PROBLEMS AND RESULTS ON RAMSEY-TURÁN TYPE THEOREMS

Preliminary Report

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We report on some work which occurred to us more than a decade ago. We seriously started to work on these problems at the international meeting on combinatorics in Calgary in 1969 [9]. In the last few weeks we formulated several new problems. We clearly had no time as yet to look at them carefully so it may turn out that some of the conjectures are false or some of our problems will be easy to solve.

We will use the notations of our paper [7]: "Some remarks on Ramsey's and Turán's theorem," Coll. Math. Soc. János Bólyai, Combinatorial Theory and Its Applications. Balatonfüred, Hungary, 1969, 395-404. To avoid the necessity of the reader having to consult [7], we will always explain a new notation the first time we use it.

In a forthcoming paper [8], we give detailed proofs of some of the results stated here.

1. Let $f(n;k_1,\ldots,k_r)$ be the largest integer for which there is a graph $G(n;e)$ of n vertices and $e=f(n;k_1,\ldots,k_r)$ edges where the largest independent set of $G(n;e)$ is less than k_r (i.e., any set of k_r vertices of $G(n;e)$ contains an edge) and the edges of our $G(n;e)$ can be coloured by $r-1$ colors so that the i^{th} color does not contain a $K(k_i)$ (a complete graph of k_i vertices).

The reader will probably recognize the connections with the classical theorems of Ramsey and Turán (for details see [7].

Trivially, $f(n;3,t) \leq \frac{nt}{2}$. In [7] we prove that if $k_1=2r+1, k_2=o(n)$ then

(1) $f(n;2r+1,o(n))=\frac{1}{2}(1-\frac{1}{r})n^2(1+o(1))$.

PROC. WEST COAST CONFERENCE ON COMBINATORICS,
GRAPH THEORY, AND COMPUTING, pp. 17-23.

Bollobás, Szemerédi, Erdös [2], [10] proved

$$(2) \quad f(n;4,o(n)) = \frac{n^2}{8}(1+o(1))$$

We could not get an asymptotic formula for $f(n;2r,o(n))$ if $r > 2$. We can easily show

$$(3) \quad \frac{2}{7}n^2 < f(n;6,o(n)) < (\frac{3}{10} + o(1))n^2$$

In [8], we prove

$$(4) \quad f(n;3,3,o(n)) = \frac{n^2}{4}(1+o(1)$$

and we conjecture

$$(5) \quad f(n;3,3,3,o(n)) = (\frac{2}{5} + o(1))n^2$$

Denote by $m_3(r)$ the largest integer for which one can color $K(m_3(r))$ by r colors so that none of the colors contains a monochromatic triangle. $m_3(2) = 5$, $m_3(3) = 16$ are well-known, but Folkman proved $m_3(4) \leq 64$. (A trivial induction gives $m_3(r+1) \leq (r+1)m_3(r)+1$ and Folkman's result shows that equality does not hold here.) The exact determination or even good estimation of $m_3(r)$ seems very difficult. It is not even known if $m_3(r)^{1/r} \to \infty$ is true.

We conjecture that

$$(6) \quad f(n;\underbrace{3,3,\ldots\ldots,3}_{r+1 \text{ times}},o(n)) = \frac{1}{2}(1 - \frac{1}{m_3(r)})\ n^2+o(n^2).$$

We prove (6) for $r=2$, (we obtain (5) for $r=3$). If (6) is true, then an easy argument gives that if we color the edges of $K(m(r)+1)$ by $r+1$ colors so that at most r colors are incident to every vertex then one of the colors must contain a triangle. This is easy to prove for $r=2$ and $r=3$, but so far we have had no success with $r=4$.

With Hajnal we also tried the following problem: Let $M_3(r)$ be the largest integer for which one can color the edges of a $K(M_3(r))$, so that there is no monochromatic triangle and every vertex is incident to at most r colors. Again a trivial induction shows

$$(7) \quad M_3(r+1) \leq (r+1)M_3(r)+1$$

Is there equality in (7) or alternatively is $M_3(r) = m_3(r)$?

Now we generalize $f(n;k_1,\dots,k_r)$ for more general graphs than complete graphs. Let $G_1,\dots,G_r$ be a set of r graphs. $f(n;G_1,\dots,G_r)$ is the largest integer for which one can color the edges of K_n by r colors so that the i^{th} color contains no G_i and the union of the first $r-1$ colors has $f(n;G_1,\dots,G_r)$ edges. In our theorems and problems G_r will be complete (it will play the role of the independent set).

In [8], we prove that for every r

(8) $f(n;K(1,r,r),o(n)) = o(n^2)$

and

(9) $f(n;K(3,3,3),o(n)) = \frac{n^2}{4}(1+o(1))$

We can not decide whether for the case between these two:

(10) $f(n;K(2,3,3),o(n)) = o(n^2)$

is true. It will perhaps help the reader if we restate the conjecture (10) in "human" language. Let $G(n)$ be a graph having n vertices and more than cn^2 edges which does not contain a complete tripartite graph on two red, three blue, and three white vertices. Is it then true that our graph must contain an independent set of at least $\varepsilon_c n$ vertices when ε_c is an absolute constant which depends only on c?

We proved that if the vertex set of G can be decomposed into two classes neither of which spans a circuit in G then

(11) $f(n;G,o(n)) < \frac{n^2}{4}(1-\varepsilon_G)$

where ε_G depends only on G. If the set of vertices of G can be decomposed into three classes so that the first two span no circuit and the third is empty then $f(n;G,o(n)) = \frac{n^2}{4}(1+o(1))$.

What are the possible values of

(12) $\lim_{n\to\infty} f(n;G,o(n))/n^2$?

We feel that an answer to (12) may be fundamental in clearing up many related questions. The value of $f(n;G,o(n))$ certainly depends on the chromatic number of G. It also depends on the value of the smallest integer t for which one can decompose the vertex set of G into t classes so that none of the classes spans a circuit. What else does it depend on?

Determine $h(c)$ where

(13) $f(n;3,3,cn) = (h(c)+o(1))n^2$.

At first we thought that (13) would be easy, but we soon observed using the coloring of the edges of $K(16)$ by three colors none of which contain a triangle that probably

$$f(n;3,3,\tfrac{n}{8}) = (\tfrac{5}{16} + o(1))n^2$$

and now we are uncertain whether it will be easy to determine $h(c)$.

Before we discuss hypergraphs, we state three more problems. Let r be a fixed integer and $n \to \infty$. Assume that $G(n)$ contains no $K(4)$ and no $K(r,r,r)$. Is it true that if the largest independent set of G is $o(n)$ then the number of its edges is $o(n^2)$? In view of (2) and (9) this seems a fascinating and perhaps difficult question. Szemerédi felt that the answer is negative, but will not be easy to obtain.

We stated in the beginning the trivial formula

(14) $f(n;3,t) \leq tn/2$.

For which values of $t = t_n$ can we have equality in (14)? B. Andrásfai and P. Erdős [1] have some results here. In particular, let t_n^* be the smallest t_n for which there is equality in (14). Is it true that

$$t_n^* > n^{\frac{1}{2}+c}$$

for a certain absolute constant c?

Perhaps it will be fruitful to study the set of integers $S(n;k_1,\ldots,k_r)$ for which if $e \in S(n;k_1,\ldots,k_r)$ then there is a $G(n;e)$ for which the largest independent set is less than k_r and the edges of our $G(n;e)$ can be colored by $r-1$ colors so that the i^{th} color does not contain a $K(k_i)$. Our "old" $f(n;k_1,\ldots,k_r)$ is the largest element of $S(n;k_1,\ldots,k_r)$. A recent theorem of Ajtai, Komlós, and Szemerédi can be fitted into the study of $S(n;k_1,\dots k_r)$. (It is, of course, not at all certain that this will help.) Their surprising theorem states: Let $G(n;kn)$ be a graph of n vertices and kn edges which contains no triangle. Then our graph contains an independent set of size greater than $\frac{cn \log k}{k}$. Observe that if we do not assume that G has no triangle then it is easy to see that the largest independent set has size cn/k and that this is best possible. The "bonus" of the condition no triangle is thus the factor $\log k$. The authors incidentally show that apart from the value of c this is best possible. Their paper which has also surprising applications in number theory will soon appear.

Assume now that our $G(n;kn)$ contains no K_4 (or K_r). Can we state that the largest independent set has size $\frac{n}{k}\phi(K)$ where $\phi(k) \to \infty$ as $K \to \infty$? At the moment nothing is known.

2. Now we give a short discussion of the problems on hypergraphs. We restrict ourselves to uniform three-graphs. A famous and well-known problem of Turán states: Denote by $f(n,K^{(3)}(4))$ the smallest integer so that every three uniform hypergraph (i.e., triple system) on n elements and $f(n;K^{(3)}(4))$ triples contains a $K^{(3)}(4)$ (i.e., all the 4 triples of a set of 4 elements). It is known and easy to see that

$$\lim_{n\to\infty} f(n;K^{(3)}(9))\Big/\binom{n}{3} = \alpha > 0$$

exists, but the value of α is not known. We are going to prove in [8] that

$$(15)\qquad f(n;K^{(3)}(4),o(n)) = (\alpha+o(1))\binom{n}{3} .$$

In other words, there is a three-uniform hypergraph on n vertices and $(\alpha+o(1))\binom{n}{3}$ hyperedges which contains no $K^{(3)}(4)$ and the largest independent set of which is $o(n)$. We prove the related result if $K^{(3)}(4)$ is replaced by $G^{(3)}(4;3)$ (i.e., the 3-uniform hypergraph of 4 vertices and three triples.

These results seem to show that the extra condition: "the largest independent set has size $o(n)$" has no effect here. This might be surprising knowing what we proved in [7] for graphs:

$$\lim_{n \to \infty} \frac{f(n;r,o(n))}{f(n;r)} = c_r < 1$$

and that the conjectured extremal hypergraph has an independent set of size $\frac{n}{3}$. This is not entirely true. Denote by $G^{(3)}(5;4)$ the hypergraph having the vertices x,y,z_1,z_2,z_3 and the edges $(x,y,z_i), i=1,2,3$ and (z_1,z_2,z_3). Clearly $f(n;G(5)/G^{(3)}(5;4)) > cn^3$ and we easily prove in [8] that

$$f(n;G^{(3)}(5,4),o(n)) = o(n^3).$$

Here are two problems which we can not do. Let $G^{(3)}(7;11)$ be the hypergraph having the vertices $x;y_1,y_2,y_3;z_1,z_2,z_3$ and the 11 triples $(x,y_i,z_j,),(y_1,y_2,y_3),(z_1,z_2,z_3)$. Is it true that

$$(16) \quad f(n;G^{(3)}(7;11),o(n)) = o(n^3) \ ?$$

Perhaps

$$(17) \quad f(n;G^{(3)}(9;30),o(n)) = o(n^3)$$

also holds. (16) and (17) seem difficult, but we had no time to study them seriously.

3. Finally we make some remarks on the Ramsey functions. Let $r(k_1,\ldots,k_r)$ be the smallest integer t for which if we color the edges of $K(t)$ by r colors then for some $i,(i=1,\ldots,r)$ the i^{th} color contains a $K(k_i)$. We conjecture

$$(18) \quad \lim_{n \to \infty}(r(3,3,n))/(r(3,n)) \to \infty$$

and

$$(19) \quad \lim_{n \to \infty}(r(3,n+1)) - r(3,n)) \to \infty .$$

It is very surprising that (18) and (19) which seem trivial at first sight should cause serious difficulties. We further expect that

$$\lim_{n \to \infty} r(3,3,n)/n^2 \to \infty$$

and perhaps $r(3,3,n) > n^{3-\epsilon}$ for every $\epsilon > 0$ if $n > n_0(\epsilon)$.

It is known that

$$(20) \qquad \frac{cn^2}{(\log n)^2} < r(3,n) < \frac{cn^2 \log\log n}{\log n}$$

The lower bound in (20) was proved by P. Erdös [4] and the upper bound by Graver and Yackel [5]. For these results see also [6]. Ajtai, Komlós, and Szemerédi in their forthcoming paper replace the upper bound of (20) by a

$$\frac{cn^2}{\log n} .$$

An asymptotic formula for $r(3,n)$ is nowhere in sight at the moment.

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