ON THE ASYMPTOTIC DENSITY OF SETS OF INTEGERS. II

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1. *Introduction and statement of results*

1.1. The object of this note is to answer (in the affirmative) the first two open problems stated at the end of a previous paper by one of us **[5]**. (For the sake of clarity we first reproduce here part of the introduction of **[5]**, to which we refer for notation and more details.)

1.2. Let $\mathbb{N}$ (resp. $\mathbb{N}^*$) denote the set of all non-negative (resp. positive) rational integers. Let A and B be a pair of direct factors of $\mathbb{N}^*$, that is, two subsets of $\mathbb{N}^*$ such that every $n \in \mathbb{N}^*$ can be written *uniquely* as

$$n = ab \qquad (a \in A,\ b \in B). \tag{1}$$

This is trivially equivalent to a decomposition of the Riemann zeta function as

$$\zeta(s) = U(s)\,V(s), \qquad U(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad V(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

where a_n and b_n equal 0 or 1, the set A (resp. B) appearing as the set of those n such that $a_n = 1$ (resp. $b_n = 1$).

It seems a rather difficult problem to describe explicitly the *structure* of all such direct factors, although the theorem demonstrated in **[5]** and our present Theorem 1 shed some light on the situation by proving the existence of their asymptotic densities. (The corresponding additive problem for $\mathbb{N}$ is much easier, and has been completely settled by de Bruijn **[1]** in 1956 and by Long **[4]** in 1967.)

Although this structure problem for A and B, as pointed out in **[5**; §9.3**]**, essentially reduces to an algebraic and combinatorial problem, it has close connections with sets of multiples (see **[3**; Chapter V**]**), and also has " multiplicative " features, in the (somewhat extended) sense of multiplicative functions. These two interesting aspects can already be foreseen from (respectively) the proof of our present Theorem 2 and the new proof of our result given in **[2]**, but will be discussed further elsewhere.

1.3. Let $S \subset \mathbb{N}^*$. As in **[5]**, denote by $d_*(S)$ [resp. $d^*(S)$] the lower asymptotic (resp. upper asymptotic) density of S, and by $d(S)$ [resp. $\delta(S)$] the asymptotic (resp. logarithmic) density of S whenever it exists. Let

$$H(S) = \sum_{n \in S} \frac{1}{n}.$$

Obviously $\max\big(H(A), H(B)\big) = \infty$, but $\min\big(H(A), H(B)\big)$ can be finite or infinite. We might refer to the case $\min\big(H(A), H(B)\big) < \infty$ as the *convergent case* and to $\min\big(H(A), H(B)\big) = \infty$ as the *divergent case.*

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It was proved in [5] that, in the convergent case, the sets A and B have asymptotic densities:

$$d(A) = 1/H(B) \qquad \text{and} \qquad d(B) = 1/H(A). \tag{2}$$

Our first theorem states that this is also true in the divergent case:

THEOREM 1. *The direct factors A and B have asymptotic densities in the divergent case $H(A) = H(B) = \infty$ as well (and (2) remains valid): $d(A) = d(B) = 0$.*

1.4. Our second theorem answers the second open problem stated at the end of [5]. (N.B. *Throughout this paper, p denotes a prime*):

THEOREM 2. *In the divergent case $H(A) = H(B) = \infty$, we have*

$$\sum_{p \in A} \frac{1}{p} = \sum_{p \in B} \frac{1}{p} = \infty.$$

2. *Proof of Theorem* 1

2.1. The proof is based on the following lemma:

LEMMA (Davenport–Erdős). *Let S be any (non-empty) subset of $\mathbb{N}^*$, and $\mathcal{M}(S)$ its "set of multiples", that is, the set of those elements of $\mathbb{N}^*$ which are divisible by at least one element of S. Then $\mathcal{M}(S)$ has logarithmic density, and $\delta(\mathcal{M}(S)) = d_*(\mathcal{M}(S))$.*

This is Theorem 12 of Chapter V of Halberstam and Roth [3].

2.2. To prove Theorem 1, clearly it suffices to prove that $d(A) = 0$, subject to the assumption

$$H(B) = \infty. \tag{3}$$

If $n \in \mathbb{N}^*$, let $\beta(n)$ be the largest divisor of n contained in B. (Clearly $\beta(n)$ exists, since $1 \in B$). Whenever $T > 0$, let

$$\mathbb{N}_T = \{n \in \mathbb{N}^* : \beta(n) < T\} \qquad \text{and} \qquad A_T = A \cap \mathbb{N}_T. \tag{4}$$

Then the *complement* $\tilde{\mathbb{N}}_T$ of $\mathbb{N}_T$ in $\mathbb{N}^*$ is $\mathcal{M}(S_T)$ where $S_T = B \cap [T, +\infty)$, hence the lemma implies that $\tilde{\mathbb{N}}_T$ and $\mathbb{N}_T$ have logarithmic densities, and

$$d^*(\mathbb{N}_T) = \delta(\mathbb{N}_T) \qquad [\text{since } d_*(\tilde{\mathbb{N}}_T) = \delta(\tilde{\mathbb{N}}_T)]. \tag{5}$$

Whenever $b \in B$, let $bA = \{k : k = ba \text{ with } a \in A\}$. Then, by (1) and (4),

$$\mathbb{N}_T \subset \bigcup_{b < T} bA. \tag{6}$$

Now, since (3) implies that $\delta(A) = 0$ (see [5; Lemma 1]), it follows from (6) that $\delta(\mathbb{N}_T) = 0$. By (5), this implies $d(\mathbb{N}_T) = 0$. Hence, in view of the second inequality (4), we obtain

$$d(A_T) = 0. \tag{7}$$

Let $\tilde{A}_T = A \cap \tilde{\mathbb{N}}_T$. Then A_T and $\tilde{A}_T$ are complements of each other in A. Therefore, whenever $x > 0$,

$$\sum_{a \leqslant x,\, a \in A} 1 = \sum_{a \leqslant x,\, a \in A_T} 1 + \sum_{a \leqslant x,\, a \in \tilde{A}_T} 1. \tag{8}$$

By (1) (and the uniqueness condition) the integers $a/\beta(a)$ (with $a \in A$) are pairwise distinct. Also, if $a \in \tilde{A}_T$, then $a/\beta(a) \leqslant a/T$. Therefore the second sum in the right side of (8) is $\leqslant x/T$. Thus, dividing both sides of (8) by x and letting successively $x \to \infty$ and $T \to \infty$, we deduce from (7) and (8) that $d(A) = 0$, as required.

2.3. *Remark.* One may observe that the method of the above proof is not applicable to the convergent case, which was settled in [5] by an entirely different method (itself not applicable to the divergent case). But we point out that the new "multiplicative" method of [2] is applicable to both cases.

3. *Proof of Theorem* 2

By contradiction. Suppose $\sum_{p \in B} p^{-1} < \infty$. We will deduce that $H(B) < \infty$.

Let L (resp. M) denote the set of those positive integers all of whose prime divisors are in A (resp. in B). Then our assumption implies that $H(M) < \infty$. Clearly

$$H(B) = \sum_{m \in M} \frac{1}{m} \sum_{l \in L,\, lm \in B} \frac{1}{l},$$

so that it suffices to show that

$$\sum_{lm \in B,\, l \in L} \frac{1}{l} \leqslant C \tag{9}$$

where C is *independent of* m. Let $x > 0$ be large (say $x \geqslant 10$). We have

$$\sum_{\substack{lm \in B \\ l \in L,\, x < l \leqslant 2x}} 1 \leqslant \Sigma_1 + \Sigma_2 \tag{10}$$

where

$$\Sigma_1 = \sum_{\substack{x < r \leqslant 2x \\ \forall p \mid r,\, p \leqslant (\log x)^2}} 1 \qquad \text{and} \qquad \Sigma_2 = \sum_{\substack{lm \in B \\ l \in L,\, l \leqslant 2x \\ \exists p \mid l \text{ with } p > (\log x)^2}} 1.$$

In the sequel we use the Vinogradov symbol $\ll$, the implied constants being *absolute* (throughout). It is well known that

$$\Sigma_1 \ll x (\log x)^{-2}. \tag{11}$$

One way to see this is to observe that for the integers r counted in Σ_1 one has $(\log x)^{2\Omega(r)} > x$ (where $\Omega(r)$ is the *total* number of prime divisors of r, that is, counting multiplicity), so that $\Omega(r) > (\log x)/(2 \log \log x)$, whence

$$\Sigma_1 \leqslant 2^{-(\log x)/(2 \log \log x)} \sum_{r \leqslant 2x} 2^{\Omega(r)}.$$

This implies (11), since $\sum_{r \leqslant 2x} 2^{\Omega(r)} \ll x \log x$.

To estimate Σ_2, observe that if l_1 and l_2 are two of the integers l counted in Σ_2, ml_1 and ml_2 belong to B, and p_1 (resp. p_2) is the largest prime divisor of l_1 (resp. l_2), then (by (1), since $p_1 \in A$ and $p_2 \in A$) the equality $ml_1/p_1 = ml_2/p_2$ can only occur if $p_1 = p_2$ and $l_1 = l_2$. Thus, to two different integers l counted in Σ_2 there correspond two different integers ml/p contained in the interval $[1, 2x(\log x)^{-2}]$. Hence

$$\Sigma_2 \leqslant 2x(\log x)^{-2}. \tag{12}$$

By (10), (11) and (12),

$$\sum_{lm \in B,\, l \in L} \frac{1}{l} \leqslant \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} \sum_{\substack{lm \in B \\ l \in L,\, 2^{k-1} < l \leqslant 2^k}} 1$$

$$\ll \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}.$$

This proves (9), and therefore Theorem 2.

References

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