THE MINIMAL REGULAR GRAPH CONTAINING A GIVEN GRAPH

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Let G be an ordinary graph of order n which is not regular and whose maximum degree is $v>0$. Let H denote any regular graph of degree v which contains a subgraph isomorphic to G. We seek the minimal order possible for H. Let x_i denote the degree of the ith vertex in G, so $v-x_i$ is the "deficiency" of that vertex; let $\sigma=\Sigma(v-x_i)$ be the sum of the deficiencies and d be the maximum deficiency

THEOREM. *The necessary and sufficient condition that $m+n$ be the minimal order possible for H is that m be the least positive integer such that*: (1) $m\geqq\sigma/v$; (2) $m^2-(v+1)m+\sigma\geqq 0$, (3) $m\geqq d$ *and* (4) $(m+n)v$ *is an even integer. The maximum value of m is n, and for each $n>3$ there exists a graph G such that $m=n$.*

Proof. Necessity. It is known that finite graphs H exist, so there is a minimal solution, say a graph H of order $m+n$, and $(m+n)v$ is clearly an even integer.

Let G' be the subgraph of H isomorphic to G and let A be the subgraph induced on the vertices of H not in G'. Then in H there are σ joins between the subgraphs G' and A. Since each of the m vertices of A receives at most v of these joins, $mv\geqq\sigma$, and clearly $m\geqq d$.

Denote by $m(A)$ the number of joins in A. The sum of the degrees of the vertices of A, as points of A, must be $mv-\sigma$, hence

(i) $m(A)=\frac{1}{2}(mv-\sigma)$.

Then from $m(m-1)/2\geqq m(A)$, it follows that

(ii) $m^2-(v+1)m+\sigma\geqq 0$,

so all four conditions are necessary.

To establish the sufficiency, let m be the least positive integer satisfying conditions (1)–(4). Define a graph H by beginning with G and m extra independent points $a_1, a_2, \cdots, a_m$. Let $p_1, p_2, \cdots, p_k$ denote the points of G with positive deficiencies $d_1, \cdots, d_k$. Let the completion of G be done in the following way. First, p_1 is completed by joins to the points $a_1, a_2, \cdots, a_{d_1}$ in succession. Then p_2 is completed by joins to successive points a_i, starting with a_{d_1+1}, which is taken cyclically to be a_1 if $d_1=m$. These completions are possible because $m\geqq d$. The degrees attained by points of A in this construction cannot differ from one another at any stage by more than one. So this is also true when the points of G are all complete.

Now let $\sigma/m=h+r/m$, where h and r are nonnegative integers and where $r<m$, and $h<v$ if $r>0$. Then when the vertices of G have been completed the set $\mathfrak{A}$ of vertices a_i, $i=1, \cdots, m$, consists of r points of degree $h+1$ and $m-r$ points of degree h. Since there are as yet no joins between points in $\mathfrak{A}$, any point of the greatest remaining deficiency $v-h$ can be completed if $v-h \leqq m-1$. But condition (2) can be written in the form

(iii) $v-\sigma/m \leqq m-1$,

from which it follows that

(iv) $v-h \leqq m-1+r/m$.

Because $0 \leqq r/m < 1$, while $v-h$ and $m-1$ are integers, (iv) implies that

(v) $v-h \leqq m-1$.

Thus there are in $\mathfrak{A}$ sufficient points so that each point individually can be completed.

Finally, the collective completion of all the points in $\mathfrak{A}$ will be possible if the sum of the deficiencies is an even integer, that is, if

(vi) $r(v-h-1)+(m-r)(v-h)=mv-\sigma$

is even. But

(vii) $mv-\sigma=mv-[nv-2m(G)]=(m+n)v-2[nv-m(G)]$.

By assumption $(m+n)v$ is even, hence $mv-\sigma$ is even and the completion of all points in $\mathfrak{A}$ is possible.

Since $\sigma < nv$, the condition $m \geqq \sigma/v$ cannot force $m > n$. Similarly $m^2-(v+1)m+\sigma \geqq 0$ always holds for $m=v+1$, and $v+1=n$. Condition (3) cannot force m to exceed $n-1$. The maximum possible value $m=n$, satisfying conditions (1) and (2) cannot be increased by condition (4), since $(m+n)v=(n+n)v$ is necessarily even. Thus in all cases $m \leqq n$.

If $n > 3$, let G be the graph obtained from a complete graph of order n by deleting one join. Then $v=n-1$ and $\sigma=2$, and the condition

(viii) $m^2-nm+2 \geqq 0$ implies that $m \geqq n$.

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