Periodica Mathematica Hungarica Vol. 6 (3), (1975), pp. 241–244.

A NOTE ON RATIONAL APPROXIMATION

by

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Recently (cf. [1], [2]) we have studied the problem of approximating reciprocals of certain entire functions by reciprocals of polynomials under the uniform norm on the positive real axis. In this connection we present here some results.

NOTATION. Let π_n denote the class of all algebraic polynomials of degree at most n. Denote by

$$\lambda_{0,n} \equiv \inf_{p \in \pi_n} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{L^\infty[0,\infty)}. \tag{1}$$

For every $r > 0$, let $\sigma_n(x; r) \in \pi_n$ denote the best Chebyshev approximation to f in $[0, r]$, i.e.,

$$\| f - \sigma_n(x; r) \|_{L^\infty[0,r]} = \inf_{\sigma_n \in \pi_n} \| f - \sigma_n \|_{L^\infty[0,r]} \equiv \delta_n(r).$$

Let $P_n(x; r) = \sigma_n(x; r) + \delta_n(r)$ for each $n \geq 0$.

THEOREM 1. *If $g(n)$ tends to infinity arbitrarily fast, then there is an entire function $f(x)$ of infinite order such that for infinitely many n*

$$\lambda_{0,n} \leq \frac{1}{g(n)}. \tag{2}$$

PROOF. Let $u_k \to \infty$ very fast and $n_k \to \infty$ even much faster. Put

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{n_k}}{u_k^{n_k}}.$$

If $n_k \to \infty$ fast enough, then $f(x)$ is an entire function of infinite order, $u_k \to \infty$ depends on $g(n)$ and n_k on u_k and on $g(n)$.

Set $0 < x \leq (u_{l+1}) \frac{1}{2}$, then

$$0 \leq \left(1 + \sum_{k=1}^{l} \frac{x^{n_k}}{u_k^{n_k}}\right)^{-1} - \frac{1}{f(x)} \leq \frac{2x^{n_{l+1}}}{u_{l+1}^{n_{l+1}}} \leq \frac{2}{2^{n_{l+1}}} < \frac{1}{g(n_l)}, \tag{3}$$

if n_{l+1} is sufficiently large.

On the other hand, let $2x > u_{l+1}$, then

$$0 \leq \left(1 + \sum_{k=1}^{l} \frac{x^{n_k}}{u_k^{n_k}}\right)^{-1} - \frac{1}{f(x)} \leq \frac{(2u_l)^{n_l}}{u_{l+1}^{n_l}} < \frac{1}{g(n_l)}, \tag{4}$$

if u_{l+1} is sufficiently large.

The result (2) follows from (3) and (4).

THEOREM 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geq 0$ $(k \geq 1)$ be any entire function of finite lower order β. Then there exists a sequence of ordinary polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $P_n \in \pi_n$ for each $n \geq 0$, such that for any $\varepsilon > 0$*

$$\varliminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L^\infty[0,\infty)} \right\}^{\frac{\beta+\varepsilon}{n}} \leq \exp\left(\frac{-1}{e+1}\right). \tag{5}$$

PROOF. By hypothesis $f(z)$ is an entire function of finite lower order β. Therefore for each $\varepsilon > 0$, we obtain

$$\varliminf_{s\to\infty} \frac{\log M(s)}{s^{\beta+\varepsilon}} = 0, \text{ where } M(s) = \operatorname*{Max}_{|z|=s} |f(z)|. \tag{6}$$

Then (6) implies that there exist arbitrary large values of s, for which

$$\frac{\log M(s)}{s^{\beta+\varepsilon}} \leq \frac{\log M(r)}{r^{\beta+\varepsilon}}, \qquad 0 < r < s. \tag{7}$$

From (7), we get with $s = re^{1/(\beta+\varepsilon}$

$$M(s) \leq \{M(r)\}^e. \tag{8}$$

We have from ([2], p. 181)

$$\left|\frac{1}{f(x)} - \frac{1}{p_n(x; r)}\right| \leq \frac{2\delta_n(r)}{f^2(0)}, \quad 0 \leq x \leq r, \tag{9}$$

and

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x; r)}\right| \leq \frac{2}{f(r)}, \qquad x \geq r. \tag{10}$$

From ([2], 3.4)) it follows that

$$\delta_n(r) \leq \sum_{k=n+1}^{\infty} a_k r^k. \tag{11}$$

Hence from (9) and (11), we get for $0 \leq x \leq r$

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x; r)}\right| \leq 2a_0^{-2} \sum_{k=n+1}^{\infty} a_k r^k \leq \tag{12}$$

$$\leq 2a_0^{-2} \exp\left(\frac{-n}{\beta+\varepsilon}\right) \sum_{k=n+1}^{\infty} a_k r^k e^{\frac{k}{\beta+\varepsilon}} \leq 2a_0^{-2} \exp\left(\frac{-n}{\beta+\varepsilon}\right) M(re^{1/(\beta+\varepsilon)}).$$

From (8) and (12), we get for all those values of r for which (7) is valid

$$\left|\frac{1}{f(x)} - \frac{1}{p_n(x; r)}\right| \leq \frac{2\{M(r)\}^e}{a_0^2 \exp\left(\frac{n}{\beta+\varepsilon}\right)} = \frac{2\{M(r)\}^{e+1}}{a_0^2 \exp\left(\frac{n}{\beta+\varepsilon}\right) M(r)}. \tag{13}$$

Now we choose here

$$\{M(r)\}^{e+1} = \exp\left(\frac{n}{\beta+\varepsilon}\right), \tag{14}$$

which is permissible because $M(r) \to \infty$ as $r \to \infty$. Then from (13) and (14), we get

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x; r)}\right| \leq \frac{2}{a_0^2 \exp\left\{\frac{n}{(e+1)(\beta+\varepsilon)}\right\}}, \quad x \in [0, r]. \tag{15}$$

From (10) and (14), we get

$$\left|\frac{1}{f(x)} - \frac{1}{P_n(x; r)}\right| \leq \frac{2}{\exp\left\{\frac{n}{(\beta+\varepsilon)(e+1)}\right\}}, \quad x \geq r. \tag{16}$$

Set $P_n(x; r) = P_n(x)$, then from (15) and (16), we get the required result (5).

Remark. Theorem 2 improves considerably a recent result of Erdős and Reddy ([1], Theorem 3). It is easy to construct an entire function, with order infinity and lower order finite, for this function clearly $\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{1/n} \nless 1$ (cf. [2], Theorem 1).

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(Received March 21, 1974)

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