ldots$ is any sequence of Lebesgue measurable subsets of the unit interval $[0, 1]$ each of Lebesgue measure at least $\eta > 0$, then there is a subsequence $\{A_{n_i}\}$ $(i = 1, 2, \ldots)$ such that the intersection $\bigcap_{i=1}^{\infty} A_{n_i}$ contains a perfect subset (and is therefore of power $2^{\aleph_0}$). They asked for what Hausdorff measure functions $\phi(t)$ is it possible to choose the subsequence to make the intersection set $\bigcap A_{n_i}$ of positive ϕ-measure. In the present note we show that the strongest possible result in this direction is true. This is given by the following theorem.

THEOREM. *Suppose $\phi(t)$ is continuous, monotonic increasing in t and such that $\lim_{t\to 0+} \phi(t) = 0$, $\lim_{t\to 0+} t^{-1}\phi(t) = +\infty$. Given any sequence $A_1, A_2, \ldots$ of Lebesgue measurable subsets of $[0, 1]$ satisfying $\limsup_{n\to\infty} |A_n| > 0$, there is a subsequence $\{A_{n_i}\}$ such that*

$$\phi - m\left(\bigcap_{i=1}^{\infty} A_{n_i}\right) = +\infty.$$

It is easy to show that the conclusion of the theorem is valid for the special sequence $\{K_q\}$ of Rademacher sets, where K_q is the set of real numbers of the form

$$a_1 2^{-1} + a_2 2^{-2} + \ldots + a_n 2^{-n} + \ldots$$

with $a_q = 0$ and $a_i = 0$ or 1 for $i \neq q$. The reason why this particular sequence of sets easily yields a subsequence with the required property is that, in a certain obvious sense, the sequence $\{K_q\}$ is "asymptotically uniformly spread" in $[0, 1]$. We cannot assume this property of a general sequence $\{A_n\}$, but the first and vital step of the proof consists in showing that there must be a subset $Q \subset [0, 1]$ and a subsequence $\{A_{n_i}\}$ which is asymptotically spread with positive minimum density throughout Q. This result is formalised in the following lemma, for which we need a definition.

DEFINITION. *If t, q are positive integers with $q \leqslant 2^t$, the closed interval $[(q-1)\,2^{-t}, q2^{-t}]$ is called a dyadic interval of order t. Any subset $E \subset [0, 1]$ which can be expressed as a finite union of dyadic intervals of order t is called a subset of order t.*

LEMMA. *Given a sequence $\{A_k\}$ of measurable subsets of $I_0 = [0, 1]$ such that $|A_k| \geqslant \eta > 0$ for all k, there exists a sequence $I_0 \supset I_1 \supset \ldots \supset I_n \supset \ldots$ such that I_n is a dyadic subset of order n, and a subsequence $\{A_{k_n}\}$ such that for all integers $r \geqslant n$,*

(i) $|A_{k_r} \cap (I_0 - I_n)| \leqslant \frac{1}{2}\eta\, |I_0 - I_n|$, (1)

(ii) *If J is a dyadic interval of order n contained in I_n,*

$$|A_{k_r} \cap J| \geqslant \tfrac{1}{2}\eta\, |J|; \tag{2}$$

(iii) $|I_n \cap A_{k_r}| \geqslant (\frac{1}{2}\eta)^2$. (3)

Further, if $Q = \bigcap_{n=0}^{\infty} I_n$, then

$$|Q| \geqslant \tfrac{1}{2}\eta. \tag{4}$$

Proof. It is clear that $|A_k \cap I_0| \geqslant \eta\, |I_0|$ for all k, so that (2) and (3) will be satisfied with $n = 0$ whatever subsequence we choose. Bisect I_0 into two dyadic intervals of order 1. Then there are two possibilities:

(i) There may be an infinite sequence of integers such that

$$|A_k \cap J| \geqslant \tfrac{1}{2}\eta\, |J| \tag{5}$$

for both the dyadic intervals J of order 1. In this case we put $I_1 = I_0$, and denote by Λ_1 the set of integers k satisfying (5).

(ii) If such a sequence cannot be found, then for (at least) one of the subintervals $J_1 \subset I_0$, there must be an infinite set of integers for which

$$|A_k \cap J_1| < \tfrac{1}{2}\eta\, |J_1|. \tag{6}$$

In this case we put $I_1 = J_2$ (the other dyadic interval of order 1) and denote by Λ_1 the set of integers k satisfying (6). Since $|A_k| \geqslant \eta$ for all k, we must have $|A_k \cap J_2| \geqslant \frac{3}{2}\eta\, |J_2| > \frac{1}{2}\eta\, |J_2|$ for $k \in \Lambda_1$.

Thus in either case we obtain a set I_1 and a sequence Λ_1 such that (5) is satisfied for all the dyadic intervals $J \subset I_1$ of order 1. We proceed by induction. Suppose we have already defined a dyadic set I_n of order n and a subsequence Λ_n such that, for $k \in \Lambda_n$,

$$|A_k \cap (I_0 - I_n)| \leqslant \tfrac{1}{2}\eta |I_0 - I_n|, \tag{7}$$

and (5) is satisfied for all the dyadic intervals $J \subset I_n$ of order n. By bisecting each of these intervals we can express I_n as a union of dyadic intervals of order $(n+1)$. Then there may be a subsequence $\Lambda_{n-1} \subset \Lambda_n$ such that (5) is satisfied for all dyadic intervals $J \subset I_n$ of order $(n+1)$. In this case define $I_{n+1} = I_n$. If this is not true, then by repeating the operation of taking a subsequence a finite number of times we can obtain a subsequence $\Lambda_{n+1} \subset \Lambda_n$ and a dyadic subset $I_{n+1} \subset I_n$, such that (5) is satisfied for all the dyadic intervals, $J \subset I_{n+1}$ of order $(n+1)$, while the other dyadic intervals J satisfy

$$|A_k \cap J| < \tfrac{1}{2}\eta |J|$$

for all $k \in \Lambda_{n+1}$. In either case we have obtained a dyadic subset $I_{n+1} \subset I_n$ and a subsequence $\Lambda_{n+1} \subset \Lambda_n$ with the desired properties.

By induction we may suppose that I_n, Λ_n have been obtained for all positive integers n. Now let $\Lambda = \{k_n\}$ be defined by taking, for k_1, the first integer in Λ_1, and for k_{n+1}, the first integer in Λ_{n+1} which is greater than k_n $(n = 1, 2, \ldots)$. It is clear that this sequence Λ satisfies conditions (1) and (2).

It follows from (7) that, for $k \in \Lambda_n$,

$$|I_n| \geqslant |I_n \cap A_k| \geqslant \eta - \tfrac{1}{2}\eta |I_0 - I_n| = \tfrac{1}{2}\eta + \tfrac{1}{2}\eta |I_n|.$$

Hence

$$|I_n| \geqslant \frac{\frac{1}{2}\eta}{1 - \frac{1}{2}\eta} > \tfrac{1}{2}\eta, \quad n = 1, 2, \ldots, \tag{8}$$

and this immediately implies (4). We can obtain (3) by applying (2) to (8), since $k_r \in \Lambda_n$ for all $r \geqslant n$. This completes the proof of the lemma.

We are now in a position to tackle the measure properties of the intersection sets. We will obtain a subsequence $\Lambda' \subset \Lambda$ such that if $E = \bigcap_{k \in \Lambda'} A_k$, then the set $E \cap Q$ has infinite ϕ-measure. The essential idea of the proof is to define a set function F which is determined for all Borel subsets of $[0, 1]$ and which is concentrated on $E \cap Q$, that is

$$F(B) = F(B \cap E \cap Q) \text{ for all Borel } B \subset [0, 1];$$

$$F(I_0) = F(E \cap Q) > 0;$$

but such that $\max \{2^n . F(J)\}$ over all dyadic intervals of order n grows slowly as n increases. We are here really using the concept of local ϕ-density of F at points of I_0, studied extensively in [**3**], but it turns out to be easier to formulate our proof independently of [**3**]. The set function

F will be obtained as a limit of a sequence of set functions defined inductively.

Proof of main theorem. There is no loss in generality in assuming that $|A_k| \geqslant \eta > 0$. for all integers k, and that the sets A_k are all closed. We first apply the lemma to obtain a sequence $\{I_n\}$ of dyadic sets and a subsequence $\Lambda = \{k_n\}$ satisfying all the conditions (1), (2), (3). Since we can define a continuous $\psi(t)$ such that $\lim_{t\to 0+} \psi(t) = 0$, $\lim_{t\to 0+} \psi(t)/\phi(t) = 0$, $\lim_{t\to 0+} t^{-1}\psi(t) = +\infty$ and $t^{-1}\psi(t)$ is monotonic for small t (an equivalent result was proved in [**2**]), there is also no loss in generality in assuming that $t^{-1}\phi(t)$ is monotonic for small positive t. Under these conditions it follows from the method of Besicovitch [**4**] that it is sufficient to show that the dyadic restricted ϕ-measure of the intersection set is infinite. Thus it will be enough to show that if

$$\bigcup_{i=1}^{\infty} J_{r,i} \supset E = \bigcap_{k\in\Lambda'} A_k,$$

where each $J_{r,i}$ is a dyadic interval of order at least r, then

$$\sum_{i=1}^{\infty} \phi(|J_{r,i}|) \geqslant \lambda_r, \quad \lambda_r \to \infty \text{ as } r\to\infty. \tag{9}$$

Our aim is to choose Λ' so that (9) is established.

Suppose $0 < \epsilon < \frac{1}{10}$. Since $\{I_n\}$ is monotonic we can choose a sequence $\{t_r\}$ of integers such that

$$|I_t - Q| < \frac{\epsilon}{2^r} \{\tfrac{1}{3}\eta\}^{r+1} \tag{10}$$

for all $t \geqslant t_r$. Because $t^{-1}\phi(t) \to +\infty$ as $t \to 0+$, we may assume that $\{t_r\}$ also increases fast enough to ensure

$$\phi(2^{-t}) \geqslant r 2^{-t} \{\tfrac{1}{3}\eta\}^{-r-1} \tag{11}$$

for all $t \geqslant t_r$.

Now put $r_1 = t_1$ and $n_1 = k_{r_1}$ so that $n_1 \in \Lambda_{t_1}$; and let

$$E_1 = A_{n_1} \cap I_{t_1}$$

By (3) we know that $|E_1| \geqslant (\frac{1}{2}\eta)^2$. Define a set function F_1 which is concentrated on E_1 by

$$F_1(B) = F_1(B\cap E_1) = \frac{|B\cap E_1|}{|E_1|} \tag{12}$$

for all Borel sets $B \subset [0, 1]$.

For each integer l, each point $x \in [0, 1]$ define

$$d_l(x, E_1) = |J_l(x) \cap E_1| \,.\, 2^l$$

where $J_l(x)$ is the dyadic interval of order l which contains x [if x is a point of the form $k.2^{-l}$ then take the dyadic interval which has x as its left-hand end point for $J_l(x)$]. By the Lebesgue density theorem it follows that, for almost all $x \in E_1$,

$$d_l(x, E_1) \to 1 \text{ as } l \to \infty.$$

If we now apply Egoroff's theorem (a similar argument was used in [**5**]) to the sequence $\{d_l(x, E_1)\}$ of measurable functions we can obtain a set $B_1 \subset E_1$ and a positive integer l_1 such that

$$d_l(x, E_1) \geqslant 1 - \tfrac{1}{6}\eta$$

for $x \in B_1$ and all $l \geqslant l_1$, and in addition

$$|E_1 - B_1| < \tfrac{1}{2}\epsilon\{\tfrac{1}{3}\eta\}^2 |E_1|. \tag{13}$$

This implies that, if J is any dyadic interval of order at least l_1 which contains a point of B_1, then

$$|J \cap E_1| \geqslant (1 - \tfrac{1}{6}\eta)|J|. \tag{14}$$

Now choose $r_2 = \max(t_2, l_1)$, $n_2 = k_{r_2}$, and let C_1 be the union of all the dyadic intervals J of order r_2 whose intersection with B_1 is not void. Let

$$D_1 = C_1 \cap A_{n_1} \cap I_{r_2}$$

Since $I_{t_1} - I_{r_2} \subset I_{t_1} - Q$ and $C_1 \supset B_1$, it follows from (10) and (13) that

$$|E_1 - D_1| \leqslant |E_1 - B_1| + |I_{t_1} - Q| < \epsilon |E_1|,$$

so that, by (12),

$$F_1(D_1) > (1 - \epsilon) F_1(E_1) = 1 - \epsilon.$$

Notice further that for any dyadic interval J, (12) and (3) imply that

$$F_1(J) = F_1(J \cap E_1) \leqslant (\tfrac{1}{2}\eta)^{-2}|J| \leqslant (\tfrac{1}{3}\eta)^{-2}|J|.$$

We now proceed by induction. Suppose $n_1, n_2, \ldots, n_q$ have been chosen with $n_i = k_{r_i}$, $r_i \geqslant t_i$, and dyadic sets $C_1, C_2, \ldots, C_q$, where C_i is of order r_{i+1}, have been obtained such that

(i) if $E_q = \bigcap_{i=1}^{q} A_{n_i} \cap \bigcap_{i=1}^{q} I_{r_i} \cap \bigcap_{i=1}^{q-1} C_i$, then

$$|J \cap E_q| \geqslant (1 - \tfrac{1}{6}\eta)|J| \tag{15}$$

for every dyadic interval J of order r_{q+1} in C_q;

(ii) $|E_q - C_q| < \dfrac{\epsilon}{2^q}(\tfrac{1}{3}\eta)^{q+1}|E_q|$; (16)

(iii) there is a set function F_q concentrated on E_q such that

$$F_q(I_0) = F_q(E_q) \geqslant 1 - 2\epsilon - \epsilon - \tfrac{1}{2}\epsilon \ldots - \epsilon 2^{-q+2} > \tfrac{1}{2}; \tag{17}$$

and

$$F_q(E_q \cap J) = F_q(J) \leqslant (\tfrac{1}{3}\eta)^{-q-1}|J| \tag{18}$$

for every dyadic interval J;

(iv) if J is any dyadic interval of order r_q, then, inside J, F_q is distributed according to the Lebesgue measure of the intersection with $J \cap E_q$, *i.e.*

$$F_q(T \cap J) = \frac{|T \cap J \cap E_q|}{|J \cap E_q|} F_q(J), \tag{19}$$

for any Borel set T.

Notice that we have already shown that the conditions (i)-(iv) are satisfied for $q = 1$.

Now put $n_{q+1} = k_{r_{q+1}}$ and define

$$E_{q+1} = C_q \cap I_{r_{q+1}} \cap A_{n_{q+1}} \cap E_q.$$

By (15) and (2) it follows that

$$|J \cap E_{q+1}| \geqslant \tfrac{1}{3}\eta |J| \tag{20}$$

for each dyadic interval J of order r_{q+1} in $C_q \cap I_{r_{q+1}}$.

We first define the set function F_{q+1} for the dyadic intervals of orders $r \leqslant r_{q+1}$ by

$$F_{q+1}(J) = F_q(J \cap C_q \cap I_{r_{q+1}}).$$

Inside each dyadic interval J of order r_{q+1} we redistribute the mass $F_{q+1}(J)$ on the set $E_{q+1} \cap J$ according to the Lebesgue measure. This is possible because, by (20), $E_{q+1} \cap J$ has positive Lebesgue measure $I_{r_{q+1}}$ for each J in $C_q \cap I_{r_{q+1}}$. Thus, for any Borel set T, and any dyadic interval J of order r_{q+1} in $C_q \cap I_{r_{q+1}}$,

$$F_{q+1}(T \cap J) = \frac{|T \cap J \cap E_{q+1}|}{|J \cap E_{q+1}|} F_{q+1}(J).$$

Since both sides of this equation are zero if J is not in $C_q \cap I_{r_{q+1}}$, we see that (19) is satisfied with q replaced by $(q+1)$.

Using (18), (16) and (10), and noting that $I_{r_q} - I_{r_{q+1}} \subset I_{r_q} - Q$, we obtain

$$\begin{aligned} F_{q+1}(I_0) &= F_q(E_q \cap C_q \cap I_{r_{q+1}}) \\ &\geqslant F_q(E_q) - F_q(E_q - C_q) - F_q(I_{r_q} - Q) \\ &\geqslant F_q(E_q) - \epsilon 2^{-q+1}, \end{aligned}$$

so that (17) is also satisfied with q replaced by $(q+1)$.

Now if J' is a dyadic interval contained in a dyadic interval J of order r_{q+1} in $C_q \cap I_{r_{q+1}}$ we have

$$F_{q+1}(E_{q+1} \cap J') = F_{q+1}(J') = \frac{|J' \cap E_{q+1}|}{|J \cap E_{q+1}|} F_q(J \cap C_q \cap I_{r_{q+1}})$$

$$\leqslant \frac{|J'|}{|J \cap E_{q+1}|} \leqslant (\tfrac{1}{3}\eta)^{-q-2} |J'|,$$

on applying (18) and (20). On the other hand if J' is a dyadic interval of order not more than r_{q+1} we have

$$F_{q+1}(J') \leqslant F_q(J').$$

It follows that (18) is satisfied with q replaced by $(q+1)$.

Since the set E_{q+1} still has positive measure [one can actually prove that $|E_{q+1}| \geqslant \frac{1}{2}(\frac{1}{3}\eta)^{q+2}$], we can again apply the Lebesgue density theorem, and Egoroff's theorem to obtain a subset $B_{q+1} \subset E_{q+1}$ and an integer l_{q+1} such that if J is any dyadic interval of order at least l_{q+1} which contains a point of B_{q+1}, then

$$|J \cap E_{q+1}| \geqslant (1 - \tfrac{1}{6}\eta) |J|$$

and

$$|E_{q+1} - B_{q+1}| < \frac{\epsilon}{2^{q+1}} (\tfrac{1}{3}\eta)^{q+2} |E_{q+1}|.$$

Put $r_{q+2} = \max(t_{q+2}, l_{q+1})$, and let C_{q+1} be the union of those dyadic intervals of order r_{q+2} which have a non-void intersection with B_{q+1}. Thus we have succeeded in extending all our conditions from q to $(q+1)$ and, by induction, we obtain the sequence $\Lambda' = \{n_q\} \subset A$ satisfying the conditions (15)–(20).

Now put $E = \bigcap_{k=\Lambda'} A_k$. By our construction

$$R = \bigcap_{q=1}^{\infty} E_q \subset E \cap Q \subset E,$$

so that it is sufficient to show that $\phi - m(R) = +\infty$. It can be shown that $F(B) = \lim_{n \to \infty} F_n(B)$ exists for each Borel set $B \subset [0, 1]$ and defines a measure concentrated on R. Further, (17) will imply that $F(I_0) > \frac{1}{2}$, and it can be shown that the upper ϕ-density of F is zero at each point of I_0. From this our conclusion would follow by [**3**]. However, we do not prove these statements as the details are somewhat complicated, and it is possible to complete our proof using the set functions F_q.

Suppose then that (9) is false, and there is a constant K such that for every integer s there is a covering $\bigcup_{i=1}^{\infty} J_{s,i} \supset R$ by dyadic intervals $J_{s,i}$ of orders $u_i \geqslant r_s$ such that

$$\sum_{i=1}^{\infty} \phi(2^{-u_i}) \leqslant K. \tag{21}$$

Let $\{v_i\}$ be a sequence of positive integers with $v_i \geqslant r_s$ such that

$$\sum_{i=1}^{\infty} \phi(2^{-v_i}) < 1. \tag{22}$$

For each integer i, let $J'_{s,i}$, $J''_{s,i}$ by dyadic intervals of order v_i contiguous to $J_{s,i}$ (one at each end). If $L_{s,i}$ denotes the interior of $J_{s,i} \cup J'_{s,i} \cup J''_{s,i}$ it is clear that the open intervals $L_{s,i}$ $(i = 1, 2, \ldots)$ cover the compact set R. Hence there is a *finite* set $\mathscr{I}$ of integers such that $R \subset \bigcup_{i \in \mathscr{I}} L_{s,i}$. Let $H_1, H_2, \ldots, H_l$ denote the dyadic intervals $J_{s,i}$, $J'_{s,i}$, $J''_{s,i}$ for $i \in \mathscr{I}$. For each i with $1 \leqslant i \leqslant l$ choose q_i so that the order w_i of the dyadic interval H_i satisfies

$$r_{q_i} \leqslant w_i < r_{q_i+1}.$$

Then provided $m = m_s$ is sufficiently large

$$r_s \leqslant r_{q_i} \leqslant w_i < r_{q_i+1} \leqslant r_m, \quad 1 \leqslant i \leqslant l. \tag{23}$$

Then by (21) and (22) we have

$$\sum_{i=1}^{l} \phi(2^{-w_i}) \leqslant K+2, \tag{24}$$

and, by (11) since $w_i \geqslant r_{q_i} \geqslant t_{q_i}$, it follows that

$$(\tfrac{1}{3}\eta)^{-q_i-1}\, 2^{-w_i} \leqslant \frac{1}{q_i}\, \phi(2^{-w_i}) \leqslant \frac{1}{s}\, \phi(2^{-w_i}). \tag{25}$$

Since R is contained in the open set $\bigcup_{i \in \mathscr{I}} L_{s,i}$ and R is the intersection of the decreasing sequence $E_1, E_2, \ldots$ of compact sets, we have $E_m \subset \bigcup L_{s,i}$ for all sufficiently large m. We now suppose that m is large enough to satisfy this condition as well as (25).

For any dyadic interval J of order u, $F_q(J)$ is monotone decreasing in q provided $r_q \geqslant u$ since F_{q+1} is obtained from F_q by first concentrating it on a subset and then redistributing the result inside J. Hence for any dyadic interval J of order u we have, by (18)

$$F_m(J) \leqslant F_q(J) \leqslant (\tfrac{1}{3}\eta)^{-q-1} |J|,$$

provided $u \leqslant r_q \leqslant r_m$. It follows now from (24) and (25) that

$$\begin{aligned} F_m(I_0) = F_m(E_m) = F_m\left(\bigcup_{i=1}^{l} H_i\right) &\leqslant \sum_{i=1}^{l} F_m(H_i) \\ &\leqslant \sum_{i=1}^{l} \{\tfrac{1}{3}\eta\}^{-q_i-2}\, 8^{-w_i} \leqslant (\tfrac{1}{3}\eta)^{-1} \frac{1}{s} \sum_{i=1}^{l} \phi(2^{-w_i}) \\ &\leqslant (\tfrac{1}{3}\eta)^{-1} (K+2) . \frac{1}{s}. \end{aligned}$$

Since s is an arbitrary integer this contradicts $F_m(I_0) > \frac{1}{2}$, when s is large enough. This contradiction establishes our theorem.

Remark. We have made no attempt to choose best possible constants at any point of the proof. If one takes care with these and adapts the ideas used in [**1**], the following apparently stronger version of our theorem can be proved.

THEOREM. *Suppose $\phi(t)$ satisfies the conditions of the previous theorem and $A_1, A_2, \ldots$ is a sequence of Lebesgue measurable subsets of $[0, 1]$ with $\limsup |A_r| \geqslant \eta > 0$. Then there is a Borel set S with $|S| \geqslant \eta$ and a sequence $q_1 < q_2 < \ldots$ such that if*

$$E = \bigcup_{j \geqslant 1} \bigcap_{r \geqslant j} A_{q_r}$$

and I is any interval for which $I \cap S$ is not void, then the ϕ-measure of $I \cap E$ is non-σ-finite.

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