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Note

On the Chromatic Index of Almost All Graphs

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Vizing has shown that if G is a simple graph with maximum vertex-degree ρ, then the chromatic index of G is either ρ or $\rho + 1$. In this note we prove that almost all graphs have a unique vertex of maximum degree, and we deduce that almost all graphs have chromatic index equal to their maximum degree. This settles a conjecture of the second author (in "Proceedings of the Fifth British Combinatorial Conference 1975").

Let G be a simple graph (that is, a graph without loops or multiple edges), and suppose that the maximum vertex-degree of G is ρ. Vizing [8] has shown that the chromatic index $\chi'(G)$ of G must be equal either to ρ (in which case we say that G is of *class one*) or to $\rho + 1$ (in which case G is of *class two*).

Limited numerical evidence seems to suggest that most graphs are of class one; for example, it is shown in [1] that of the 143 connected graphs with not more than six vertices, only eight are of class two. The object of this note is to prove this result in general.

THEOREM. *Almost all graphs are of class one.*

(In this note, to say that "almost all graphs have a given property" means that if $P(n)$ is the probability that a random graph with n vertices has that property, then $P(n) \to 1$ as $n \to \infty$; in other words, if U_n is the number of graphs with n vertices having that property, and if V_n is the total number of graphs with n vertices, then $U_n/V_n \to 1$ as $n \to \infty$. Further results on random graphs may be found in [2, 3].)

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In order to establish this theorem, we shall need the following lemma, which is of considerable interest in its own right:

LEMMA. *Almost all graphs have a unique vertex of maximum degree.*

Proof of lemma. We first prove the corresponding result for labeled graphs.

If G is a random labeled graph with n vertices, then the probability that a given vertex has degree k is $\binom{n-1}{k} 2^{1-n}$, since each edge of G appears with probability $\frac{1}{2}$.

If $k = \frac{1}{2}(n-1) + t$ (say), then by a standard asymptotic argument for the binomial distribution (see, for example, [4, pp. 179–180]), we have

$$\binom{n-1}{k} 2^{1-n} = (1 + o(1))(\tfrac{1}{2}\pi n)^{-1/2} e^{-2t^2/n}. \tag{1}$$

It follows from the Inclusion–Exclusion Principle (as used, for example, in [6, pp. 71–72]) that for almost all graphs G, the maximum vertex-degree of G is equal to

$$\tfrac{1}{2}(n-1) + \tfrac{1}{2}(n \log n)^{1/2} + o(n \log n)^{1/2}$$

and hence that G almost surely has a vertex of degree at least

$$\tfrac{1}{2}(n-1) + \tfrac{1}{2}\{(1-\epsilon)\, n \log n\}^{1/2}, \tag{2}$$

for any given $\epsilon > 0$.

To prove the lemma for labeled graphs, it now suffices to prove that if

$$k > \tfrac{1}{2}(n-1) + \tfrac{1}{2}\{(1-\epsilon)\, n \log n\}^{1/2}, \tag{3}$$

then G almost surely does not have two vertices both of degree k.

But the probability that two given vertices both have degree k is $(1 + o(1))\binom{n-1}{k}^2 2^{2-2n}$, and so it is enough to prove that

$$n^2 \sum{}' \binom{n-1}{k}^2 2^{2-2n} \to 0, \qquad \text{as } n \to \infty; \tag{4}$$

where the prime indicates that the summation extends only over those values of k satisfying (3). But (4) follows from (1) by a simple calculation, and so the lemma is proved for labeled graphs.

To deduce the corresponding result for unlabeled graphs is now a simple matter. Since almost all unlabeled graphs with n vertices can be labeled in $n!$ ways, and since (by a result of Pólya) the number of unlabeled graphs with n

vertices is asymptotically equal to $(n!)^{-1}$ times the number of labeled graphs with n vertices (that is, $2^{n(n-1)/2}$), it follows that every property which is true for almost all labeled graphs is simultaneously true for almost all unlabeled graphs, and conversely. (This is the "Metatheorem" in [5, Chapter 9], to which the reader is referred for a further discussion of this type of argument.) The result for unlabeled graphs therefore follows from the result for labeled graphs, thereby completing the proof of the lemma.

To deduce the theorem from the lemma, it is sufficient to prove that if a graph G has only one vertex of maximum degree, then G is necessarily of class one. But this follows immediately from a result of Vizing [9] which states that every graph of class two has at least three vertices of maximum degree. This completes the proof of the theorem.

We conclude this note with the following corollary:

COROLLARY. (i) *Almost all connected graphs are of class one.*

(ii) *Almost all 2-connected graphs are of class one.*

(iii) *Almost all Hamiltonian graphs are of class one.*

Proof. (i) follows since almost all graphs are connected [5, p. 206].

(ii) follows since almost all graphs are 2-connected [5, p. 207].

(iii) follows since almost all graphs are Hamiltonian [7].

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