On Some Applications of Graph Theory, II

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1. In the first paper of this sequence we dealt with applications of graphs in the Michigan sense; in this paper we are dealing with m-graphs, i.e. structures in which the fundamental elements beside the vertices are not edges $P_\mu P_\nu$ ($1 \leqslant \mu < \nu \leqslant n$) but m-tuples

$$P_{\mu_1} P_{\mu_2} \dots P_{\mu_m} \qquad (= \text{'}m\text{-edges'}),$$
$$1 \leqslant \mu_1 < \mu_2 < \dots < \mu_m \leqslant n.$$

For $m = 2$ we get the graphs. n is again the order of the m-graph; the meaning of m-subgraphs, complete m-graphs is obvious. We shall deal especially with the case $m = 3$, i.e. with the case of trigraphs, though the geometrical problems we are dealing with have natural analogues (and also problems of new type) for higher dimensions, and these involve m-graphs with general m's.

Let A_1, A_2, A_3 be three distinct points of the plane and let

$$f(A_1, A_2, A_3), \qquad g(A_1, A_2, A_3)$$

two non-negative triangle functions which are continuous functions of the vertices. Such choices of f and g are e.g.

$$f = \text{perimeter of the } \Delta A_1 A_2 A_3$$

$$g = \text{radius of the inscribed circle of } \Delta A_1 A_2 A_3 \tag{1.1}$$

or

$$f = g = \text{area of } \Delta A_1 A_2 A_3 \tag{1.2}$$

or

$$f = g = \text{perimeter of } \Delta A_1 A_2 A_3. \tag{1.3}$$

Let D be a constant such that

$$f(A_1, A_2, A_3) \leqslant 1 \text{ implies } g(A_1, A_2, A_3) \leqslant D; \tag{1.4}$$

here $D = +\infty$ is also admitted. We shall deal with finite sets of points $A_1, A_2, \ldots, A_k$ in the plane subjected to the restriction

$$\max f(A_{i_1}, A_{i_2}, A_{i_3}) = 1 \tag{1.5}$$

where the max refers to

$$1 \leqslant i_1 < i_2 < i_3 \leqslant k;$$

we shall call them normalized point sets. Suppose that, for a certain integer $l \geqslant 3$, there is constant $D_l < D$ such that for all normalized point sets

$$(A_1, A_2, \ldots, A_l)$$

$$\min g(A_{i_1}, A_{i_2}, A_{i_3}) \leqslant D_l \tag{1.6}$$

(where the min refers to $1 \leqslant i_1 < i_2 < i_3 \leqslant l$) holds. Then we are going to prove the

THEOREM I. *If $n \geqslant l$, then for all normalized point sets $(P_1, P_2, \ldots, P_n)$ at least*

$$\binom{l}{3}^{-1} \binom{n}{3}$$

triangles $P_{i_1} P_{i_2} P_{i_3}$ $(1 \leqslant i_1 < i_2 < i_3 \leqslant n)$ satisfy the inequality

$$g(P_{i_1}, P_{i_2}, P_{i_3}) \leqslant D_l. \tag{1.7}$$

In other words, in all normalized point sets $(P_1, P_2, \ldots, P_n)$, at least $\binom{l}{3}^{-1}$th part of all $P_{i_1} P_{i_2} P_{i_3}$ triangles satisfies (1.7).

In very general cases (to which (1.1), (1.2) and (1.3) belong)

$$\max \min_{1 \leqslant i_1 < i_2 < i_3 \leqslant l} g(A_{i_1}, A_{i_2}, A_{i_3})$$

exists, even for all $l \geqslant 3$; here the maximum refers of course to all normalized systems of l points in the plane. These can be called—by analogy with paper I of this sequence of papers or with the lecture of one of us in Proceedings† of the Combinatorial Colloquium in Calgary held June 1–14, 1969—the 'packing constants' δ_l of the problem; in such cases D_l in (1.7) can be replaced of course by δ_l. As it was shown l.c. in the case of problems concerning distances, the packing constants defined *there* lead to several *best possible* inequalities. This is not so much the case with triangle problems though, as will be indicated in some cases later, our graph–theoretical method can also lead to best possible results here. Nevertheless this method is perhaps the only one available at

† In course of publication.

present which yields estimates in *general* problems of the combinatorial geometry of triangles; and it is possible that the use of more appropriate theorems on trigraphs will considerably improve present results. If physicists should ever introduce potentials depending on the interaction of three (not two) particles, then results of the above type would gain an additional interest.

In the case of $l = 5$ the value $\binom{l}{3}^{-1}$ is $\frac{1}{10}$. A better result in this case is given by the

THEOREM II. *Suppose the inequality*

$$\min_{1 \leqslant i_1 < i_2 < i_3 \leqslant 5} g(A_{i_1}, A_{i_2}, A_{i_3}) \leqslant D_5 (< D) \tag{1.8}$$

holds for all normalized point sets $(A_1, A_2, A_3, A_4, A_5)$. *Then for all* $n \geqslant 7$ *and normalized point sets* $(P_1, P_2, \ldots, P_n)$ *at least*

$$\frac{1}{7}\binom{n}{3}$$

triangles $P_{i_1} P_{i_2} P_{i_3}$ $(1 \leqslant i_1 < i_2 < i_3 \leqslant n)$ *satisfy the inequality*

$$g(P_{i_1}, P_{i_2}, P_{i_3}) \leqslant D_5. \tag{1.9}$$

In other words, for $n \geqslant 7$ the inequality (1.9) holds with a probability $\geqslant \frac{1}{7}$ in all normalized point sets $(P_1, P_2, \ldots, P_n)$.

2. In order to give Theorem II effective geometrical applications, let us consider first the case (1.2). Then $\delta_3 = 1$ and choosing the four vertices of a square, we obtain $\delta_4 = 1$. As we shall see in the Appendix

$$\delta_5 = \frac{\sqrt{5} - 1}{2}; \tag{2.1}$$

thus Theorem II gives the

COROLLARY I. *Having on the plane* $n \geqslant 7$ *points* $P_1, \ldots, P_n$ *so that the maximal area of all* $P_{i_1} P_{i_2} P_{i_3}$ *triangles is* 1, *then at least* $\frac{1}{7}\binom{n}{3}$ *such triangles have an area* $\leqslant \dfrac{\sqrt{5} - 1}{2}$.

In other words if, for $n \geqslant 7$, the maximal area of

$$\Delta P_{i_1} P_{i_2} P_{i_3} \qquad (1 \leqslant_{i_1} <_{i_2} <_{i_3} \leqslant_n)$$

is 1, then the inequality

$$\text{area } \Delta P_{i_1} P_{i_2} P_{i_3} \leqslant \frac{\sqrt{5}-1}{2} \tag{2.2}$$

holds with a probability $\geqslant \frac{1}{7}$. Since this does not depend on n, a trivial passage to the limit gives the corresponding theorem for bounded closed measurable point sets in the plane with a plausible interpretation of the probability.

Putting m points 'near' to each vertices of a square (m large) we see at once (with $n = 4m$) that in the Corollary I the constant $\frac{1}{7}$ certainly cannot be replaced by any constant $> \frac{5}{8}$; the same example shows that the 'bad' triangles, ($4m^3$ in number, with area $>(\sqrt{5}-1)/2$) are actually 'very bad', i.e. with area $> 1-\varepsilon$. We conjectured that this is best possible in the following sense. There exists a constant

$$0 < \theta_1 < 1$$

with the following property. Having $n = 4m$ points $P_1, \ldots, P_n$ on the plane such that

$$\max_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} \text{area } \Delta P_{i_1} P_{i_2} P_{i_3} = 1,$$

then, taking any $(4m^3 + 1)$ triangles $P_{i_1} P_{i_2} P_{i_3}$ out of these, one of them at least has an area

$$\leqslant \theta_1. \tag{2.3}$$

This special problem was solved by a special argument by B. Bollobás;† he did not specify the value of θ_1 (though this would be of interest).

On putting m points near to each vertex of a regular hexagon (m large) one can see after a little reflection that in the Corollary I the constant $\frac{1}{7}$ cannot be replaced by $\frac{11}{18}$. In order to push the constant down from $\frac{11}{18}$ it would be reasonable to study the distribution of triangle areas generated by the point system of a regular n-gon.

What does Theorem II give in the case (1.3)? Whereas in this case we have again $\delta_3 = \delta_4 = 1$, we can only show that $\delta_5 < 1$. Hence this theorem gives the

COROLLARY II. *There is a constant* $\delta^* < 1$ *such that, if the points* $P_1, P_2, \ldots, P_n$ $(n \geqslant 7)$ *have the property*

$$\max_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} \text{perimeter of } \Delta P_{i_1} P_{i_2} P_{i_3} = 1,$$

then at least $\dfrac{1}{7}\dbinom{n}{3}$ *triangles have perimeter* $\leqslant \delta^*$.

† Oral communication.

It would be of interest to determine δ^*; probably it is

$$2\,\frac{1+\cos\frac{1}{5}\pi}{1+4\cos\frac{1}{5}\pi}\sim 0{\cdot}85 \tag{2.4}$$

which occurs in the case of a regular pentagon. Again, as before, Corollary II implies that we may assert with a probability $\geqslant \frac{1}{7}$ that the perimeter of a random triangle with vertices in a given bounded closed measurable point set does not exceed the δ^*-th part of the maximal possible perimeter value of such triangles.

3. Concerning possible improvements of our theorem, we make the following observation. The applications of graph theory in the first paper were partly based on the following theorem (see (**1**)). If $3 \leqslant l \leqslant n$ and

$$n \equiv h \bmod (l-1), \qquad 0 \leqslant h \leqslant l-2, \tag{3.1}$$

then the maximal number of edges in a graph of order n and not containing complete subgraphs of order l is

$$\frac{l-2}{2(l-1)}\,(n^2-h^2)+\binom{h}{2}, \tag{3.2}$$

equality being attained if and only if the vertices are distributed into $(l-1)$ disjoint classes 'possibly uniformly' so that two vertices are connected by an edge if and only if they belong to different classes. The problem to generalize the theorem (3.1)–(3.2) to m-graphs was already raised in the same paper; the partial reason for the imperfection of our theorems is the fact that such a generalization does not exist up to now for $3 \leqslant m < l \leqslant n$, even asymptotically. A conjecture in this direction asserts for the case

$$(m-1)/(l-1) \tag{3.3}$$

that an 'extremal m-graph' can be obtained by distributing the vertices into

$$\frac{l-1}{m-1} \tag{3.4}$$

disjoint classes 'possibly uniformly' and taking all m-edges with *not all* vertices in the same class. If this were true our proof of Theorem I would for $\varepsilon > 0$ and $n > n_0(\varepsilon)$ yield the quantity

$$\left(\frac{2}{l-1}\right)^2-\varepsilon \tag{3.5}$$

in the place of $\binom{l}{3}^{-1}$. Hence the constant $\frac{1}{7}$ would be replaced by $(\frac{1}{4}-\varepsilon)$ in

Corollaries I and II. However, in the last case if A and B are two points with $\overline{AB} = \frac{1}{2}$ and if k points (k large) are taken close to A and B, then the maximal perimeter of the triangles is 1 while the number of such triangles whose perimeter is less than δ^* is

$$2\binom{k}{3}.$$

Moreover

$$\frac{2\binom{k}{3}}{\binom{2k}{3}} \to \tfrac{1}{4} \qquad \text{for } k \to \infty.$$

Hence if the m-graph conjecture (3.3)–(3.4)–(3.5) is correct then the graph–theoretical approach can lead to results, which are in a sense best possible in triangle problems too. As to this particular problem, B. Bollobás† succeeded in proving by a special method without conjectures that there is a constant $\delta^{**} < 1$ such that for arbitrarily small $\varepsilon > 0$ and $n > n_0(\varepsilon)$ having n points $P_1, P_2, \ldots, P_n$ on the plane such that

$$\max_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} \text{perimeter of } \Delta P_{i_1} P_{i_2} P_{i_3} = 1,$$

at least $(\frac{1}{4} - \varepsilon)\binom{n}{3}$ of these triangles have a perimeter $\leqslant \delta^{**}$. Again, nobody knows at present which one is greater, δ^* or δ^{**}.

Not having the theorem corresponding to (3.1)–(3.2), we could use only the strongest existing theorem in the required direction due to Gy. Katona, T. Nemetz, M. Simonovits (see (**2**)) according to which in an m-graph of order n the existence of more than

$$\left\{1 - \frac{1}{\binom{l}{m}}\right\}\binom{n}{m} \tag{3.6}$$

m-edges implies the existence of a complete m-subgraph of order l if only $3 \leqslant m < l \leqslant n$. But for $m = 3, l = 5$ we shall need a slightly stronger result; we shall use the

LEMMA. *If in a trigraph of order $n \geqslant 7$ we have more than*

$$\frac{6}{7}\binom{n}{3} \tag{3.7}$$

triedges, then the trigraph contains a complete subtrigraph of order 5.

†Oral communication.

4. We shall prove only Theorem I; the proof of Theorem II goes through analogously with (3.7) in place of (3.6).

Let $l \geqslant 3$ be fixed and let our (normalized) point set P consist of the points $P_1, P_2, \ldots, P_n$ in the plane ($n \geqslant l$). We attach to this point set P the trigraph G with the vertices $P_1', P_2', \ldots, P_n'$ so that the triedge $P_{i_1}' P_{i_2}' P_{i_3}'$ occurs in G if and only if

$$g(P_{i_1}', P_{i_2}', P_{i_3}') > D_l. \tag{4.1}$$

Suppose that the number of our triangles with the property (4.1) is greater than

$$\left\{1 - \frac{1}{\binom{l}{3}}\right\} \binom{n}{3} \overset{\text{def}}{=} U. \tag{4.2}$$

Then the trigraph G would contain more than U triedges and hence, owing to (3.6), it would contain a complete trisubgraph of order l. But this means again the existence of l points in P such that *all* triangles from these points satisfy (4.1). But this contradicts (1.6). Hence (4.1) holds for U triangles at most, and thus the inequality (1.7) holds for at least

$$\frac{1}{\binom{l}{3}} \binom{n}{3}.$$

triangles.

5. We have to prove the lemma. Denoting the maximal number of triedges in a trigraph of order n not containing a complete trisubgraph of order 5 by $N(n)$, we see at once that $N(6) \leqslant 18$ (since if the triedge $P_4 P_5 P_6$ were the *only* missing one then $(P_1, P_2, P_3, P_4, P_5)$ would be a complete pentagon). The inequality $N(6) \geqslant 18$ is clear by removing from the complete trigraph of order 6 the triedges $P_1 P_2 P_3$ and $P_4 P_5 P_6$ and evidently this is the *only* type of trigraphs of order 6 with $N(6) = 18$. As to $N(7)$, we assert that

$$N(7) = 30. \tag{5.1}$$

That $N(7) \geqslant 30$ is again evident by considering the trigraph obtained by omitting from the complete trigraph of order 7 all triedges from (P_1, P_2, P_3, P_4) and the triedge (P_5, P_6, P_7). Let N_i stand for the number of triangles in the trigraph obtained from our given graph of order 7 by omitting P_i. Then from $N(6) = 18$ we have on one hand

$$\sum_{i=1}^{7} N_i \leqslant 7 \cdot 18 = 126 \tag{5.2}$$

and on the other hand

$$\sum_{i=1}^{7} N_i = 4N(7) \tag{5.3}$$

since any fixed triedge occurs in exactly four N_i's. Hence

$$N(7) \leqslant 31$$

and, if $N(7) = 31$, we have

$$\sum_{i=1}^{7} N_i = 124,$$

i.e.

$$N_7 \overset{\text{def}}{=} \max_i N_i \geqslant 18$$

and hence

$$N_7 = 18. \tag{5.4}$$

But then the structure of N_7 is given above and the number of triedges containing P_7 as a vertex is 13, i.e. from the $\binom{6}{2} = 15$ triedges only two are missing. This gives altogether six configurations and one can check easily that each contains a complete trisubgraph of order 5 which is a contradiction, and hence (5.1) is true. But, as remarked in (**2**),

$$N(n) \Big/ \binom{n}{3}$$

is monotonically non-increasing; hence we have for $n \geqslant 7$

$$\frac{N(n)}{\binom{n}{3}} \leqslant \frac{N(7)}{\binom{7}{3}} = \frac{6}{7},$$

as required.

Appendix

6. We shall show that if P_1, P_2, P_3, P_4 and P_5 are 5 points in the plane so that

$$\max_{1 \leqslant i_1 < i_2 < i_3 \leqslant 5} \text{area } \Delta P_{i_1} P_{i_2} P_{i_3} = 1,$$

then

$$V \overset{\text{def}}{=} \min_{1 \leqslant i_1 < i_2 < i_3 \leqslant 5} \text{area } \Delta P_{i_1} P_{i_2} P_{i_3} \leqslant \frac{\sqrt{5}-1}{2}. \tag{6.1}$$

This inequality is best possible as is shown by the regular pentagon. (Somewhat longer proof would also show that *all* cases of equality are given by affine regular pentagons but we shall not go into details of this.)

Case I. The smallest convex polygon K of the P_ν's has at most four vertices. If K is a triangle, we have at once $V \leqslant \frac{1}{5}$; if K is a quadrilateral then the area of K is $\leqslant 2$ and hence

$$V \leqslant \tfrac{2}{4} = \tfrac{1}{2} < \frac{\sqrt{5}-1}{2}.$$

Hence we may suppose that our five points form a convex pentagon. Let $\Delta P_1 P_2 P_3$ be one of its triangles with

$$\text{area } \Delta P_1 P_2 P_3 = 1. \tag{6.2}$$

Case II. Two sides of $\Delta P_1 P_2 P_3$ *are on the perimeter of the convex polygon.*

Let the point Q be such that

$$P_3 Q \parallel P_1 P_2 \qquad P_2 Q \parallel P_1 P_3 \tag{6.3}$$

i.e.

$$\text{area } \Delta P_2 P_3 Q = 1.$$

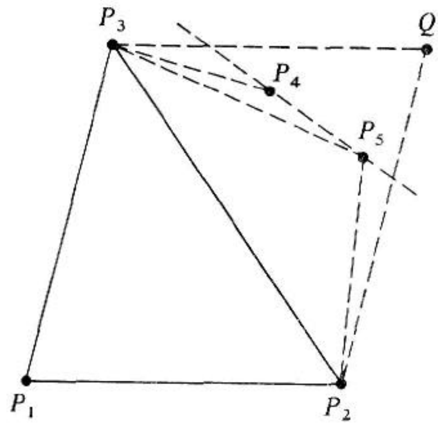

FIGURE 1

Owing to the maximality of $\Delta P_1 P_2 P_3$ both points P_4 and P_5 are in $\Delta P_2 P_3 Q$ and owing to the convexity of $P_1 P_2 P_3 P_4 P_5$ the line $P_4 P_5$ does meet within the triangle $P_2 P_3 Q$ the sides $P_2 Q$ and $P_3 Q$ only (see Fig. 1). Since the triangles $P_2 P_3 P_5$ and $P_3 P_4 P_5$ are disjoint and the sum of their areas is $\leqslant 1$ owing to (6.3), we have in this case

$$V \leqslant \tfrac{1}{2} < \frac{\sqrt{5}-1}{2}.$$

Case III. Only one side $\Delta P_1 P_2 P_3$ is on the smallest convex polygon; without loss of generality let this be $P_1 P_2$.

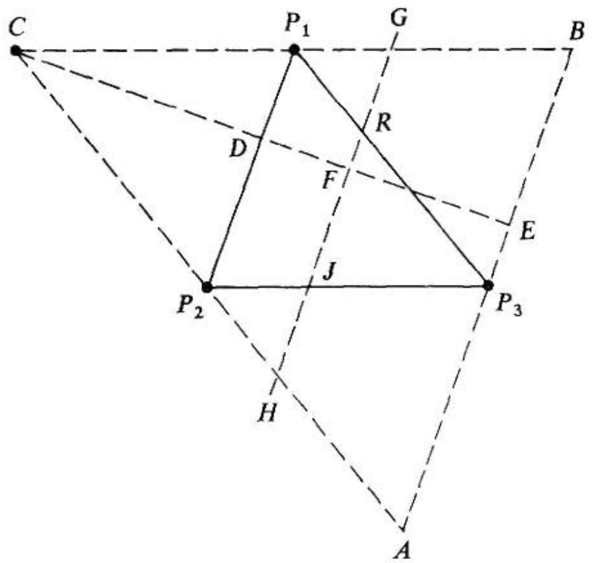

FIGURE 2

Let (see Fig. 2)

$$P_1 B \,||\, P_2 P_3 \qquad P_2 A \,||\, P_1 P_3$$

i.e.

$$\overline{AP_3} = \overline{P_3 B}.$$

Owing to (6.2) and the definition of Case III

$$P_4 \in \Delta P_1 P_3 B, \qquad P_5 \in \Delta P_2 P_3 A. \tag{6.4}$$

Let C be the point of intersection of $P_1 B$ and $P_2 A$ further

$$CD \perp P_1 P_2, \qquad \overline{CD} = \tfrac{1}{2}\overline{CE} = m. \tag{6.5}$$

Let $0 < \theta < 1$ to be determined later and F be such that

$$\overline{DF} = \theta m \qquad \text{and} \qquad GF \,||\, P_1 P_2. \tag{6.6}$$

Let finally the points R, J and H be defined as the points of intersection of GF with $P_1 P_3, P_2 P_3$ and CA respectively. If P_4 is between the parallels $P_1 P_2$ and GF, then

$$\begin{aligned} \text{area } \Delta P_1 P_2 P_4 &\leqslant \text{area } \Delta P_1 P_2 G \\ &= \theta \text{ area } \Delta P_1 P_2 P_3 = \theta \end{aligned} \tag{6.7}$$

and analogously with P_5 instead of P_4. Hence we have to investigate only the case when

$$P_4 \in BGRP_3 \quad \text{and} \quad P_5 \in P_3\, JHA. \tag{6.8}$$

In this case let us investigate the area of $\Delta P_3 P_4 P_5$. If P_4 is not on the broken line BGR then, shifting it along $P_3 P_4$, the area of $\Delta P_3 P_4 P_5$ is increased; and analogously for P_5; hence we may suppose that

$$\begin{aligned} &P_4 \text{ is on the broken line } BGR \\ &P_5 \text{ is on the broken line } AHJ. \end{aligned} \tag{6.9}$$

Suppose that P_4 is on the segment BG, but not at G. We move P_4 parallel to $P_3 P_5$ to a position P_4' inside $BGRP_3$; this process preserves the area of $\Delta P_3 P_4 P_5$. P_4' may be chosen to lie on $P_3\, G$. If, next, P_4 is moved along $P_3\, G$ from P_4' to G, the area of $\Delta P_3 P_4 P_5$ is increased. Performing an analogous operation on P_5, we see that

$$\text{area } \Delta P_3 P_4 P_5 \leqslant \text{area } \Delta P_3 GH. \tag{6.10}$$

But

$$\overline{GH} = (1+\theta)\,\overline{P_1 P_2}, \qquad \overline{FE} = (1-\theta)\,\overline{DE}$$

and thus from (6.2)

$$\text{area } \Delta P_3 GH = (1-\theta^2) \quad \text{area } \Delta P_1 P_2 P_3 = 1-\theta^2. \tag{6.11}$$

Hence, in the Case *III*, (6.7) and (6.10)—(6.11) give

$$V \leqslant \max(\theta, 1-\theta^2)$$

for all $0 < \theta < 1$. Choosing θ so that

$$\theta = 1 - \theta^2,$$

(6.1) is proved.

References

1. Turán, P. Egy gráfelméleti szélsöértékfeladatról. *Matematikai Lapok* **48** (1941), 436–452 (With German abstract).

2. Katona, Gy, Nemetz, T. and Simonovits, M. Egy Turán-féle gráfproblémáról. *Matematikai Lapok* **15** (1964), 228–238 (With Russian and English abstracts).