SIMPLE ONE-POINT EXTENSIONS OF TOURNAMENTS

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1. *Introduction.* A *tournament* $\mathscr{T} = \langle T, \to \rangle$ is a relational structure on the non-empty set T such that for $x, y \in T$ exactly one of the three relations

$$x \to y, x = y, y \to x$$

holds. Here $x \to y$ expresses the fact that $\{x, y\} \in \to$ and we sometimes write this in the alternative form $y \leftarrow x$. Extending the notation to subsets of T we write $A \to B$ or $B \leftarrow A$ if $a \to b$ holds for all pairs a, b with $a \in A$ and $b \in B$. $\mathscr{T}' = \langle T,' \to' \rangle$ is a *subtournament* of $\mathscr{T}$, and $\mathscr{T}$ is an *extension* of $\mathscr{T}'$, if $T' \subset T$ and $\to'$ is the restriction of $\to$ to T'; we will usually write $\langle T', \to \rangle$ instead of $\langle T', \to' \rangle$. In particular, if $|T - T'| = k$, we call $\mathscr{T}$ a *k-point extension* of $\mathscr{T}'$.

A *convex subset* of $\mathscr{T}$ is a set $K \subset T$ such that either $K \to \{x\}$ or $\{x\} \to K$ for every $x \in T - K$. Equivalently, K is convex if, and only if, $x, y \in K$, $z \in T$, $x \to z \to y$ implies $z \in K$. The convex set K is *non-trivial* if $K \neq T$ and $|K| > 1$. A tournament $\mathscr{T}$ is *simple*† if it has no non-trivial convex subset.

In [1] we showed with Fried that any tournament $\mathscr{T}$ of order $|\mathscr{T}| = |T| \neq 2$ has a simple 2-point extension. This result is best possible in the sense that there is no simple tournament of order 4 and if $\mathscr{T}$ is an odd chain then it does not have a simple 1-point extension. We stated in [1] that we did not know how to characterize those tournaments which do have simple 1-point extensions. Moon [2] settled this problem for finite tournaments by showing that the only finite exceptions are the ones we had already noted. We now extend Moon's result to general tournaments and prove the following theorem.

Theorem. *If the tournament $\mathscr{T} = \langle T, \to \rangle$ is not a finite odd chain and $|T| \neq 3$, then it has a simple 1-point extension.*

Our proof of this result for the finite case (§3) is different from Moon's proof.

2. *Notation and preliminary lemmas.* Let $\mathscr{T} = \langle T, \to \rangle$ be a tournament. If $x \in T$ we define $\mathscr{T}(x, \to) = \{y \in T : x \to y\}$ and $\mathscr{T}(x, \leftarrow) = \{y \in T : x \leftarrow y\}$. To uniformize our notation, whenever we say that $\mathscr{T}^* = \langle T^*, \to \rangle$ is a 1-point extension of $\mathscr{T}$ we shall always denote the added point by z, i.e. $T^* - T = \{z\}$. The extension $\mathscr{T}^*$ is then uniquely determined by specifying the set $B = \mathscr{T}^*(z, \to)$ and we denote this 1-point extension of $\mathscr{T}$ by $\mathscr{T}(B)$. An element $x \in T$ is *extremal* if either $\{x\} \to T - \{x\}$ or $\{x\} \leftarrow T - \{x\}$. $C \subset T$ is a *chain* of $\mathscr{T}$ if $\to$ is transitive on C (i.e. $\langle C, \to \rangle$ is a simple order). We shall write $C = \{x_1, \ldots, x_n\}_\to$ to indicate that C is the chain in which $x_i \to x_j$ for $1 \leqslant i < j \leqslant n$. We denote by $C_2(\mathscr{T})$ the set $\{X \subset T : |X| = 2, X \textit{ convex in } \mathscr{T}\}$ and by $G(\mathscr{T})$ the graph with vertex set T and edge set $C_2(\mathscr{T})$. The valency of a point $x \in T$ in the graph $G(\mathscr{T})$ will be denoted by $\rho(x)$.

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† In other words, there is no non-trivial equivalence relation θ on T such that $x \to y$, $x \equiv x'$ (mod θ), $y \equiv y'$ (mod θ) $\Rightarrow x' \to y'$.

LEMMA 1. *$\rho(x) \leqslant 2$ for all $x \in T$.*

Proof. Suppose the lemma is false. Then there are distinct points x, y_1, y_2, $y_3 \in T$ such that $\{x, y_i\} \in C_2(\mathscr{T})(1 \leqslant i \leqslant 3)$. By symmetry we can assume that $y_1 \to y_2 \to y_3$. Now by the convexity of $\{x, y_1\}$ and $\{x, y_3\}$ we obtain the contradictory relations $x \to y_2$ and $y_2 \to x$.

LEMMA 2. *A path in $G(\mathscr{T})$ is a chain in $\mathscr{T}$.*

Proof. Suppose $C = \{x_1, \ldots, x_n\}$ is a path of length n in $G(\mathscr{T})$, i.e. $\{x_i, x_{i+1}\} \in C_2(\mathscr{T})(1 \leqslant i < n)$. Assume that $x_1 \to x_2$. Let $2 \leqslant j < n$ and suppose that we have already established that $\{x_1, \ldots, x_j\}_\to$ is a chain. Since $x_i \to x_j$ and $\{x_j, x_{j+1}\}$ is convex, we have that $x_i \to x_{j+1}(1 \leqslant i < j)$. Also $x_j \to x_{j+1}$ since $x_{j-1} \to x_{j+1}$ and $\{x_{j-1}, x_j\}$ is convex. Thus $\{x_1, \ldots, x_{j+1}\}_\to$ is a chain. It follows by induction that C is a chain. A similar argument applies if $x_1 \leftarrow x_2$ or if C is a 1-way or 2-way infinite path of $G(\mathscr{T})$.

LEMMA 3. *$G(\mathscr{T})$ is circuit free.*

Proof. Suppose $\{x_1, \ldots, x_n\}$ is a circuit of length $n \geqslant 3$ in $G(\mathscr{T})$. Assume that $x_1 \to x_2$. Then, by Lemma 2, $\{x_1, \ldots, x_n\}_\to$ is a chain in $\mathscr{T}$. Also, $\{x_2, \ldots, x_n, x_1\}_\to$ is a chain. This gives the contradictory relation $x_2 \to x_1$.

An immediate deduction from Lemmas 1 and 3 is the

COROLLARY. *$G(\mathscr{T})$ is the union of disjoint paths.*

For brevity we shall write $\mathscr{T} \in \mathscr{E}$ if $\mathscr{T}$ has a simple 1-point extension. The next lemma is due to Moon **[2]**.

LEMMA 4. *If $\mathscr{T}$ is simple and $|\mathscr{T}| \geqslant 4$, then $\mathscr{T} \in \mathscr{E}$.*

Proof. Since $2^{|T|} \geqslant 2|T| + 2$, there is a set $B \subset T$ such that $B \neq \varnothing$, $B \neq T$, $B \neq \mathscr{T}(x, \to)$ and $B \neq \{x\} \cup \mathscr{T}(x, \to)$ for any $x \in T$. Consider the 1-point extension $\mathscr{T}(B) = \langle T \cup \{z\}, \to \rangle$. Suppose K is a non-trivial convex subset of $\mathscr{T}(B)$. Since B is a proper, non-empty subset of T, $K \neq T$. Therefore, since $\mathscr{T}$ is simple and $K \cap T$ is convex in $\mathscr{T}$, $K = \{x, z\}$ for some $x \in T$. Since $T - B \to \{z\} \to B$, we have that $T - B - \{x\} \to \{x\} \to B - \{x\}$ contrary to the definition of B. This shows that $\mathscr{T}(B)$ is simple and that $\mathscr{T} \in \mathscr{E}$.

Our main lemma is the following.

LEMMA 5. *Let $|\mathscr{T}| \geqslant 4$, and suppose there is an element $x \in T$ such that* (i) $\rho(x) \leqslant 1$, (ii) *x is not an extremal element of $\mathscr{T}$, and* (iii) $\mathscr{T}_1 = \langle T - \{x\}, \to \rangle \in \mathscr{E}$. *Then $\mathscr{T} \in \mathscr{E}$.*

Proof. By Lemma 4 we can assume that $\mathscr{T}$ is not simple. By (iii) there is $B \subset T - \{x\}$ such that $\mathscr{T}_1(B) = \langle (T - \{x\}) \cup \{z\}, \to \rangle$ is simple. If there is a $y \in T$ by (i) there is at most one such that $\{x, y\} \in C_2(\mathscr{T})$ and if $y \notin B$, then we put $A = B \cup \{x\}$. Otherwise, put $A = B$. Then the 1-point extension $\mathscr{T}(A)$ of $\mathscr{T}$ is simple.

To see this, suppose that K is a non-trivial convex subset of $\mathscr{T}(A)$. Since $\mathscr{T}_1(B)$ is a simple subtournament it follows that either (a) $K = (T - \{x\}) \cup \{z\}$ or (b) $K = \{x, z\}$ or (c) $K = \{x, y\}$ for some $y \in T - \{x\}$. If (a) holds, then x is extremal in $\mathscr{T}$. If (b) holds, then $\mathscr{T}(x, \to) = B$, $\mathscr{T}$ is isomorphic to $\mathscr{T}_1(B)$ and therefore simple. If (c) holds, then y is the unique point such that $\{x, y\} \in C_2(\mathscr{T})$ and, by the definition of A, either $y \to z \to x$ or $y \leftarrow z \leftarrow$, *i.e.* $\{x, y\}$ is not convex in $\mathscr{T}(A)$. Thus in each case we obtain a contradiction.

3. *Proof of the theorem for finite tournaments.* We need two additional lemmas which are true only for finite tournaments.

LEMMA 6. *If $\mathscr{T}$ is a finite tournament, then $G(\mathscr{T})$ cannot have exactly two components.*

Proof. Suppose that $T = C_1 \cup C_2$, where C_1, C_2 are components of the graph $G(\mathscr{T})$. Then by Lemma 2 and the Corollary after Lemma 3, C_1, C_2 are paths in $G(\mathscr{T})$ and chains in $\mathscr{T}$. Suppose that $C_1 = \{x_1, \ldots, x_m\}_\to$, $C_2 = \{y_1, \ldots, y_n\}_\to$ and that $x_1 \to y_1$. Using the fact that $\{x_i, x_{i+1}\}$ and $\{y_j, y_{j+1}\}$ are convex in $\mathscr{T}$, we immediately deduce that $C_1 \to C_2$. Thus $\mathscr{T}$ is a chain and $G(\mathscr{T})$ has only one component.

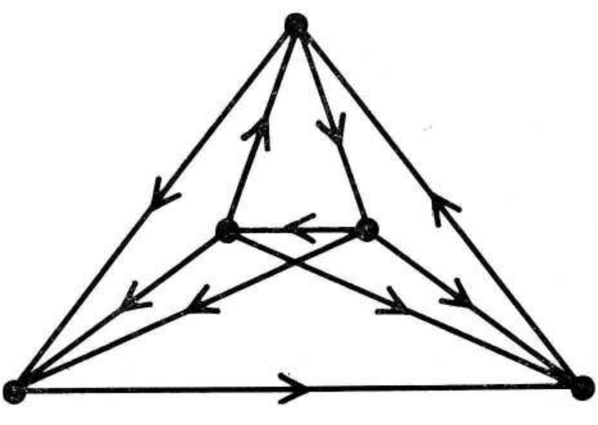
LEMMA 7. *Suppose that $\mathscr{T}$ is a finite tournament, $|\mathscr{T}| \geqslant 4$ and that $\mathscr{T}$ is not a chain. Then there is $x \in T$ such that* (i) $\rho(x) \leqslant 1$, (ii) *x is not an extremal point and* (iii) *$T - \{x\}$ is not a chain.*

Proof. Since $|\mathscr{T}| \geqslant 4$, it follows from Lemma 6 and the Corollary after Lemma 3 that there are at least four points $x \in T$ which satisfy (i). Since there are at most two extremal points, there are two points, say x_1 and x_2, which satisfy both the conditions (i) and (ii). We may assume that $T - \{x_1\} = \{y_1, \ldots, y_r, x_2, y_{r+1}, \ldots, y_n\}_\to$ and $T - \{x_2\} = \{y_1, \ldots, y_s, x_1, y_{s+1}, \ldots, y_n\}_\to$ are both chains and that $1 \leqslant r \leqslant s \leqslant n$. Since $\mathscr{T}$ is not a chain we can assume further that $r < s$ and that $x_1 \to x_2$. It is obvious that (ii) and (iii) are satisfied with $x = y_{r+1}$ and a simple matter to verify that (i) also holds.

We now conclude the proof for finite tournaments.

If $|\mathscr{T}| = 1$ or 2, then trivially $\mathscr{T} \in \mathscr{E}$. Also $\mathscr{T} \in \mathscr{E}$ if $|\mathscr{T}| = 4$ since every tournament of order 4 is isomorphic to a subtournament of the simple tournament $\mathscr{T}^*$ of order 5 illustrated.

$\mathscr{T}^*$:



We now assume that $\mathscr{T}$ is a finite tournament of order $|\mathscr{T}| \geqslant 5$ and use induction on $|\mathscr{T}|$. By hypothesis $\mathscr{T}$ is not an odd chain. If $\mathscr{T}$ is the even chain $\{x_1, \ldots, x_{2n}\}_\to$, then it is easy to verify that $\mathscr{T}(\{x_{2i+1} : 0 \leqslant i < n\})$ is a simple 1-point extension.

Therefore, we may suppose that $\mathcal{T}$ is not a chain. By Lemma 7, there is $x \in T$ such that $\rho(x) \leqslant 1$, x is not extremal and $\mathcal{T}_1 = \langle T - \{x\}, \rightarrow \rangle$ is not a chain. By the induction hypothesis $\mathcal{T}_1 \in \mathcal{E}$ and therefore $\mathcal{T} \in \mathcal{E}$ by Lemma 5.

4. *Proof of the theorem for infinite tournaments.* We first need some results about chains. Let $\mathcal{T} = \langle T, \rightarrow \rangle$ be a chain. Then $\mathcal{T}$ is a simply ordered set in which *a precedes b* if and only if $a \rightarrow b$. The order type of $\mathcal{T}$ with this ordering will be denoted by $\mathrm{tp}\mathcal{T}$. As usual, ω denotes the least infinite ordinal number and ω^* is the reverse order type. A set $X \subset T$ is *cofinal* (*coinitial*) in $\mathcal{T}$ if whenever $t \in T$ then either $t \in X$ or there is $x \in X$ such that $t \rightarrow x$ $(x \rightarrow t)$. For $a, b \in T$ we denote by $[a, b]$ the closed interval $\{a, b\} \cup \{x \in T : a \rightarrow x \rightarrow b \text{ or } b \rightarrow x \rightarrow a\}$. Let $\mathcal{I}(\mathcal{T})$ be the set of all the non-trivial closed intervals $[a, b] \subset T$ (i.e. with $a \neq b$). Also, for $x \in T$ we define

$$E(x) = \{y \in T : [x, y] \textit{ is finite}\}.$$

Clearly, $E(x)$ is a sub-interval of $\mathcal{T}$ which is either finite or infinite and having order type ω or ω^* or $\omega^* + \omega$.

We showed in **[1]** that, if $\mathcal{T}$ is a chain, then $\mathcal{I}(\mathcal{T})$ has the Bernstein property, *i.e.* there is a set $B \subset T$ such that

$$B \cap I \neq \varnothing \quad \textit{and} \quad I - B \neq \varnothing \qquad (I \in \mathcal{I}(\mathcal{T})). \tag{1}$$

(This result was also proved by Hausdorff **[3]** for the case when $\mathcal{T}$ is a densely ordered set.)

Of course, if $\mathcal{T}$ is a finite chain $\{x_1, \ldots, x_n\}_{\rightarrow}$ then B is one of the two sets $\{x_1, x_3, \ldots\}$ or $\{x_2, x_4, \ldots\}$. Note that if B satisfies (1) then so also does the complementary set $T - B$. Consequently, there is $B \subset T$ which satisfies (1) and

$$B \textit{ is coinitial in } \mathcal{T}. \tag{2}$$

Unless $\mathcal{T}$ is an odd chain we assert further that there is $B \subset T$ such that (1) and (2) hold and also

$$T - B \textit{ is cofinal in } \mathcal{T}. \tag{3}$$

For suppose that B satisfies (1) and (2) but not (3). Then there is a final element $a \leftarrow T - \{a\}$ and $a \in B$. It is easy to see that (1) and (3) hold with

$B_1 = (E(a) - B) \cup (B - E(a))$ in place of B. Now, if (2) is false for B_r, then there is an initial element $b \rightarrow T - \{b\}$ and $b \in T - B_1$. It follows that $b \in E(a) \cap B$ and that $T = E(a)$ is an odd chain.

We need the following stronger result.

LEMMA 8. *Suppose that $\mathcal{T}$ is a chain and that* $\mathrm{tp}\mathcal{T} \nleq \omega + \omega^*$. *Then there are two distinct sets $B \subset T$ satisfying* (1), (2) & (3).

Proof. By the above, there is one set $B \subset T$ such that (1), (2) and (3) hold. Let U denote the set of extremal points of $\mathcal{T}$ (*i.e.* $|U| = 0, 1$ or 2). Put $A = \bigcup_{x \in U} E(x)$ and $B_1 = (A \cap B) \cup (T - A \cup B)$. It is easy to see that the set B_1 satisfies (1), (2) and (3). Also, since $\mathrm{tp}\mathcal{T} \nleq \omega + \omega^*$, it follows that $A \neq T$ and hence that $B_1 \neq B$.

LEMMA 9. *Let $\mathcal{T}$ be a chain and let $B \subset T$ satisfy* (1), (2) *and* (3). *Then $\mathcal{T}(B)$ is a simple* 1*-point extension of $\mathcal{T}$.*

Proof. Suppose that K is a non-trivial convex subset of $\mathscr{T}(B)$. Then $K \cap T$ is a sub-interval of $\mathscr{T}$. If $z \notin K$, then $|K \cap T| > 1$ and there is $I \in \mathscr{I}(\mathscr{T})$ such that $I \subset K$. Therefore, by (1), there are $a, b \in I$ such that $a \in B$, $b \notin B$. Hence, $a \leftarrow z \leftarrow b$, and this contradicts the assumption that K is convex. Therefore, $z \in K$. $K \cap T$ is coinitial in $\mathscr{T}$. Otherwise, there is $x \in B$ such that $x \rightarrow K \cap T$ and we have the contradiction that $z \rightarrow x \rightarrow K - \{z\}$. Similarly, $K \cap T$ is cofinal in $\mathscr{T}$. Since $K \cap T$ is an interval in $\mathscr{T}$, $K = T \cup \{z\}$ and this is a contradiction.

LEMMA 10. *If A is a maximal chain of the tournament $\mathscr{T} = \langle T, \rightarrow \rangle$ and $a \in A$, $b \in T - A$, then $\{a, b\}$ is not convex in $\mathscr{T}$.*

Proof. Suppose $\{a, b\}$ is convex. Then $\mathscr{T}(a, \leftarrow) \cap A \rightarrow \{a, b\} \rightarrow \mathscr{T}(a, \rightarrow) \cap A$ and so $A \cup \{b\}$ is a chain.

LEMMA 11. *Let $\mathscr{T} = \langle T, \rightarrow \rangle$ be an infinite tournament and suppose that $T = \bigcup_{\nu<\lambda} A_\nu$, where A_μ is a maximal infinite chain of the subtournament $\langle \bigcup_{\mu \leqslant \nu < \lambda} A_\nu, \rightarrow \rangle$ ($\mu < \lambda$). Let $B_\nu \subset A_\nu$ be a set satisfying the conditions* (1), (2) *and* (3) *for the infinite chain $\langle A_\nu, \rightarrow \rangle$ ($\nu < \lambda$), and let $B = \bigcup_{\nu<\lambda} B_\nu$. If X is a non-trivial convex subset of $\mathscr{T}(B)$, then there is an ordinal α such that $0 < \alpha < \lambda$ and $X = \{z\} \cup \bigcup_{\alpha \leqslant \nu < \lambda} A_\nu$.*

Proof. By Lemma 9, for each $\nu < \lambda$, either $A_\nu \cup \{z\} \subset X$ or

$$|(A_\nu \cup \{z\}) \cap X| \leqslant 1.$$

Suppose that $z \notin X$. Then there are β, γ such that $\beta < \gamma < \lambda$ and

$$|X \cap A_\beta| = |X \cap A_\gamma| = 1.$$

Then $X \cap (A_\beta \cup A_\gamma)$ is convex in $\langle A_\beta \cup A_\gamma, \rightarrow \rangle$, a contradiction against Lemma 10. Therefore, $z \in X$. Let α be the least ordinal such that $X \cap A_\alpha \neq \varnothing$. Then $A_\alpha \subset X$. If $\alpha < \beta < \lambda$ and $y \in A_\beta$, then by the maximality of A_α there are $x, x' \in A_\alpha$ such that $x \rightarrow y \rightarrow x'$ and so $y \in X$. This proves that $X = \{z\} \cup \bigcup_{\alpha \leqslant \nu < \lambda} A_\nu$. Also, since X is a non-trivial, $\alpha > 0$.

LEMMA 12. *Suppose that $\mathscr{T}$ and $A_\mu (\mu < \lambda)$ satisfy the same conditions as in Lemma* 11. *Suppose also that* $\mathrm{tp}\langle A_0, \rightarrow \rangle \not\leqslant \omega + \omega^*$. *Then $\mathscr{T} \in \mathscr{E}$.*

Proof. Let $B_\nu (\nu < \lambda)$ satisfy conditions (1), (2) and (3) for the chain $\langle A_\nu, \rightarrow \rangle$. By Lemma 8, there is $B_0' \subset A_0$ such that $B_0' \neq B_0$ and such that B_0' also satisfies (1), (2) and (3) for $\langle A_0, \rightarrow \rangle$. Put $B = \bigcup_{\nu<\lambda} B_\nu$, $B' = B_0' \cup \bigcup_{0<\nu<\lambda} B_\nu$. Suppose that $\mathscr{T}(B)$ is not simple. Then by Lemma 11, there is α such that $0 < \alpha < \lambda$ and $X = \bigcup_{\alpha \leqslant \nu < \lambda} A_\nu \cup \{z\}$ is convex in $\mathscr{T}(B)$. Similarly, if $\mathscr{T}(B')$ is not simple, there is β such that $0 < \beta < \lambda$ and $Y = U_{\beta \leqslant \nu < \lambda} A_\nu \cup \{z\}$ is convex in $\mathscr{T}(B')$. Let $\gamma = \max\{\alpha, \beta\}$. Suppose there is an element $y \in B_0 - B_0'$. Since $z \rightarrow y$ in $\mathscr{T}(B)$, we have that $A_\gamma \rightarrow y$ in $\mathscr{T}$. Similarly, since $z \leftarrow y$ in $\mathscr{T}(B')$, $A_\gamma \leftarrow y$ in $\mathscr{T}$. This contradiction shows that $B \subset B'$. By symmetry, we also have that $B' \subset B$, and this contradicts the fact that $B \neq B'$.

We now conclude the proof of our theorem.

Let $\mathscr{T} = \langle T, \rightarrow \rangle$ be an infinite tournament. If X is a chain of $\mathscr{T}$, then there is a maximal infinite chain $A_0 \supset X$. Hence there is a partition of T,

$$T = A_0 \cup A_1 \cup \ldots \cup A_\lambda, \tag{4}$$

where $\lambda(\geqslant 1)$ is an ordinal, A_λ is finite (possibly empty), $A_0 \supset X$ and $A_\mu(\mu < \lambda)$ is a maximal infinite chain in the sub-tournament $\langle \bigcup_{\mu \leqslant \nu < \lambda} A_\nu, \rightarrow \rangle$. We shall write $T' = T - A_\lambda$ and $\mathscr{T}' = \langle T', \rightarrow \rangle$.

We shall first prove, by induction on $|A_\lambda|$, that if $\mathscr{T}' \in \mathscr{E}$ then $\mathscr{T} \in \mathscr{E}$. If $A_\lambda = \varnothing$ there is nothing to prove. Suppose $A_\lambda \neq \varnothing$. By Lemma 10, there is no edge of the graph $G(\mathscr{T})$ of the form $\{x, y\}$ with $x \in A_\lambda$ and $y \notin A_\lambda$. Therefore, by the Corollary after Lemma 3, there is an element $x \in A_\lambda$ such that $\rho(x) \leqslant 1$. This x is not extremal in $\mathscr{T}$ by the maximality of A_0. By the induction hypothesis $\langle T - \{x\}, \rightarrow \rangle \in \mathscr{E}$ and hence $\mathscr{T} \in \mathscr{E}$ by Lemma 5.

We will now assume that $\mathscr{T} \notin \mathscr{E}$ and obtain a contradiction. If there is a partition (4) of T which is such that $\mathrm{tp}\langle A_0, \rightarrow \rangle \nleqslant \omega + \omega^*$, then $\mathscr{T}'$ (and hence $\mathscr{T}$) $\in \mathscr{E}$ by Lemma 12. Therefore, we can assume that

$$\mathrm{tp}\langle X, \rightarrow \rangle \leqslant \omega + \omega^* \tag{5}$$

whenever X is a chain in $\mathscr{T}$. Consider any partition of T of the form (4). Let $B_\nu \subset A_\nu$ be a set which satisfies the conditions (1), (2) and (3) for the chain $\langle A_\nu, \rightarrow \rangle$ $(\nu < \lambda)$ and let $B = \bigcup_{\nu < \lambda} B_\nu$. By our assumption $\mathscr{T}' \notin \mathscr{E}$ and so $\mathscr{T}'(B)$ is not simple. Therefore, by Lemma 11, there is α such that $0 < \alpha < \lambda$ and $\bigcup_{\alpha \leqslant \nu < \lambda} A_\nu \cup \{z\}$ is convex in $\mathscr{T}'(B)$. Since $A_0 - B_0 \rightarrow z \rightarrow B_0$ in $\mathscr{T}'(B)$, we have that

$$A_0 - B_0 \rightarrow A_\alpha \rightarrow B_0$$

in $\mathscr{T}$. Since A_α, B_0 are infinite chains and A_α precedes B_0, it follows from (5) that $\mathrm{tp}\langle A_\alpha, \rightarrow \rangle = \omega$. Similarly, $A_0 - B_0$ and A_α are infinite chains and $A_0 - B_0$ precedes A_α, and hence $\mathrm{tp}\langle A_\alpha, \rightarrow \rangle = \omega^*$ by (5). This contradiction completes the proof.

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