AN EXTREMAL PROBLEM IN GRAPH THEORY

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To Bernhard Hermann Neumann on his 60th birthday

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$G(n; l)$ will denote a graph of n vertices and l edges. Let $f_0(n, k)$ be the smallest integer such that there is a $G(n; f_0(n, k))$ in which for every set of k vertices there is a vertex joined to each of these. Thus for example $f_0(3, 2) = 3$ since in a triangle each pair of vertices is joined to a third. It can readily be checked that $f_0(4, 2) = 5$ (the extremal graph consists of a complete 4-gon with one edge removed). In general we will prove: Let $n > k$, and

$$(1) \qquad f(n, k) = (k-1)n - \binom{k}{2} + \left[\frac{n-k}{2}\right] + 1;$$

then $f_0(n, k) = f(n, k)$.

It will be convenient to say that the vertices $x_1, \ldots, x_k$ of G are visible from x_{k+1}, if all the edges (x_i, x_{k+1}), $i = 1, \cdots, k$ occur in G. A graph is said to have property P_k if every set of k of its vertices is visible from another vertex. G_n will denote a graph of n vertices (the number of edges being unspecified) and $G(m)$ denotes a graph having m edges. Let $G_n^{(0)} = (Gn; f(n; k))$ be defined as follows: the vertices of $G_n^{(0)}$ are $x_1, \cdots, x_n$. The vertices x_i, $i = 1, \cdots, k-1$ are joined to every other vertex and our $G_n^{(0)}$ has $[n-k+2/2]$ further edges which are as disjoint as possible. In other words if $n-k+1$ is even $G_n^{(0)}$ has the further edges (x_{k+2j}, x_{k+2j+1}), $j = 0, \cdots, [n-k-1/2]$, if $n-k+1$ is odd the edges are (x_k, x_{k+1}), (x_k, x_{k+2}), (x_{k+j+1}, x_{k+j+2}), $j = 1, \cdots, [n-k-2/2]$. It is easy to see that $G_n^{(0)}$ has property P_k. Now we prove

THEOREM 1. *A graph $G(n; f(n, k))$ has property P_k if and only if it is our graph $G_n^{(0)}$.*

Theorem 1 is vacuous for $n \leqq k$ and it is trivial for $n = k+1$, thus we can assume $n \geqq k+2$. Clearly Theorem 1 implies (1). To see this it suffices to observe that if a $G(n; f(n, k)-1)$ would have property P_k we could add to it a new edge so that the resulting $G(n; f(n, k))$ would not be a $G_n^{(0)}$.

Since $G_n^{(0)}$ has property P_k we only have to prove that a $G(n; f(n, k))$ has property P_k then it must be our $G_n^{(0)}$. Before we give the somewhat complicated proof we outline a simple proof of (1) for $k = 2$.

LEMMA. *Let G_n have property P_k then every pair of its vertices is visible from at least $k-1$ vertices.*

Assume that the Lemma is false. Then say x_1 and x_2 are visible from only $y_1, \ldots, y_l$, $l \leqq k-2$. But then the set of $l+2 \leqq k$ vertices $x_1, x_2, y_1, \ldots, y_l$ would not be visible from any vertex of G_n, which contradicts our assumption.

Let now x_i, $i = 1, \ldots, n$ be the vertices of G_n and assume that v_i is the valency of x_i (i.e. x_i is joined to v_i vertices of G). Our Lemma implies

$$\sum_{i=1}^{n} \binom{v_i}{2} \geqq (k-1) \binom{n}{2} \tag{2}$$

since the number of pairs of vertices visible from x_i is $\binom{v_i}{2}$.

From (2) it is easy to deduce (1) for $k = 2$. To see this observe that the number of edges of a graph is $\frac{1}{2} \sum_{i=1}^{n} v_i$.

By (2) $\sum_{i=1}^{n} \binom{v_i}{2} \geqq \binom{n}{2}$ and thus by a simple argument $\frac{1}{2} \sum_{i=1}^{n} v_i$ will be at least as large as in the case that one v_i say v_1 is as large as possible i.e. $v_1 = n-1$, and $v_2, \ldots, v_n$ are as small as is consistent with (2). Now it is easy to see that P_2 implies $v_i \geqq 2$ for all i. Hence

$$\frac{1}{2} \sum_{i=1}^{n} v_i \geqq \frac{1}{2}(n-1+2(n-1)) = \frac{3}{2}(n-1) \tag{3}$$

which agrees with (1) for $k = 2$ if n is odd. If n is even a similar but somewhat more complicated argument proves (1).

It does not seem easy to deduce (1) from (2) for $k > 2$. One could easily obtain

$$f(n, k) = (k-\tfrac{1}{2})n+O(1)$$

but a more precise estimation seems difficult. Hence to prove (1) and Theorem 1 we shall use a different method.

We say that $G(m)$ has property θ_t if it contains a set S of t vertices $x_1, \ldots, x_t$ each of which is joined to some vertex of $G(m)$ not in S. $\bar{G}$ is the complementary graph of G i.e. two vertices are joined in $\bar{G}$ if and only if they are not joined in G.

Put $n = k+t-1$. Then

$$\binom{n}{2} - f(n, k) = \binom{t}{2} - \left[\frac{t+1}{2}\right].$$

Now a simple argument shows that the fact that $G(n; f(n, k))$ does not have property P_k is equivalent to $\bar{G}(n; f(n, k)) = G\left(\binom{t}{2} - [(t+1)/2]\right)$ having property θ_{t-1}. Thus Theorem 1 is equivalent to the following

THEOREM 2. *Every $G\left(\binom{t}{2} - [(t+1)/2]\right)$ has property θ_{t-1} except if it is a $\bar{G}_n^{(0)}$.*

Clearly our $\bar{G}_n^{(0)}$ is a $G\left(t, \binom{t}{2} - [(t+1)/2]\right)$ where the missing $[(t+1)/2]$ edges are as disjoint as possible.

Theorem 2 is vacuous for $t < 2$ and trivial for $t \leqq 3$. Henceforth assume $t \geqq 4$.

To prove Theorem 2 let $G\left(\binom{t}{2} - [(t-1)/2]\right) = G$ be any graphs which does not have property θ_{t-1}. We will show that it must be a $\bar{G}_n^{(0)}$. First of all we can assume that all vertices of our G have valency $\leqq t-2$. For if not then say x_1 is joined to $y_1, \ldots, y_{t-1}$ which shows that G has property θ_{t-1} which contradicts our assumption.

Assume next that G has a vertex x of valency $t-2$ (this will be the critical case). Denote by $y_1, \ldots, y_{t-2}$ the vertices joined to x and let $z_1, \ldots$ be the other vertices of G. Clearly no two z's can be joined. For if (z_1, z_2) would be an edge of G then $z_1, y_1, \ldots, y_{t-2}$ are $t-1$ vertices each of them are joined to a vertex not in the set, or G has property θ_{t-1}. Also no y can be joined to two z's. For if y_1 is joined to z_1 and z_2 then the $t-1$ vertices $z_1, z_2, y_2, \ldots, y_{t-2}$ would show that G has property θ_{t-1}.

Next we show that at least $t-3$ y's are joined to some z (as we know each y can be joined to at most one z). Assume that u y's are joined to some $z (u < t-3)$. Clearly ($v(G)$ denotes the number of edges of G)

$$(4) \qquad v(G) = \binom{t}{2} - \left[\frac{t+1}{2}\right] = u + \binom{t-1}{2} - N \text{ or } u - N = \left[\frac{t}{2}\right] - 1,$$

where N is the number of the edges of the complete graph spanned by $y_1, \ldots, y_{t-2}$ which do not occur in G. Now clearly

$$(5) \qquad N \geqq \left[\frac{u+1}{2}\right]$$

since a y joined to a z cannot be joined to all the other y's (since otherwise lts valency would be $t-1$), hence a missing edge (i.e. an edge not in G) is incident to every y which is joined to a z and this proves (5). From (4) and (5) we have

$$(6) \qquad \left[\frac{u}{2}\right] \geqq \left[\frac{t}{2}\right] - 1$$

(6) clearly implies $u \geqq t-3$ as stated.

Hence either $u = t-3$ or $u = t-2$. (4) and $u \leqq t-2$ implies that we must have equality in (5) i.e. $N = [(u+1)/2]$.

First we prove Theorem 2 if $u = t-3$. (6) implies that if $u = t-3$, t is odd and since $N = [(u+1)/2] + [u/2] = [(t-2)/2]$ and every y which is joined to a z must be adjacent to a missing edge we obtain that the $[u/2]$ missing edges must be isolated. In other words we can assume that our G contains all the edges of the complete graph spanned, by $x, y_1, \ldots, y_{t-2}$ with the exception of the edges (y_{2i}, y_{2i+1}), $i = 1, \ldots, [(t-2)/2]$. Further every y_i, $i = 2, \ldots, t-2$ is joined to exactly one z. If all these z's coincide then G is spanned by $x, y_1, \ldots, y_{t-2}, z$ and is clearly our $\bar{G}_n^{(0)}$ and Theorem 2 is proved in this case.

To complete our proof of the case $u = t-3$ assume that y_1 is joined to z_i and y_j to z_j, $(z_i \neq z_j)$, $2 \leqq i < j \leqq t-2$. But then the $t-1$ vertices $x, z_i, z_j, \{y_l\}$ $1 \leqq l \leqq t-2, l \neq i, l \neq j$ show that our G has property θ_{t-1} (x and z_i are joined to y_i, z_j is joined to y_j and every other y_l $l \neq i, l \neq j$ is joined to y_i or y_j [since the missing edges were isolated]). This contradiction completes the proof of Theorem 2 if $u = t-3$.

Assume next $u = t-2$. Then each y is incident to at least one missing edge and since the number of missing edges is $[(u+1)/2] = [(t-1)/2]$ we obtain that for even t there are $(t-2)/2$ isolated missing edges. Just as in the case $u = t-3$ we see that all the $t-2$ y's must be joined to the same z. But then we again obtain our $\bar{G}_n^{(0)}$. This disposes of the case $u = t-2$, t even.

Assume next $u = t-2$, t odd. These are $[(t-1)/2]$ missing edges and since each y is incident to one of them we can assume without loss of generality that the missing edges are (y_1, y_2), (y_1, y_3), (y_{2l}, y_{2l+1}), $l = 2, \ldots, [(t-2)/2]$. If all the y's are joined to the same z we again get our $\bar{G}_n^{(0)}$. Thus we can assume that not all the y's are joined to the same z. Now to complete our proof we have to distinguish two cases. Assume first that there is a z say z_i which is joined to only one y say y_i. This case can immediately be disposed of since the set of $t-1$ vertices x, z_i, $\{y_l\}$, $1 \leqq l \leqq t-2, l \neq i$ shows that our G has property θ_{t-1} (x and z_i are joined to y_i and all other y's are by our assumption joined to a z different from z_i). This contradiction proves Theorem 2 in this case.

Assume finally that every z is joined to more than one y and there are at least two z's. Let, say, z_1 be joined to y_i and y_j and z_2 to y_r. Observe now that either every y is joined in G to one of the two vertices y_i and y_r or every y is joined to one of the two vertices y_j and y_r (this follows from the fact that the missing edges are either isolated or have at most one vertex of valency two). Assume thus that every y is joined either to y_i or to y_r. But then the set of $t-1$ vertices $x, z_1, z_2, \{y_l\}$, $1 \leqq l \leqq t-2, l \neq i, l \neq r$ show that our G has property θ_{t-1} (x and z_1 are joined to y_i, y_2 to y_r and every y_l, $l \neq i, l \neq r$ is joined either to y_i or y_r). This contradiction completes the proof of Theorem 2 if G has a vertex of valency $\geqq t-2$.

Assume now that all vertices of $G = G\left(\binom{t}{2} - [(t+1)/2]\right)$ have valency $< t-2$. We will show by induction with respect to t that then our G must have property θ_{t-1} and this will complete the proof of Theorems 2 and 1 and also of (1).

Assume that the maximum valency of a vertex of our G is $r < t-2$. Let x be joined to $y_1, \ldots, y_r$. Denote as before by $z_1, \ldots$ the other vertices of G and let u be the largest number of z's joined to a y. Assume that y_1 is joined to $z_1, \ldots, z_u$. We evidently have

$$u \leqq \min (t-r-1, r-1). \tag{7}$$

To prove (7) observe that $u \geqq r$ would imply $v(y_1) > r$ and $u \geqq t-r$ would imply that G satisfies θ_{t-1} (consider the vertices $y_2, \ldots, y_r, z_1, \ldots, z_u$).

Denote by u_i the number of z's joined to y_i $(u_1 = u)$ and by w_i the number of y's joined to y_i. By (7) $v(y_i) = 1+u_i+w_i \leqq r-1$. Thus by (7) the number E of edges incident to the vertices $x, y_1, \ldots, y_r$ equals

$$E = r + \sum_{i=1}^{r} (u_i + \tfrac{1}{2}w_i) \leqq r(u+1) + \frac{r(r-u-1)}{2} = \frac{r(r+u+1)}{2} \leqq r^2. \tag{8}$$

(8) follows from the fact that E is evidently maximal if all the u_i are $u = r-1$ (i.e. they are all as large as possible) and if $w_i = r-u-1 = 0$. From (7) we have (G_1 is the graph spanned by the z's)

$$v(G_1) \geqq \binom{t}{2} - \left[\frac{t+1}{2}\right] - r^2. \tag{9}$$

Assume first $r \leqq t/2$. Then we obtain from (9)

$$v(G_1) > \binom{t-r}{2} - \left[\frac{t-r+1}{2}\right]. \tag{10}$$

Hence by our induction assumption G_1 has property θ_{t-r-1} i.e. it contains a set of vertices $z_1, \ldots, z_{t-r-1}$ each of which is joined to some z_j, $j > t-r-1$. But then the $t-1$ vertices $z_1, \ldots, z_{t-r-1}, y_1, \ldots, y_r$ show that G has property θ_{t-1}, which proves Theorem 2 if $r \leqq t/2$.

Assume next $t/2 < r \leqq t-3$. From (7) we have $u_i \leqq t-r-1$ and by (8) E is maximal if all the u_i are $t-r-1$ and $w_i = r-1-u_i = 2r-t$. But then by (8)

$$E \leqq r + r(t-r-1) + \frac{r}{2}(2r-t) = \frac{rt}{2}. \tag{11}$$

From (11) we have

$$v(G_1) \geqq \binom{t}{2} - \left[\frac{t+1}{2}\right] - \frac{rt}{2} > \binom{t-r}{2} - \left[\frac{t-r+1}{2}\right].$$

Thus the proof can be completed as in the previous case, and the proof of Theorem 2 is complete.

Denote by $f_0(n, k, r)$ the smallest integer for which there is a $G(n; f_0(n, k, r))$ in which every set of k vertices are visible from at least r vertices. We say that a graph has property $P_{k,r}$ if every set of k of its vertices is visible from at least r vertices. Just as in our Lemma we can show that if G_n has property $P_{k,r}$ then every pair of its vertices is visible from at least $k+r-2$ vertices (our old property P_k is $P_{k,1}$).

Thus we obtain as in (2) that if G_n has property $P_{k,r}$ then if $k > 1$

$$\sum_{i=1}^{n} \binom{v_i}{2} \geqq (k+r-2)\binom{n}{2}. \tag{2'}$$

From (2') we can deduce that if $n < n_0(k, r)$ then

$$f_0(n, k, r) = f_0(n, k+r-1) = f(n, k+r-1). \tag{12}$$

(12) certainly does not hold for every n, k and r. It is easy to see that $f_0(10, 2, 6) = 40$ but $f(10, 7) = 41$. Our Theorem 1 states that (12) always holds for $r = 1$ and perhaps it always holds for $r = 2$ also if $k > 1$. For $k = 1$ every G_n each vertex of which has valency $\geqq r$ clearly has property $P_{1,r}$, thus $f_0(n, 1, r) = [(rn+1)/2]$, in other words if $k = 1$, $r > 1$ then (12) is not true. We hope to return to these questions on another occasion.

Finally we can ask the following question: Denote by $F(n, k)$ the smallest integer for which there exists a directed graph $G(n; F(n, k))$ so that to every k vertices $x_1, \ldots, x_k$ of our G there is a vertex y of G so that all the edges (y, x_i) $i = 1, \ldots, k$ occur in G and are directed away from y. It is easy to see that for $n \geqq 3$, $F(n, 1) = n$ (for $n \leqq 2$ there clearly is no solution). It is not hard to show that for $n \geqq 7$, $F(n, 2) = 3n$ and for $n < 7$ there is no solution. For $k \geqq 3$, we only have crude inequalities for $F(n, k)$. We say that G_n has property S_k (after Schütte who posed the problem) if for every set of k nodes $(x_1, \ldots, x_k)$ there is at least one node y in G_n so that all the edges (y, x_i), $i = 1, \ldots, k$ occur in G and are directed away from y. Denote by $f(k)$ the smallest value of n for which an S_k-graph of n vertices exists. We have

$$(k-1)2^k+3 \leqq f(k) < ck^2 2^k. \tag{13}$$

(13) is due to P. Erdös, E. Szekeres and G. Szekeres (Math. Gazette 47 p. 220 and 49 p. 290). We can show that for $n > n_0(k)$

$$f(k-1) \cdot n \leqq F(n, k) \leqq f(k) \cdot n. \tag{14}$$

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