PROBLEMS AND RESULTS IN COMBINATORIAL ANALYSIS

P. ERDÖS

This review of some solved and unsolved problems in combinatorial analysis will be highly subjective. I will only discuss problems which I either worked on or at least thought about. The disadvantages of such an approach are obvious, but the disadvantages are perhaps counterbalanced by the fact that I certainly know more about these problems than about others (which perhaps are more important). I will mainly discuss finite combinatorial problems. I cannot claim completeness in any way but will try to refer to the literature in some cases; even so many things will be omitted. $|S|$ will denote the cardinal number of S; $c, c_1, c_2, \ldots$ will denote absolute constants not necessarily the same at each occurrence.

I. I will start with *some problems dealing with subsets of a set*. Let $|S|=n$. A well-known theorem of Sperner [57] states that if $A_i \subset S$, $1 \leq i \leq m$, is such that no A_i contains any other, then max $m = \binom{n}{[n/2]}$. The theorem of Sperner has many applications in number theory; as far as I know these were first noticed by Behrend [2] and myself [8].

I asked 30 years ago several further extremal problems about subsets which also have number theoretic consequences. Let $A_i \subset S$, $1 \leq i \leq m_1$, assume that there are no three distinct A's so that $A_i \cup A_j = A_r$. I conjectured that

$$\max m_1 = (1+o(1))\binom{n}{[n/2]},$$

but could not even prove max $m_1 = o(2^n)$. This latter result was proved by Sárközi and Szemerédi but was never published because it was superseded by the result of Kleitman [44], who first of all proved that max $m < 2^{3/2}\binom{n}{[n/2]}$ and recently (in this volume) that

$$\max m_1 < \binom{n}{[n/2]}\left(1+c\left(\frac{\log n}{n}\right)^{1/2}\right)$$

which is in fact stronger than my conjecture. It would be of interest to determine max m_1; maybe this question has no simple solution, but perhaps an asymptotic formula for

$$\max m_1 - \binom{n}{[n/2]}$$

is not quite hopeless.

The second problem I asked was: Let $A_i \subset S$, $1 \leq i \leq m_2$. Assume that there are not four distinct A's say A_i, A_r, A_s, A_t satisfying

$$A_i \cup A_r = A_s, \qquad A_i \cap A_r = A_t.$$

Kleitman proved $\max m_2 < c_1 2^n/n^{1/4}$ and I showed that $\max m_2 > c_2 2^n/n^{1/4}$. Presumably

$$\max m_2 = (c+o(1))2^n/n^{1/4}$$

but as far as I know this has not yet been proved. These results are not yet published (they will appear in Proc. Amer. Math. Soc.).

Here I would like to mention a question which goes back to Dedekind: How many families of subsets of S are there where no set of a family contains any other? Denote the number of such families by $f(n)$. There may not be a simple explicit formula for $f(n)$, but Kleitman, sharpening previous results of several authors, proved (not yet published)

$$\log f(n) = (1+o(1))\binom{n}{[n/2]} \log 2.$$

It would be interesting to give an asymptotic formula for $f(n)$ but this is probably rather difficult.

Kleitman proved several other conjectures of mine involving subsets, some of which have not yet been published.

Rota observed that Dilworth's theorem [6] implies Sperner's theorem and many other results in combinatorial analysis.

Ko, Rado and I [26] proved that if $n \geq 2k$, $A_i \subset S$, $|A_i| = k$, $A_i \cap A_j \neq \emptyset$, $1 \leq i < j \leq m$, then

$$\max m = \binom{n-1}{k-1}.$$

Assume now $|A_i \cap A_j| \geq r$. Put $\max m = f(n, k, r)$. We proved that for $n > n_0(k, r)$,

$$f(n, k, r) = \binom{n-r}{k-r}. \tag{1}$$

Min [26] observed (in the same paper) that (1) is not true in general; the determination of $f(n, k, r)$ in general seems to be a difficult problem. We conjectured

$$f(4k, 2k, 2) = \frac{1}{2}\left(\binom{4k}{2k} - \binom{2k}{k}^2\right), \tag{2}$$

but we could not decide whether (2) is true.

We also observed that if $A_i \subset S$, $A_i \cap A_j \neq \varnothing$, $1 \leq i < j \leq m$, then $\max m = 2^{n-1}$. It does not seem to be easy to determine the number of families $A_i \subset S$, $A_i \cap A_j \neq \varnothing$, $1 \leq i < j \leq 2^{n-1}$. We could not even get an asymptotic formula for the number of these families.

Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq k$. What is the smallest value of k so that there should always be three A's any two of which have the same union? This question like some other problems in this chapter has connections with number theory. (More generally we can ask what is the smallest value of $k = k_r$ so that there always are r A's any two of which have the same union.)

II. **Some geometric problems.** Let $z_i \geq 1$, $1 \leq i \leq n$. Consider all the sums $\sum_{i=1}^{n} \varepsilon_i z_i$, $\varepsilon_i = \pm 1$. I [9] proved as an easy application of Sperner's theorem that the number of sums which fall into the interior of an interval of length 2 is at most $\binom{n}{[n/2]}$, with equality if $z_i = 1$, $1 \leq i \leq n$. I conjectured that if the z_i are complex numbers satisfying $|z_i| \geq 1$, then every circle of radius 1 contains at most $\binom{n}{[n/2]}$ sums $\sum_{i=1}^{n} \varepsilon_i z_i$ (this would sharpen a result of Littlewood and Offord); more generally I conjectured that the above result may remain true if the z_i are vectors in Hilbert space or even in a Banach space.

Katona [40] and Kleitman [43] independently and almost simultaneously proved my conjecture in the plane by giving an interesting generalization of Sperner's theorem, and Kleitman [43] also proved that my conjecture holds in k-dimensional space if $n > n_0(k)$, but the general conjecture has not yet been settled. Sárközi, Szemerédi and I have the following conjecture: Let $|z_i| \leq 1$, $1 \leq i \leq n$, then there are at least $c2^n/n$ summands $\sum_{i=1}^{n} \varepsilon_i z_i$, $\varepsilon_i = \pm 1$, which are of absolute value $\leq \sqrt{2}$ (it is easy to see that $\sqrt{2}$ cannot be diminished; let an odd number of z's be 1 and an odd number i). The order of magnitude $c2^n/n$ is easily seen to be best possible if true. Analogous conjectures can easily be made for higher dimensions.

Sárközi and Szemerédi proved that if $-1 \leq z_i \leq 1$ then there are at least $\binom{n}{[n/2]}$ sums $\sum_{i=1}^{n} \varepsilon_i z_i$, $\varepsilon_i = \pm 1$, which are less than 1 in absolute value. It is easy to see that $\binom{n}{[n/2]}$ is best possible.

Sárközi and Szemerédi further observed that our conjecture in the plane is true if the following purely combinatorial result holds: Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq k$, $B_j \subset S$, $1 \leq j \leq l$ (the A's and B's are all distinct). Assume $|A_{i_1} \cap A_{i_2}| \geq 2$, $1 \leq i_1 \leq i_2 \leq k$; $|B_{j_1} \cap B_{j_2}| \geq 2$, $1 \leq j_1 < j_2 \leq l$; $|A_i \cap B_j| \geq 1$, $1 \leq i \leq k$; $1 \leq j \leq l$. Then

$$k + l \leq 2^{n-1} - c2^n/n.$$

Miss E. Klein raised in 1932 the following problem: Let $f(n)$ be the smallest integer with the property that from $f(n)$ points in the plane one can always select vertices of a convex n-gon. Miss Klein proved that $f(4) = 5$, Makai and Turán proved $f(5) = 9$. Szekeres and I [31], [32] proved

$$2^{n-2} < f(n) \leq \binom{2n-4}{n-2}$$

(our proof of the lower bound contained an inaccuracy which was corrected by Kalbfleisch). It seems likely that $f(n) = 2^{n-2} + 1$ but this is not known for $n \geq 6$.

Let there be given 2^n points in the plane. Szekeres and I [32] proved that these points always determine an angle greater than $\pi(1-1/n)$, an earlier result of Szekeres [58] states that to every ε there are 2^n points so that every angle is less than $\pi(1-1/n)+\varepsilon$. Thus for 2^n points the problem of minimizing the maximum of the greatest angle is completely solved. It is not impossible that if $n>n_0$ then already $2^{n-1}+1$ points always determine an angle $>\pi(1-1/n)$, but we only proved that 2^n-1 points always determine an angle $\geq\pi(1-1/n)$. In higher dimensions sharp results are known only for special values of n, thus Danzer and Grünbaum [5] proved that 2^n+1 points in n-dimensional space always determine an angle $>\pi/2$.

Sylvester conjectured and Gallai first proved that if we have n points in the plane not all on a line then there is at least one line which goes through exactly two of the points. Denote by $f(n)$ the minimum number of such lines. N. G. de Bruijn and I conjectured that $f(n)\to\infty$ as $n\to\infty$. This was proved by Motzkin [47]. Kelly and Moser [41] proved that $f(n)\geq 3n/7$ and this is best possible for $n=7$. Motzkin conjectured that $f(n)\geq[n/2]$ and showed that for infinitely many n this is best possible.

Let there be given n points not all on a line, I observed that it easily follows from Gallai's result that these points determine at least n lines. G. Dirac conjectures that one of the n points is such that it is connected with the other points by more than cn distinct lines.

Assume now that the n points are such that not more than $n-k$ of them are on a line. I conjectured that these points determine at least ckn lines. If k is fixed and $n>n_0(k)$ then Kelly and Moser [41] determined the minimum number of lines which these points determine.

Let there be given n points in the plane, not all on a circle. I conjectured that these points determine always at least $\binom{n-1}{2}$ circles. B. Segre disproved this conjecture for $n=8$, but Elliott [7] proved it for $n>n_0$.

One can pose the following general problem: Let $a_1,\ldots,a_n$ be n elements, $A_1,\ldots,A_t$, $t>1$, be sets whose elements are the a's. Assume $|A_i|\geq r$, $1\leq i\leq t$, and that each r-tuple is contained in precisely one of the A's. Put $\min t=f(n;r)$. Hanani, Szekeres, de Bruijn and I [3] proved that $f(n;2)=n$ and Hanani proved (see Erdős, [16])

$$c_1n^{3/2} < f(n;3) < c_2n^{3/2}.$$

Thus for $r=2$ the combinatorial and geometric problem has the same solution (in the geometric problem the A's are the lines joining the points) but for $r=3$ this is no longer the case (for $r=3$ the r's are circles). The cases $r>3$ have not been investigated.

Further geometric problems and results of a combinatorial nature can be found in [10], [12], [42]. Many very interesting problems on combinatorial geometry are found in the lithographed notes of Croft. Further I would like to refer to two books, Hadwiger and Debrunner [35] and [42].

III. A well-known *theorem of Ramsey* [50] states that if $|S|\geq\aleph_0$ and we split the i-tuples of S into two classes then there is an infinite set all of whose i-tuples are in the same class. Many extensions and generalizations of Ramsey's theorem have been published in the last few years (see my remarks under Ramsey [50]). Here we will only be connected with the finite version of Ramsey's theorem. Denote by $f(i; k, l)$ the smallest integer so that if $|S| = f(i; k, l)$ and we split the i-tuples of S into two classes then there either is a subset of k elements all whose i-tuples are in the first class or a subset of l elements all whose i-tuples are in the second class. Ramsey was the first who obtained upper bounds for $f(i; k, l)$. Szekeres and I proved [31]

$$f(2; k, l) \le \binom{k+l-1}{k-1} \tag{3}$$

and I [11] proved that

$$f(2; k, k) > 2^{k/2}. \tag{4}$$

It would be very nice to prove that

$$\lim_{k=\infty} f(2; k, k)^{1/k}$$

exists and to determine its value.

(4) was proved not by an explicit construction but by a simple probabilistic reasoning. It would be very desirable to obtain a good lower bound for (4) by an explicit construction.

I [15] proved by a more complicated probabilistic reasoning

$$f(2; 3, l) > cl^2/(\log l)^2 \tag{5}$$

and Graver and Yackel [33] recently showed that

$$f(2; k, l) < cl^{k-1} \log \log l/\log l. \tag{6}$$

My method which I used to prove (5) very likely will also give (k fixed, $l \to \infty$)

$$f(2; k, l) > c_1 l^{k-1}/(\log l)^{c_2} \tag{7}$$

but I have not worked out the formidable details.

Very little is known about the exact values of $f(2; k, l)$. Trivially $f(2; 2, l) = l$ and $f(2; k, 2) = k$. Further we have (see Graver and Yackel, [33])

$$f(2; 3, 3) = 6, \quad f(2; 3, 4) = 9, \quad f(2; 3, 5) = 14, \quad f(2; 3, 6) = 18,$$
$$f(2; 3, 7) = 23, \quad f(2; 4, 4) = 18.$$

As far as I know nothing is known about the exact values of $f(i; k, l)$ for $i \ge 3$. Hajnal, Rado and I [25] proved that $f(i; k, l)$ is less than an $(i-1)$-times iterated exponential and greater than an $(i-2)$-times iterated exponential.

We can generalize the Ramsey numbers by division into more than two classes—even less is known about these than about division into two classes (Greenwood and Gleason [34]).

The following question which is related to Ramsey's theorem is perhaps of some interest: Let K_n be the complete graph of n vertices. Denote its vertices by $X_1, \ldots, X_n$. To the edge (X_i, X_j) we make correspond $\varepsilon_{i,j}$ where $\varepsilon_{i,j} = \pm 1$. Put

$$F(n) = \min \max \sum \varepsilon_{i,j}$$

where the maximum is taken over the edges of all the complete subgraphs of K_n and the minimum over all the $2^{C_{n,2}}$ (C is the binomial coefficient) choices of the $\varepsilon_{i,j}$. I can only prove [18]

$$n/4 < F(n) < cn^{3/2}.$$

It would be desirable to obtain better estimates for $F(n)$.

ADDED IN PROOF. J. Spencer and I proved $F(n) > c_1 n^{3/2}$.

IV. Miscellaneous combinatorial problems. Miller [46] in the course of some investigations in set theory introduced the following concept: A family of sets $\{A_\alpha\}$ is said to have property B if there is a set S which has a nonempty intersection with every A_α and does not contain any of the A_α's.

Hajnal and I [23] continued Miller's investigations and also asked the following question about finite sets: What is the smallest integer $m(n)$ for which there is a family of sets $\{A_k\}$, $|A_k| = n$, $1 \le k \le m(n)$, which does not have property B? Trivially $m(2) = 3$. It is not difficult to see that $m(3) = 7$. The value of $m(4)$ is unknown.

Schmidt [55] and I [19] proved

$$2^n(1+4/n)^{-1} < m(n) < n^2 2^{n+1}.$$

It would be of interest to give an asymptotic formula for $m(n)$ and to compute $m(4)$. Perhaps no simple formula for $m(n)$ exists.

Gallai asked: Does there exist a family of sets $\{A_k\}$, $1 \le k \le m_1(n)$, not having property B and satisfying $|A_k| \le n$ and $|A_{k_1} \cap A_{k_2}| \le 1$, $1 \le k_1 \le k_2 \le m_1(n)$? A priori it is not obvious that $m_1(n)$ is finite, but Hajnal and I proved that $m_1(n) < c^n$ for every n [24]. We cannot prove that $\lim_{n \to \infty} m_1(n)^{1/n}$ exists.

Rado and I investigated the following question: A family of sets is called a Δ-system if every two members of the family have the same intersection. Denote by $F(k, l)$ the smallest integer for which if $\{A_i\}$, $1 \le i \le F(k, l)$, is a family of sets each having k elements, then it always contains a subfamily A_{i_r}, $1 \le r \le l$, which is a Δ-system. We proved

$$(l-1)^k < F(k, l) < k!(l-1)^k\left(1 - \frac{1}{2!(l-1)} - \cdots - \frac{k-1}{k!(l-1)^{k-1}}\right).$$

We believe that

$$F(k, l) < c^k l^k \tag{8}$$

holds. (8) would have many number-theoretic applications but is also of great intrinsic interest.

We investigated the problem of Δ-systems also if $k + l \ge \aleph_0$. But this set-theoretical problem is much simpler than the combinatorial one and we have determined $F(k, l)$ in this case [29].

I conjectured that if $a_1, \ldots, a_{2^k}$ is a sequence of length 2^k where a_i is one of the integers $0, 1, \ldots, k-1$ then there are always two consecutive blocks containing each of the integers the same number of times. This is obvious for $k=2$ and de Bruijn proved it for $k=3$. But for $k=4$ de Bruijn and I disproved it. Later Croft constructed a sequence of length 50 for $k=4$ without such consecutive blocks and he suggested that for $k=4$ there probably is an infinite sequence without two such consecutive blocks. With the help of the Atlas computer Churchhouse constructed such a sequence of length about 1700 which gives a strong support to the conjecture of Croft.

Steiner conjectured that if $n=6k+1$ or $6k+3$ then there exists a system of triplets of n elements so that every pair is contained in one and only one triplet of our system. It is obvious that if n is not of the above form then such a system does not exist.

Steiner's conjecture was first proved by Reiss [51]. More generally the following question can be asked: For which values of n is there a system of combinations taken s at a time formed from n elements so that every r-tuple is contained in one and only one of our s-tuples. Hanani [37], [38] settled the cases $r=3$, $s=4$, $r=2$, $s=4$ and $r=2$, $s=5$. The general problem seems very difficult.

Finally I would like to call attention to an old conjecture of van der Waerden which seems surprisingly difficult: The permanent of an n by n doubly stochastic matrix is $\geq n!/n^n$. Equality only if all elements of the matrix are $1/n$.

V. Some problems in combinatorial number theory. Van der Waerden [59] proved the following theorem: If we split the integers into two classes at least one of them contains arbitrarily long arithmetic progressions. Here we are more concerned with the following finite form of van der Waerden's theorem: Let $f(n)$ be the smallest integers so that if we split the integers not exceeding $f(n)$ into two classes at least one of them contains an arithmetic progression of n terms. Van der Waerden's proof gives a very poor upper bound for $f(n)$. Sharpening a previous result of Rado and myself, Schmidt proved $f(n) > 2^{n-c(n \log n)^{1/2}}$ [54]. I understand that recently Belrekamp proved $f(n) > 2^n$ (Canad. Math. Bull. **11** (1968), 409–414). It would be very desirable to obtain better lower and especially upper bounds for $f(n)$. Undoubtedly $\lim_{n \to \infty} f(n)^{1/n}$ exists—I expect the limit to be infinite. One could try to estimate $f(n, m)$ where $f(n, m)$ is the smallest integer so that if we split the integers not exceeding $f(n, m)$ into two classes either the first class contains an arithmetic progression of n terms or the second an arithmetic progression of m terms. Also one can consider splittings into more than two classes. R. Schneider and R. Ecks have certain results in these directions.

Define $f_\epsilon(n)$ to be the smallest integer so that if $g(m) = \pm 1$, $1 \leq m \leq f_\epsilon(n)$, is any number-theoretic function then there is an arithmetic progression of n terms

$$0 < a < a+d < \cdots < a+(n-1)d \leq f_\epsilon(n)$$

for which

$$\left| \sum_{k=0}^{n-1} g(a+kd) \right| > \epsilon n.$$

I proved $f_\varepsilon(n) > (1+\eta)^n$, $\eta = \eta(\varepsilon)$ [18]. The proof is probabilistic and similar to my proof with Rado. I would guess that $f_\varepsilon(n) < (1+\eta_1)^n$ and perhaps $\eta_1 \to 1$ as $\varepsilon \to 0$ but I cannot disprove

$$\lim_{n=\infty} f_\varepsilon(n)^{1/n} = \infty$$

for every $\varepsilon > 0$.

Roth [52], proved that if $g(m) = \pm 1$, $1 \le m \le n$, there always is an arithmetic progression $1 \le a < \cdots < a+kd \le n$ for which (for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$)

$$\left| \sum_{l=0}^{k-1} g(a+ld) \right| > n^{1/4-\varepsilon} \tag{9}$$

and he conjectured that in (9) $n^{1/4-\varepsilon}$ can be replaced by $n^{1/2-\varepsilon}$. I proved [22] that there is a constant C and a $g(m) = \pm 1$ so that for every progression

$$\left| \sum_{l=0}^{k-1} g(a+ld) \right| < Cn^{1/2} \tag{10}$$

and I conjecture that (10) holds for every $C > 0$ if $n > n_0(C)$.

ADDED IN PROOF. This conjecture was just proved by J. Spencer.

An old conjecture of mine states that if $g(m) = \pm 1$, $1 \le m \le \infty$, then to every c there is a d and an m so that

$$\left| \sum_{k=1}^{m} g(kd) \right| > c. \tag{11}$$

The proof of (11) seems to present great difficulties.

Let $a_1 < \cdots < a_k$ be k distinct real numbers. Denote by $f(n; a_1, \ldots, a_k)$ the number of solutions of

$$n = \sum_{i=1}^{k} \varepsilon_i a_i, \qquad \varepsilon_i = 0 \text{ or } 1.$$

Moser and I [21] proved that

$$f(n; a_1, \ldots, a_k) < c2^k/k^{3/2}(\log k)^{3/2} \tag{12}$$

and we conjectured that in (12) $(\log k)^{3/2}$ can be omitted. This conjecture was proved by Sárközi and Szemerédi [53].

Moser and I further conjectured that if $k = 2l+1$ then

$$f(n; (a_1, \ldots, a_{2l+1})) \le f(n; -l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l). \tag{13}$$

As far as I know (13) has not yet been proved. Van Lint [45] found an asymptotic formula for $f(n; -l, \ldots, l)$.

We further conjectured that the number of solutions of

$$n = \sum_{i=1}^{k} \varepsilon_i a_i, \qquad \sum_{i=1}^{k} \varepsilon_i = t$$

is for every t less than $c2^k/k^2$.

Donald Newman conjectured that for every n and m there is a function $h_{n,m}$ having the following properties: $h_{n,m}$ is defined for $1 \le i \le n$ and

$$h_{n,m}(i) \neq h_{n,m}(j), \qquad 1 \le i < j \le n, \quad m < h_{n,m}(i) \le m+n,$$
$$(i, h_{n,m}(i)) = 1, \qquad 1 \le i \le n.$$

Baines and Daykin [1] proved that for $n=m$ such a function exists, but the general case is not yet settled. One would think that Hall's theorem can be applied here, but this seems to present great difficulties.

Schur [56] proved the following result: Denote by $H(n)$ the least integer so that if we split the integers from 1 to $H(n)$ into two classes, the equation $x+y=z$ is solvable in at least one class. Schur proved $H(n) \le [en!]$. It would be very interesting to decide whether $\lim_{n\to\infty} H(n)^{1/n}$ is finite or not.

Saunders in his dissertation written under Ore proved the following result: To every k there is an $F(k)$ so that if we split the integers from 1 to $F(k)$ into two classes there are always k integers $a_1 < \cdots < a_k$ so that all the $2^k - 1$ sums

$$\sum_{i=1}^{k} \varepsilon_i a_i, \qquad \varepsilon_i = 0 \text{ or } 1 \quad (\text{not all } \varepsilon_i = 0), \tag{14}$$

are in the same class. Graham and Rothschild then asked the following question: Split the set of all the integers into two classes. Does there then exist an infinite sequence $a_1 < \cdots$ so that all the sums

$$\sum_{i=1}^{\infty} \varepsilon_i a_i, \qquad \varepsilon_i = 0 \text{ or } 1, \tag{15}$$

are in the same class, where in (15) not all the ε_i are 0 but only a finite number of them are different from 0?

I do not know the answer to this question. It easily follows from Ramsey's theorem that there is an infinite sequence where all the sums

$$\sum_{i=1}^{\infty} \varepsilon_i a_i, \quad \varepsilon_i = 0 \text{ or } 1, \qquad \sum_{i=1}^{\infty} \varepsilon_i = t,$$

are in the same class. But I cannot decide the following question: Does there exist an infinite sequence $a_1 < \cdots$ where all the a_i, $1 \le i < \infty$, and all the sums $a_i + a_j$, $1 \le i < j < \infty$, belong to the same class. This would of course follow from Graham's conjecture. The following weakening of Graham's conjecture also does not seem to be completely trivial: There is an infinite sequence $a_1 < \cdots$ so that all the sums

$$\sum_{i=1}^{\infty} \varepsilon_i a_i, \quad \varepsilon_i = 0 \text{ or } 1, \qquad \sum_{i=1}^{\infty} \varepsilon_i = t, \quad 1 \le t < \infty,$$

belong to the same class but the class may depend on t.

Finally I would like to ask the following question which I could not decide even if we assume the continuum hypothesis. Split the real numbers into two classes. Does there then exist a set of power $\aleph_1$, $\{a_\alpha\}$, $1 \le \alpha < \omega_1$, so that all the sums

$$a_{\alpha_1} + a_{\alpha_2}, \qquad 1 \le \alpha_1 < \alpha_2 < \omega_1,$$

belong to the same class?

For further problems of combinatorial number theory I refer to [22] and my first paper on extremal problems in number theory [17]. See also [20].

VI. Finally I would like to call attention to some curious results and *problems* of Czipszer, Hajnal and myself [4] *which are partly graph-theoretic and partly analytic*—in fact they deal with Tauberian theorems.

Let G be an infinite graph whose vertices are the integers and $g(n)$ the number of edges of G both vertices of which do not exceed n. A monotone path of length k is a sequence of integers $i_1 < \cdots < i_{k+1}$ where i_j and i_{j+1}, $j = 1, \ldots, k$, are joined by an edge. We conjecture that if for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$

$$g(n) > n^2\left(\frac{1}{4} - \frac{1}{4k} + \varepsilon\right)$$

then G contains infinitely many monotone paths of length k.

We proved this conjecture for $k = 2$ and $k = 3$, but could not settle the general case.

For $k = 2$ we proved the following stronger theorem: Assume that for $n > n_0(\varepsilon)$

$$g(n) > \frac{n^2}{8} + \left(\frac{1}{32} + \varepsilon\right) \frac{n^2}{(\log n)^2},$$

then G contains infinitely many monotone paths of length k. This result is best possible since it fails if we only assume

$$g(n) = \frac{n^2}{8} + \frac{n^2}{32(\log n)^2} + o\left(\frac{n}{(\log n)^2}\right).$$

We further proved that there is a G for which

$$\liminf_{n=\infty} g(n)/n^2 > \tfrac{1}{4}$$

but G does not contain an infinite monotone path. On the other hand we showed that there is an $\alpha > 0$ so that if

$$\liminf_{n=\infty} g(n)/n^2 > \tfrac{1}{2} - \alpha$$

then G contains an infinite monotone path. We do not know the largest value of α which will insure this conclusion.

Several further problems of a combinatorial nature can be found in my two papers [13], [14] on unsolved problems.

References

1. D. E. Daykin and M. J. Baines, *Coprime mappings between sets of consecutive integers*, Mathematika **10** (1963), 132–136. MR **29** #1176.

2. F. Behrend, *On sequences of numbers not divisible one by another*, J. London Math. Soc. **10** (1935), 42–44.

3. N. G. de Bruijn and P. Erdős, *On a combinatorial problem*, Nederl. Akad. Wetensch. Proc. **51** (1948), 1277–1279 = Indag. Math. **10** (1948), 421–423. MR **10**, 424.

4. J. Czipszer, P. Erdős and A. Hajnal, *Some external problems on infinite graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 441–457. MR **27** #744.

5. L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdös und von V. L. Klee*, Math. Z. **79** (1962), 95–99. MR **25** #1488.

6. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) **51** (1950), 161–166. MR **11**, 309.

7. P. D. T. A. Elliott, *On the number of circles determined by n points*, Acta Math. Acad. Sci. Hungar. **18** (1967), 181–188. MR **35** #4793.

8. P. Erdős, *On the density of some sequences of numbers*. II, J. London Math. Soc. **12** (1937), 7–11.

9. ———, *On a lemma of Littlewood and Offord*, Bull. Amer. Math. Soc. **51** (1945), 898–902. MR **7**, 309.

10. ———, *On sets of distances of n points*, Amer. Math. Monthly **53** (1946), 248–250. MR **7**, 471.

11. ———, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294. MR **8**, 479.

12. ———, *On some geometrical problems*, Mat. Lapok **8** (1957), 86–92. (Hungarian) MR **20** #6056.

13. ———, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300. MR **20** #5157.

14. ———, *Some unsolved problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 221–254. MR **31** #2106.

15. ———, *Graph theory and probability*. II, Canad. J. Math. **13** (1961), 346–352. MR **22** #10925.

16. ———, *On some elementary geometrical problems*, Köz. Mat. Lapok **24** (1962), 193–201. (Hungarian)

17. ———, *Remarks on number theory*. IV. *Extremal problems in number theory*. I, Mat. Lapok **13** (1962), 228–255. (Hungarian) MR **33** #4020.

18. ———, *On combinatorial questions connected with a theorem of Ramsey and van der Waerden*, Mat. Lapok **14** (1963), 29–37. (Hungarian) MR **34** #7409.

19. ———, *On a combinatorial problem*. II, Acta Math. Acad. Sci. Hungar. **15** (1964), 445–447. MR **29** #4700.

20. ———, *Some recent advances and current problems in number theory*, Lectures on Modern Mathematics, vol. III. Wiley, New York, 1965, pp. 196–244. MR 31 #2191.

21. ———, *Extremal problems in number theory*, Proc. Sympos. Pure Math., vol. VIII, Amer. Math. Soc., Providence, R.I., 1965, pp. 181–189. MR **30** #4740.

22. ———, *Remarks on number theory*. V. *Extremal problems in number theory*. II, Mat. Lapok **17** (1966), 135–155. (Hungarian) MR **36** #133.

23. P. Erdős and A. Hajnal, *On a property of families of sets*, Acta Math. Acad. Sci. Hungar. **12** (1961), 87–123. MR **27** #50.

24. ———, *On chromatic number of graphs and set-systems*, Acta Math. Acad. Sci. Hungar. **17** (1966), 61–99, see 94–99. MR **33** #1247. We use probabilistic methods. Hales and Jewett [36] prove the finiteness of $m_1(n)$ in a completely different setting by a direct construction.

25. P. Erdős, A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Acad. Sci. Hungar. **16** (1965), 93–196. MR **34** #2475.

26. P. Erdős, Chao Ko and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320. MR **25** #3829.

27. P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255. MR **12**, 322.

28. ———, *Combinatorial theorems on classifications of subsets of a given set*, Proc. London Math. Soc. (3) **2** (1952), 417–439. MR **16**, 455.

29. P. Erdös and R. Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. **35** (1960), 85–90. MR **22** #2554.

30. ———, *Intersection theorems for systems of sets*. II, J. London Math. Soc. **44** (1969), 467–479.

31. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470.

32. ———, *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **3–4** (1960/61), 53–62. MR **24** #A3560.

33. J. E. Graver and James Yackel, *Some graph theoretic results associated with Ramsey's theorem*, J. Combinatorial Theory **4** (1968), 125–175; $f(2,3,6)$ was first determined by Gerson and Kalbfleisch. MR **37** #1278.

34. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–7. MR **16**, 733.

35. H. Hadwiger and H. Debrunner, *Kombinatorische Geometrie in der Ebene*, Inst. Math., Univ. Genève, Geneva, 1960; English transl., Holt, Rinehart and Winston, New York, 1964. MR **22** #11210; MR **29** #1577.

36. A. Hales and R. I. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222–229. MR **26** #1265.

37. H. Hanani, *On quadruple systems*, Canad. J. Math. **12** (1960), 145–157. MR **22** #2558.

38. ———, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist. **32** (1961), 361–386. MR **29** #4161.

39. S. Hansen, *A generalization of a theorem of Sylvester on the lines determined by a finite point set*, Math. Scand. **16** (1965), 175–180. MR **34** #3411.

40. Gy. Katona, *On a conjecture of Erdös and a stronger form of Sperner's theorem*, Studia Sci. Math. Hungar. **1** (1966), 59–63. MR **34** #5690.

41. L. M. Kelly and W. O. J. Moser, *On the number of ordinary lines determined by n points*, Canad. J. Math. **1** (1958), 210–219. MR **20** #3494.

42. V. L. Klee (editor), *Convexity*, Proc. Sympos. Pure Math., vol. VII, Amer. Math. Soc., Providence, R.I., 1963.

43. D. Kleitman, *On a lemma of Littlewood and Offord on the distrbution of certain sums*, Math. Z. **90** (1965), 251–259. MR **32** #2336.

44. ———, *On a combinatorial problem of Erdös*, Proc. Amer. Math. Soc. **17** (1966), 139–141. MR **32** #2337.

45. J. H. van Lint, *Representation of* 0 *as* $\sum_{k=-N}^{N} \varepsilon_k k$, Proc. Amer. Math. Soc. **18** (1967), 182–184. MR **34** #5789.

46. E. W. Miller, *On a property of families of sets*, C.R. Soc. Sci. Varsovie **30** (1937), 31–38.

47. Th. Motzkin, *The lines and planes connecting the points of a finite set*, Trans. Amer. Math. Soc. **70** (1951), 451–464. This paper contains many further interesting problems some of which have been settled in the meantime. See, e.g., Hansen [39]. MR **12**, 849.

48. C. St. J. A. Nash-Williams, *On well-quasi-ordering transfinite sequences*, Proc. Cambridge Philos. Soc. **61** (1965), 33–39. MR **30** #3850.

49. R. Rado, *Studien zur Kombinatorik*, Math. Z. **36** (1933), 424–480.

50. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30** (1929), 264–286, also Collected Papers, 82–111. See also Erdös and Rado [28]. For generalizations of Ramsey's Theorem see Erdös and Rado [27] and Nash-Williams [48]. I would further like to mention the following unpublished result of Galvin: Let F be a family of finite subsets of the integers so that every infinite set contains a set in F. Then every infinite subset S of the integers contains an infinite subset S_1, so that every $S_2 \subset S_1$, $|S_2| = \aleph_0$ has an initial segment in F.

51. M. Reiss, *Über eine Steinersche kombinatorische Aufgabe*, J. Reine Angew. Math. **56** (1859), 326–344.

52. K. F. Roth, *Remark concerning integer sequences*, Acta Arith. **9** (1964), 257–260. MR **29** #5806.

53. A. Sárközi and E. Szemerédi, *Über ein Problem von Erdös und Moser*, Acta Arith. **11** (1966), 205–208. MR **32** #102.

54. W. Schmidt, *Two combinatorial theorems on arithmetic progressions*, Duke Math J. **29** (1962), 129–140. MR **25** #1125.

55. ———, *Ein kombinatorisches Problem von P. Erdös*, Acta Math. Acad. Sci. Hungar. **15** (1964), 373–374. MR **29** #4701.

56. I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jber. Deutsch. Math.-Verein. **25** (1916), 114–117. See also Rado [49].

57. A. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Z. **27** (1928), 544–548.

58. G. Szekeres, *On an extremum problem in the plane*, Amer. J. Math. **63** (1941), 208–210. MR **2**, 263.

59. B. L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wisk. (2) **15** (1927), 212–216. See also R. Rado [49].

University of Colorado

Hungarian Academy of Sciences