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On the number of prime factors of integers

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1. Throughout this paper, we use the following notations:

$c_1, c_2, \dots$ denote positive absolute constants. The number of elements of a finite set S is denoted by $|S|$. We write $p^\alpha \| n$ if $p^\alpha | n$ but not $p^{\alpha+1} \nmid n$;

$d(n)$ denotes the number of positive divisors of n: $d(n)=\sum_{d|n} 1$;

$\nu(n)$ denotes the number of prime factors of n counted with multiplicities: $\nu(n)=\sum_{p^\alpha \| n} \alpha$;

$\varkappa(n)$ denotes the number of distinct prime factors of n: $\varkappa(n)=\sum_{p|n} 1$;

$\pi_i(x)$ denotes the number of integers n satisfying $n \leqq x$ and $\nu(n)=i$;

$\varrho_i(x)$ denotes the number of integers n satisfying $n \leqq x$ and $\varkappa(n)=i$;

$P(n)$ and $p(n)$ denote the greatest and least prime factor of n, respectively.

2. In [2], the authors asserted that for any $\omega>0$, there exists a constant $c_1=c_1(\omega)$ such that for all sufficiently large x and $1 \leqq i \leqq \omega \log\log x$, we have

$$\pi_i(x) < c_1(\omega) \frac{x}{\log x} \frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \omega \log\log x. \tag{1}$$

(There was also a missprint: $1 \leqq i \leqq \omega \log x$ was printed instead of $1 \leqq i \leqq \omega \log\log x$.) We attributed this theorem to Hardy and Ramanujan (referring to [4]), and we used it (with $\omega=100$) to prove that for all $\varepsilon>0$ and large k,

$$\sum_{0 \leqq i \leqq z \log\log k} \pi_i(k) < \frac{k}{(\log k)^{\varphi(z)-\varepsilon}} \tag{2}$$

and

$$\sum_{(1+z)\log\log k < i} \pi_i(k^2) < c_2 \frac{k^2}{(\log k)^{\varphi(z)-\varepsilon}} \tag{3}$$

(see (25) and (33) in [2]) where

$$\varphi(x) = 1 + x \log x - x \tag{4}$$

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and z denotes the single real root of the equation $\varphi(x)=\varphi(1+x)$; a simple computation shows that

$$0{,}54 < z < 0{,}55. \tag{5}$$

The first author used (1) also in [1], in order to prove that for all $\varepsilon>0$ and $x>x_0(\varepsilon)$, we have

$$\sum_{i>\frac{\log\log x}{\log 2}} \pi_i(x) < \frac{x}{(\log x)^{1-\varepsilon}}(e\log 2)^{\frac{\log\log x}{\log 2}} \tag{6}$$

(see (3) in [1]).

However, (1) is *false* in the form stated above (as K. K. Norton pointed out it in a letter written to the authors). In fact, Hardy and Ramanujan proved (1) with $\varrho_i(x)$ in place of $\pi_i(x)$:

$$\varrho_i(x) < c_3(\omega)\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \omega\log\log x. \tag{7}$$

Furthermore, they proved in [4] that for all $\delta>0$, (1) holds with $\omega=\frac{10}{9}-\delta$:

$$\pi_i(x) < \frac{c_4}{\delta}\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad 1 \leqq i \leqq \left(\frac{10}{9}-\delta\right)\log\log x.$$

Sathe [6] extended this result by proving that for all $\delta>0$, we have

$$\pi_i(x) < c_5(\delta)\frac{x}{\log x}\frac{(\log\log x)^{i-1}}{(i-1)!} \quad \text{for} \quad x \geqq 3,\ 1 \leqq i \leqq (2-\delta)\log\log x. \tag{8}$$

Selberg [7] gave a different proof of Sathe's result and showed that for all $\delta>0$, we have

$$\pi_i(x) \sim c_6(x\log x)2^{-i} \quad \text{for} \quad (2+\delta)\log\log x \leqq i \leqq c_7\log\log x. \tag{9}$$

This result shows that (1) *does not hold* for $i\geqq(2+\delta)\log\log x$ (while we used (1) with $\omega=100$ in order to prove (3)); in fact, the right hand side of (8) is greater than the right hand side of (1). (See also [3] and [5].)

The aim of this paper is to correct the papers [1] and [2] by deducing an upper estimate for $\pi_i(x)$ which is slightly weaker than the best possible but which holds for *all* i:

Theorem 1. *For all* $\delta>0$, *we have*

$$\pi_i(x) < \begin{cases} c_8(\delta)\dfrac{x}{\log x}\dfrac{(\log\log x)^{i-1}}{(i-1)!} & \textit{for} \quad 1 \leqq i \leqq (2-\delta)\log\log x \\[2mm] c_9 i^4\dfrac{x\log x}{2^i} & \textit{for} \quad 1 \leqq i \end{cases} \tag{10}$$

and for all $x\geqq 3$.

Sections 3 and 4 will be devoted to the proof of this theorem. In Section 5, we prove two corollaries of Theorem 1. In Section 6, we show that in fact, (2), (3) and (6) can be deduced easily from these corollaries.

3. In order to prove Theorem 1, we need two lemmas.

Lemma 1. *For all non-negative real numbers Z and A, let $G(Z, A)$ denote the number of positive integers n satisfying $n \leqq Z$ and $\varkappa(n) \geqq A$. Then there exists an absolute constant c_{10} such that for all Z and A, we have*

$$G(Z, A) \leqq c_{10} 2^{-A} Z \log(Z+2). \tag{11}$$

Proof. If $\varkappa(n) \geqq A$ then we have

$$d(n) = \prod_{p^\alpha \| n} d(p^\alpha) \geqq \prod_{p^\alpha \| n} 2 = \prod_{p | n} 2 = 2^{\varkappa(n)} \geqq 2^A$$

thus

$$\sum_{n \leqq Z} d(n) \geqq \sum_{\substack{n \leqq Z \\ \varkappa(n) \geqq A}} d(n) \geqq \sum_{\substack{n \leqq Z \\ \varkappa(n) \geqq A}} 2^A = 2^A G(Z, A). \tag{12}$$

On the other hand, it is well-known that for $Z \to +\infty$,

$$\sum_{n \leqq Z} d(n) \sim Z \log Z$$

thus for all $Z (\geqq 0)$, we have

$$\sum_{n \leqq Z} d(n) \leqq c_{11} Z \log(Z+2). \tag{13}$$

(12) and (13) yield (11).

Lemma 2. *For a positive real number y and a non-negative integer α, write*

$$F(y, \alpha) = \sum_{\substack{P(n) \leqq y \\ \nu(n) = \alpha}} \frac{1}{n}.$$

Then there exists an absolute constant c_{12} such that for $y \geqq 2$ and all α, we have

$$F(y, \alpha) \leqq c_{12} (\alpha+1) 2^{-\alpha} (\log y)^2. \tag{14}$$

Proof. Let us write

$$f(t) = \prod_{p \leqq y} \sum_{k=0}^{\alpha} \left(\frac{t}{p}\right)^k = \sum_{i=0}^{m} a_i t^i$$

(where $m = \alpha \pi(y)$). Then obviously, all the coefficients a_i are non-negative and we have $F(y, \alpha) = a_\alpha$. Thus

$$f(2) = \sum_{i=0}^{m} a_i 2^i \geqq a_\alpha 2^\alpha = 2^\alpha F(y, \alpha). \tag{15}$$

On the other hand, by the definition of $f(t)$ and using the Mertens-formula, we obtain that

$$(16)\quad f(2) = \prod_{p\le y}\sum_{k=0}^{\alpha}\left(\frac{2}{p}\right)^k = (\alpha+1)\prod_{3\le p\le y}\sum_{k=0}^{\alpha}\left(\frac{2}{p}\right)^k \le (\alpha+1)\prod_{3\le p\le y}\sum_{k=0}^{+\infty}\left(\frac{2}{p}\right)^k =$$

$$= (\alpha+1)\prod_{3\le p\le y}\frac{1}{1-\frac{2}{p}} = (\alpha+1)\left(\prod_{3\le p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{3\le p\le y}\left(1-\frac{1}{p}\right)^2\left(1-\frac{2}{p}\right)^{-1} =$$

$$= (\alpha+1)\left(\prod_{3\le p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{3\le p\le y}\frac{(p-1)^2}{(p-2)p} < (\alpha+1)\left(\prod_{p\le y}\frac{1}{1-\frac{1}{p}}\right)^2\prod_{n=3}^{+\infty}\frac{(n-1)^2}{(n-2)n} <$$

$$< (\alpha+1)(c_{13}\log y)^2\cdot 2 = c_{14}(\alpha+1)(\log y)^2.$$

(15) and (16) yield (14).

4. Completion of the proof of Theorem 1. If $1\le i\le(2-\delta)\log\log x$ then the first inequality in (10) holds by the Sathe—Selberg formula (8). Thus it is sufficient to prove that

$$(17)\quad \pi_i(x) < c_9 i^4\frac{x\log x}{2^i}\quad\text{for all}\quad x\ge 3\ \text{and}\ 1\le i.$$

Let us fix a real number $x\ge 3$ and a positive integer i. Let S denote the set of the positive integers n satisfying $n\le x$ and $v(n)=i$ (so that $\pi_i(x)=|S|$). Furthermore, let S_1 denote the set of the positive integers n for which $n\le x$ and there exists a positive integer t such that $t>2^i$ and t^2/n. Write $S_2=S-S_1$. Then we have

$$(18)\quad S\subset S_1\cup S_2$$

and

$$(19)\quad S_1\cap S_2=\emptyset.$$

(18) implies that

$$(20)\quad \pi_i(x)=|S|\le|S_1|+|S_2|.$$

Obviously, we have

$$(21)\quad |S_1| = \sum_{t=2^i+1}^{+\infty}\sum_{\substack{n\le x\\ t^2|n}}1 = \sum_{t=2^i+1}^{+\infty}\left[\frac{x}{t^2}\right] < x\sum_{t=2^i+1}^{+\infty}\frac{1}{t^2} <$$

$$< x\sum_{t=2^i+1}^{+\infty}\frac{1}{(t-1)t} = x\sum_{t=2^i+1}^{+\infty}\left(\frac{1}{t-1}-\frac{1}{t}\right) = \frac{x}{2^i}.$$

In order to estimate $|S_2|$, let us write all $n\in S$ in the form $n=n_1n_2$ where

$$P(n_1) \leqq 2^i, \quad p(n_2) > 2^i. \tag{22}$$

If there exists a prime number p such that $p>2^i$ and $p^2|n_2$ then by the definition of S_1, we have $n\in S_1$ thus by (19), $n\notin S_2$. In other words, for all $n\in S_2$, n_2 is squarefree thus

$$\varkappa(n_2) = \nu(n_2) = \nu(n)-\nu(n_1) = i-\nu(n_1). \tag{23}$$

If $n\in S_2$ and we put $\nu(n_1)=\alpha$ then by (23),

$$0 \leqq \alpha = i-\varkappa(n_2) \leqq i. \tag{24}$$

By (22), (23) and (24), we have

$$|S_2| = \sum_{\substack{n_1n_2\leqq x\\ P(n_1)\leqq 2^i,\ p(n_2)>2^i\\ \nu(n_1)+\varkappa(n_2)=i}} 1 = \sum_{\alpha=0}^{i} \sum_{\substack{n_1\leqq x\\ P(n_1)\leqq 2^i\\ \nu(n_1)=\alpha}} \sum_{\substack{n_2\leqq x/n_1\\ p(n_2)>2^i\\ \varkappa(n_2)=i-\alpha}} 1 \leqq \sum_{\alpha=0}^{i} \sum_{\substack{n_1=x\\ P(n_1)\leqq 2^i\\ \nu(n_1)=\alpha}} \sum_{\substack{n_2\leqq x/n_1\\ \varkappa(n_2)\geqq i-\alpha}} 1.$$

In order to estimate the inner sum, we use Lemma 1 with $Z=x/n_1$ and $A=i-\alpha$ We obtain that

$$|S_2| \leqq \sum_{\alpha=0}^{i} \sum_{\substack{n_1\leqq x\\ P(n_1)\leqq 2^i\\ \nu(n_1)=\alpha}} c_{10}2^{-i+\alpha}\frac{x}{n_1}\log\left(\frac{x}{n_1}+2\right) <$$

$$< c_{10}\sum_{\alpha=0}^{i} \sum_{\substack{n_1\leqq x\\ P(n_1)\leqq 2^i\\ \nu(n_1)=\alpha}} 2^{-i+\alpha}\frac{x}{n_1}\log(x+2) <$$

$$< c_{15}\sum_{\alpha=0}^{i} 2^{-i+\alpha}x\log x \sum_{\substack{P(n_1)\leqq 2^i\\ \nu(n_1)=\alpha}} \frac{1}{n_1} = c_{15}\sum_{\alpha=0}^{i} 2^{-i+\alpha}x\log x F(2^i,\alpha)$$

where $F(y,\alpha)$ is defined in Lemma 2. By using Lemma 2, we obtain that

$$|S_2| < c_{15}\sum_{\alpha=0}^{i} 2^{-i+\alpha}x\log x\cdot c_{12}(\alpha+1)2^{-\alpha}(\log 2^i)^2 < \tag{25}$$

$$< c_{16}i^2 2^{-i}x\log x\sum_{\alpha=0}^{i}(\alpha+1) < c_{17}i^4 2^{-i}x\log x.$$

(20), (21) and (25) yield that

$$\pi_i(x) \leqq |S_1|+|S_2| < 2^{-i}x+c_{17}i^4 2^{-i}x\log x <$$

$$< c_{18}i^4 2^{-i}x\log x$$

which proves (17) and this completes the proof of Theorem 1.

5. It can be deduced easily from Theorem 1 that

Corollary 1. *If*

$$\delta > 0 \quad and \quad 1 < y < 2-\delta \tag{26}$$

then we have

$$\sum_{i \geqq j} \pi_i(x) < c_{19}(\delta) \frac{1}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \tag{27}$$

$$for \quad y \leqq j/\log\log x \leqq 2-\delta, \quad x > x_0(y, \delta);$$

furthermore, we have

$$\sum_{i \geqq j} \pi_i(x) < c_{20}\, j^4 \frac{x \log x}{2^j} \quad for\ all\ j\ and \quad x \geqq 3. \tag{28}$$

Proof. First we prove (28). By Theorem 1, we have

$$\sum_{i \geqq j} \pi_i(x) < \sum_{i \geqq j} c_9 i^4 \frac{x \log x}{2^i} = c_9 x \log x \sum_{i \geqq j} \frac{i^4}{2^i}. \tag{29}$$

Obviously, for $i > i_0$ we have

$$\frac{(i+1)^4}{2^{i+1}} < \frac{2}{3} \frac{i^4}{2^i}$$

thus for $j \geqq i_0$,

$$\sum_{i \geqq j} \frac{i^4}{2^i} < \frac{j^4}{2^j} \sum_{t=0}^{+\infty} \left(\frac{2}{3}\right)^t = 3 \cdot \frac{j^4}{2^j}$$

hence

$$\sum_{i \geqq j} \frac{i^4}{2^i} < \max\left\{\sum_{i=1}^{+\infty} \frac{i^4}{2^i} \left(\max_{1 \leqq t \leqq i_0} 2^t t^{-4}\right), 3\right\} \cdot \frac{j^4}{2^j} = c_{21} \frac{j^4}{2^j} \tag{30}$$

for all j. (29) and (30) yield (28).

Now we prove (27). The function $\varphi(x) = 1 + x \log x - x$ is increasing for $x > 1$, thus writing

$$\eta = \eta(\delta) = \frac{\varphi(2) - \varphi(2-\delta)}{2 \log 2},$$

we have $0 < \eta$. Thus Theorem 1 and (28) yield (with respect to (26)) that for

$x > x_0(y, \delta)$,

(31)
$$\sum_{i \geqq j} \pi_i(x) = \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} \pi_i(x) + \sum_{[(2-\eta)\log\log x]+1 \leqq i} \pi_i(x) <$$

$$< \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} c_8(\eta(\delta)) \frac{x}{\log x} \frac{(\log\log x)^{i-1}}{(i-1)!} +$$

$$+ c_{20} \frac{([(2-\eta)\log\log x]+1)^4}{2^{[(2-\eta)\log\log x]+1}} x \log x <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{j \leqq i \leqq [(2-\eta)\log\log x]} \frac{(\log\log x)^{i-j}}{j(j+1)\dots(i-1)} +$$

$$+ c_{21} \frac{(2\log\log x)^4}{2^{(2-\eta)\log\log x}} x \log x <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{j \leqq i} \left(\frac{\log\log x}{j}\right)^{i-j} +$$

$$+ c_{23} x \frac{(\log\log x)^4}{(\log x)^{(2-2\eta)\log 2 + \eta\log 2 - 1}} <$$

$$< c_{22}(\delta) \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} \sum_{t=0}^{+\infty} y^{-t} + \frac{x}{(\log x)^{(2-2\eta)\log 2 - 1 + \eta/2}} =$$

$$= c_{22}(\delta) \frac{y}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} + \frac{x}{(\log x)^{\varphi(2-\delta)+\eta/2}} <$$

$$< c_{23}(\delta) \frac{1}{y-1} \frac{x}{\log x} \frac{(\log\log x)^{j-1}}{(j-1)!} + \frac{x}{(\log x)^{\varphi(2-\delta)+\eta/2}}.$$

By the Stirling-formula, we have

(32)
$$\frac{1}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} = \frac{1}{\log x} \frac{k}{\log\log x} \frac{(\log\log x)^k}{k!} \sim$$

$$\sim c_{24} \frac{1}{\log x} \frac{k}{\log\log x} \left(\frac{e\log\log x}{k}\right)^k k^{-1/2} =$$

$$= c_{24} \frac{k^{1/2}}{\log\log x} (\log x)^{-1+(1-\log(k/\log\log x))k/\log\log x} =$$

$$= c_{24} \frac{k^{1/2}}{\log\log x} (\log x)^{-\varphi(k/\log\log x)} \quad \text{for} \quad x \geqq 3 \text{ and } k \to +\infty.$$

Thus with respect to (26), for $y \leqq j/\log\log x \leqq 2-\delta$, $x > x_1(y, \delta, \eta) = x_1(y, \delta, \eta(\delta)) = x_2(y, \delta)$ we have

$$c_{23}(\delta)\frac{1}{y-1}\frac{x}{\log x}\frac{(\log\log x)^{j-1}}{(j-1)!} > \tag{33}$$

$$> c_{25}(y, \delta)x(\log\log x)^{-1/2}(\log x)^{-\varphi(j/\log\log x)} >$$

$$> c_{25}(y, \delta)x(\log\log x)^{-1/2}(\log x)^{-\varphi(2-\delta)} > x(\log x)^{-\varphi(2-\delta)-\eta/2}.$$

(31) and (33) yield (27) and this completes the proof of Corollary 1.

Corollary 2. *If* $y > 1$ *and* $\varepsilon > 0$ *then for* $y\log\log x \leqq j$, $x > x_0(\varepsilon)$ *we have*

$$\sum_{i \geqq j} \pi_i(x) < \begin{cases} \dfrac{x}{(\log x)^{\varphi(y)-\varepsilon}} & \text{if} \quad 1 < y < 2 \\ \dfrac{x}{(\log x)^{(1-\varepsilon)y\log 2 - 1}} & \text{if} \quad 2 \leqq y. \end{cases} \tag{34}$$

Proof. If $1 + (\log x)^{-\varepsilon/2} < y \leqq 2 - \varepsilon/2$ then (27) (with $\varepsilon/2$ in place of δ) and (32) yield that

$$\sum_{i \geqq j} \pi_i(x) < c_{19}(\varepsilon/2)\frac{1}{y-1}x\frac{1}{\log x}\frac{(\log\log x)^{j-1}}{(j-1)!} <$$

$$< c_{26}(\varepsilon)(\log x)^{\varepsilon/2}x\frac{j^{1/2}}{\log\log x}(\log x)^{-\varphi(j/\log\log x)} < x(\log x)^{-\varphi(y)+\varepsilon}$$

for $x > x_1(\varepsilon)$, while if $1 < y \leqq 1 + (\log x)^{-\varepsilon/2}$ then (34) holds trivially for $x > x_2(\varepsilon)$ since we have

$$\lim_{t \to 1} \varphi(t) = \varphi(1) = 0$$

and, for all j,

$$\sum_{i \geqq j} \pi_i(x) \leqq x.$$

If $2 - \varepsilon/2 < y$ then by (28), we have

$$\sum_{i \geqq j} \pi_i(x) < c_{20}j^4\frac{x\log x}{2^j} < \frac{x\log x}{2^{(1-\varepsilon/4)j}} \leqq \tag{35}$$

$$\leqq \frac{x\log x}{2^{(1-\varepsilon/4)y\log\log x}} = \frac{x}{(\log x)^{(1-\varepsilon/4)y\log 2-1}}$$

for $x > x_3(\varepsilon)$. If $2 \leqq y$ then this yields (34). Finally, if $2-\varepsilon/2 < y \leqq 2$ then we obtain from (35) that

$$\sum_{i \geqq j} \pi_i(x) < \frac{x}{(\log x)^{(1-\varepsilon/4)y\log 2-1}} =$$

$$= \frac{x}{(\log x)^{(1+y\log y-y)+y(\log 2-\log y)-(2-y)-(\varepsilon y \log 2)/4}} =$$

$$= \frac{x}{(\log x)^{\varphi(y)+y(\log 2-\log y)-(2-y)-(\varepsilon y\log 2)/4}} <$$

$$< \frac{x}{(\log x)^{\varphi(y)-\varepsilon/2-\varepsilon/2}} = \frac{x}{(\log x)^{\varphi(y)-\varepsilon}}$$

which completes the proof of (34).

6. In this section, we correct the proofs of (2), (3) and (6). In the proof of (2), we used (1) only for $i \leqq z \log\log k$. Thus we need (1) with $\omega = z < 0{,}55 < 10/9$ but in this case, (1) holds by the classical Hardy—Ramanujan result.

Now we are going to prove (3). Let $\delta = \delta(\varepsilon)$ denote a small positive number such that we have

$$\varphi(1+z-\delta) > \varphi(1+z) - \varepsilon/2 = \varphi(z) - \varepsilon/2$$

(note that $\varphi(1+z) = \varphi(z)$ by the definition of z). By using Corollary 2 with $1+z-\delta$, $\varepsilon/2$, k^2 and $[(1+z-\delta)\log\log k^2]+1$ in place of y, ε, x and j, respectively, we obtain that

$$\sum_{(1+z)\log\log k < i} \pi_i(k^2) < \sum_{[(1+z-\delta)\log\log k^2]+1 \leqq i} \pi_i(k^2) <$$

$$< \frac{k^2}{(\log k^2)^{\varphi(1+z-\delta)-\varepsilon/2}} < \frac{k^2}{(\log k)^{\varphi(z)-\varepsilon/2-\varepsilon/2}} = \frac{k^2}{(\log k)^{\varphi(z)-\varepsilon}}$$

for $k > k_0(\varepsilon)$ which proves (3).

Finally, note that the right hand side of (6) can be rewritten in the form

$$\frac{x}{(\log x)^{1-\varepsilon}} (e \log 2)^{\frac{\log\log x}{\log 2}} = \frac{x}{(\log x)^{1-\varepsilon-(1+\log\log 2)/\log 2}} = \frac{x}{(\log x)^{\varphi(1/\log 2)-\varepsilon}}$$

so that (6) can be obtained from Corollary 2 with $1/\log 2 \; (<2)$ in place of y.

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