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MORE RESULTS ON RAMSEY—TURÁN TYPE PROBLEMS

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The paper deals with common generalizations of classical results of Ramsey and Turán. The following is one of the main results. Assume $k\geqq 2$, $\varepsilon>0$, G_n is a sequence of graphs of n-vertices and at least $\frac{1}{2}\left(\frac{3k-5}{3k-2}+\varepsilon\right)n^2$ edges, and the size of the largest independent set in G_n is $o(n)$. Let H be any graph of arboricity at most k. Then there exists an n_0 such that all G_n with $n>n_0$ contain a copy of H. This result is best possible in case $H=K_{2k}$.

1. Introduction. Notation. Statement of the main results

In her paper [9] the third author raised a general scheme of new problems. These problems can be considered as common generalizations of the problems treated in the classical results of Ramsey and Turán. Since 1969 she and the first author have published a sequence of papers on the subject [5], [6], [4]. This work is a continuation of the above sequence.

We are going to define the Ramsey—Turán function $RT(...)$ below. Our main aim is to give reasonable estimates for this function in some special cases. However before doing this we have to say a few words about notation. We hope that in general these will be standard and self-explanatory, but we *do not* stick to the special notation used in the earlier papers mentioned above.

In what follows the letters k, l, m, n, r, s, t denote non-negative integers. We set $\mathbf{n}=\{0, 1, ..., n-1\}$. For arbitrary sets A, B let $[A]^n=\{X\subset A\colon |X|=n\}$, $[A]^{\geqq n}=\{X\subset A\colon |X|\geqq n\}$, etc. Further, let $[A, B]^{n,m}=\{X\subset A\cup B\colon |X\cap A|=$ $=n\wedge|X\cap B|=m\}$.

For an arbitrary sequence $k_0, ..., k_{r-1}$ the Ramsey function $R_s(k_0, ..., k_{r-1})$ equals to the minimal n such that for all s-partitions $[\mathbf{n+1}]^s=\bigcup_{i<r} E_i$ of length r of $\mathbf{n+1}$ there are $i<r$ and $A\subset\mathbf{n+1}$ such that $|A|\geqq k_i$ and A is homogeneous for the partition in the class E_i, i.e. $[A]^s\subset E_i$.

Definition 1.1. (*The Ramsey—Turán function* $RT(n; k_0, ..., k_{r-1}; l)$).

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Assume $n, k_0, \ldots, k_{r-1}, l$ are such that $n \leqq R_2(k_0, \ldots, k_{r-1}, l)$. Then there exists a largest integer e satisfying the following condition:

(1.2) There exists a graph $G=\langle V_G, E_G\rangle=\langle V, E\rangle$ with $|V|=n$, $|E|=e$ and there is a partition $E=\bigcup_{i<r} E_i$ of E such that there is no independent set of size l for G and none of the graphs $G_i=\langle V, E_i\rangle$, $i<r$ contains a complete k_i graph.

Let $RT(n; k_0, \ldots, k_{r-1}; l)=e$ for the above e.

Note that this is not the most general problem one can raise here. First of all the function just defined is really $RT_s(n; k_0, \ldots, k_{r-1}; l)$ for $s=2$, but we do not give here any results for hypergraphs corresponding to the cases $s>2$.

It should be also clear that it is just a convenience for us to give an a priori different role to the number l, and not to speak about 2-partitions of length $r+1$ of a set of size n. In our results the r'th class plays a special role.

There is one more remark in place here. Let $\tilde{RT}(n; k_0, \ldots, k_{r-1}; l)$ be the set of all those numbers e for which there is a graph $G=\langle V, E\rangle$ satisfying (1.2) with $|E|=e$.

The investigation of this set $\tilde{RT}(n; k_0, \ldots, k_{r-1}; l)$ in general could be quite interesting and relevant to several problems considered in the literature (e.g. size Ramsey numbers).

It is a quite intrigueing question if this set is always an interval. We have no counterexample to this statemant. However in the cases we are going to investigate $RT(n; k_0, \ldots, k_{r-1}; l)$ sufficiently characterizes $\tilde{RT}(n; k_0, \ldots, k_{r-1}; l)$ and we do not discuss this problem any further.

Finally we would like to state two obvious formulas showing explicitly the connection of the RT function and the Ramsey and Turán functions:

$$\{(n, k_0, \ldots, k_{r-1}, l)\colon RT(n; k_0, \ldots, k_{r-1}; l) \text{ is defined}\} =$$

$$\{(n, k_0, \ldots, k_{r-1}, l)\colon n \leqq R_2(k_0, \ldots, k_{r-1}, l)\}$$

and $RT(n; k+1; n+1)$ is the Turán number of the complete k-graph. Our main results in this paper concern the case $r=1$, and we start to discuss them now.

The classical result of P. Turán yields that for $k \geqq 2$

$$RT(n;\ k+1;\ n+1) = \frac{1}{2}\left(1-\frac{1}{k}\right) n^2(1+o(1)) \tag{1.3}$$

As to the function $RT(n; k; l)$ most of the known results are asymptotic estimates in case l is replaced by a function of n which is $o(n)$. We will continue in this tradition and we will freely use the symbol $RT(n; k; o(n))$.

The earliest result on the subject stated in [4] says that for $k \geqq 2$

$$RT(n;\ 2k-1;\ o(n)) = \frac{1}{2}\left(1-\frac{1}{k-1}\right) n^2(1+o(1)) \tag{1.4}$$

The case of even second entries turned out to be much harder. It was proved only much later in [1] and [10] that

$$RT(n, 4, o(n)) = \frac{n^2}{8}(1+o(1)) \tag{1.5}$$

[10] gives the upper estimate and [1] the counterexample.

One of the main aims of this paper is to generalize this and prove that for $k \geqq 2$

$$RT(n, 2k, o(n)) = \frac{1}{2}\left(\frac{3k-5}{3k-2}\right) n^2(1+o(1)) \tag{1.6}$$

The lower estimates are all obtained using the only important genuine example given in [1]. This will be done in Section 4.

First we give a technical definition to restate (1.4) and (1.6) in one formula

Definition 1.7. *For* $l \geqq 3$ *let*

$$a_l = \frac{1}{2}\frac{l-3}{l-1} = \frac{1}{2}\cdot\frac{3l-9}{3l-3} \quad \text{in case } l \text{ is odd,}$$

$$a_l = \frac{1}{2}\cdot\frac{3l-10}{3l-4} \quad \text{in case } l \text{ is even.}$$

The sequence $a_3, a_4, a_5, a_6, \ldots = 0, \frac{1}{8}, \frac{1}{4}, \frac{2}{7}, \ldots$ is strictly increasing.

Now the common generalization of (1.4) and (1.6) is that for $l \geqq 3$

$$RT(n, l, o(n)) = a_l n^2(1+o(1)) \tag{1.8}$$

The upper estimate will be a corollary to our main Theorem 1 which is an Erdős—Stone type generalization of (1.8).

First we introduce a convenient symbolism for stating this result.

Definition 1.9. Assume n, l and the graph H are such that there is a graph $G = \langle \mathbf{n}, E\rangle$ not containing H as a subgraph and having no independent set of size l. We denote by $RT(n; H; l)$ the largest integer e for which a graph G described above exists with $|E| = e$.

Note that $RT(n; k; l) = RT(n; K_k; l)$ for the complete k-graph K_k.

(1.10) Let Chrom (k) denote the class of graphs G with chromatic number at most k.

Taking into consideration that $RT(n, G, n+1)$ is the Turán number of G, a classical result of [7] generalizes (1.3) to

(1.11) *For* $k \geqq 2$, *and for each* $G \in \mathrm{Chrom}\,(k+1)$, $RT(n, G, n+1) \leqq \leqq \frac{1}{2}\left(1-\frac{1}{k}\right)n^2(1+o(1))$.

We now remind the reader that a graph $G = \langle V, E\rangle$ is said to have arboricity number at most k iff V can be written as the union of sets $V = \bigcup_{i<k} V_i$ in such a way that $G(V_i)$, the subgraph of G spanned by V_i, is a forest for $i<k$.

Now in our generalization of (1.8) the arboricity plays a role similar to that of the chromatic number in (1.11). However, the analogy is not complete and we need some more definitions to state the result.

Definition 1.12. Let $l \geqq 3$ and set $k = \left[\frac{l}{2}\right]$. Let $\mathrm{Arb}\,(l) = \{G = \langle V, E\rangle$: There is a sequence $(V_i : i \leqq k)$ such that $V = \bigcup_{i \leqq k} V_i$, $G(V_i)$ are forest for $i < k$, $G(V_k)$ has no edge and $V_k = \emptyset$ for even $l\}$.

Note that for even l, $\mathrm{Arb}\,(l)$ is the class of graphs of arboricity $\leqq \frac{l}{2}$, while for odd l, $\mathrm{Arb}\,(l)$ consists of graphs whose vertex set is the union of an independent set and of a subset spanning a subgraph of arboricity at most $\left[\frac{l}{2}\right]$.

Note that $K_l \in \mathrm{Arb}\,(l)$ for $l \geqq 3$. Now we are in a position to state our main

Theorem 1. *For* $l \geqq 3$ *and* $G \in \mathrm{Arb}\,(l)$ $RT(n; G, o(n)) \leqq a_l n^2 (1 + o(1))$.

Note that because of $K_l \in \mathrm{Arb}\,(l)$ this yields the upper estimate needed for (1.8). The proof of Theorem 1 will be given in Section 4.

We would like to point out an interesting phenomenon.

Definition 1.13. By the above result, for each G there is a smallest real number c, $0 \leqq c < \frac{1}{2}$ such that $RT(n; G; o(n)) \leqq cn^2(1 + o(1))$. Let us denote this smallest c by $c_{RT}(G) = c(G)$. We call it the critical number of G.

There are some graphs for which we can not determine the critical number. Such is the two by two Turán graph $K_{2,2,2} = G_1$. By Theorem 1, we know that $c(G_1) \leqq \frac{1}{8}$ but we have no other information.

On the other hand for all graphs G for which we can determine $c(G)$ the critical number turns out to be one of the a_l; $l = 3, 4, \ldots$. We do not venture to conjecture that this is true in general, but we point out that our results imply that $c(G)$ can not be arbitrary.

In Section 5 we shall prove

(1.14) For all graphs G, $c(G) \in [a_l, a_{l+1}]$ for some odd l.
Hence e.g. there is no graph G with $\frac{1}{8} < c(G) < \frac{1}{4}$.

The main tool of the proof of Theorem 1 is a judicious application of Szemerédi's regularity lemma invented in [11] and improved in [12]. In Section 4 we will restate this lemma for the convenience of the reader. We will use this lemma in Section 6 to solve a problem stated in [6] as well. The other tools of the proof are the tree building lemma (to be given in Section 2) and a generalization of Turán's theorem for some discrete weight functions. This will be given in Section 3.

Finally, in Section 6 we will give the usual mess of miscellaneous unsolved problems.

2. The tree building lemma

First we restate an easy lemma of [8] which will serve as a basis of most of the computations.

Lemma 2.1. *Given $c, \varepsilon>0$ and r there is an s such that for all sufficiently large n and for all set system $\mathcal{F}\subset[\mathbf{n}]^{\geqq cn}$ with $|\mathcal{F}|\geqq s$ there is a subsystem $\mathcal{F}'\subset\mathcal{F}$ with $|\mathcal{F}'|\geqq r$ and $|\cap\mathcal{F}'|\geqq(c^r-\varepsilon)n$.*

The graphs defined below will be used for constructing subtrees of a graph.

Definition 2.2. For $r', r''\geqq 1$, a graph $H=\langle V, E\rangle$ is said to be an *(r', r'')-graph with root x* $(x\in V)$ if $|V|\leqq(r')^{r''}+1$ and for each $y\in V$ which has distance at most r' from x the degree of y is at least r''. The edges adjacent to the root x of H will be called the special edges of H.

The following is obvious.

(2.3) For $r\geqq 1$ an arbitrary tree of $r+1$ vertices is a subgraph of each (r, r)-graph. The following lemma deals with situations when we are given a graph $G=\langle V, E\rangle$, $|V|=n$, an integer $p\geqq 1$, sets V_i, $i<p$ $|V_i|=n$ and mappings $f_i\colon E\to[V_i]^{\geqq cn}$ for $i<p$. The aim of the lemma is to find a large subtree T of G such that, for all $i<p$, $\bigcap\limits_{e\in T} f_i(e)$ is still large. Here is a formal statement of the lemma.

Lemma 2.4. *(The tree building lemma.) Given $c>0$, $r', r'', p\geqq 1$ there exist $c'>0$ and s such that for all graphs $G=\langle V, E\rangle$ with $|V|=n$, $|E|\geqq sn$ and for all mappings $f_i\colon E\to[V_i]^{\geqq cn}$, $|V_i|=n$, for $i<p$, for all sufficiently large n one can find an (r', r'') subgraph $H\subset E$ with*

$$|\cap\{f_i(e)\colon e\in H\}|\geqq c'n \quad \textit{for} \quad i<p.$$

(Note that in the statement of the lemma we have identified the (r', r'')-subgraph in question to its set of edges H, the vertex set being $\cup H$.)

Proof. We write down the proof for $p=1$, the rest being a mere technicality. We prove a stronger statement by induction on r'. Namely we prove that given $c>0$, for every r'', t there are $c'>0$ and s such that for all $G=\langle V, E\rangle$ with $|E|\geqq sn$ and for all $f\colon E\to[V_1]^{\geqq cn}$, $|V_1|=n$, there are at least tn edges of G which are special edges of some (r', r'')-subgraph $H\subset E$ satisfying

$$|\cap\{f(e)\colon e\in H\}|\geqq c'n.$$

For $r'=1$ this follows from Lemma (2.1). Assume that the lemma is true for r' for all $\bar r''$, $\bar t''$, $\bar c>0$ with suitable $\bar c'>0$ and $\bar s$.

Let $G=\langle V, E\rangle$ and $f\colon E\to[V_1]^{\geqq cn}$, $|V|=|V_1|=n$ be given and let n be sufficiently large. By the induction hypothesis, we can arrange that for

$$\bar E=\{e\colon e \text{ is a special edge of some } (r', r''+1)\text{-graph}$$
$$\bar H(e)\subset G \text{ with } \bar f(e)=\cap\{f(e')\colon e'\in\bar H(e)\}\wedge|\bar f(e)|\geqq\bar c'n\}$$

we have $|\bar E|\geqq\bar t n$. For $e\in\bar E$, let $g(e)$ denote the endpoint of e, which is *not* the root of $\bar H(e)$.

Apply Lemma 2.1 for the graph $\bar{G}=\langle V, \bar{E}\rangle$ and for the mapping $\bar{f}: \bar{E} \to [V_1]^{\geq \bar{c}' n}$.

In case $\bar{l}$ and n are large enough and $c'>0$ is small enough we will get that for

$$\bar{\bar{E}} = \{e \in \bar{E}: \text{There is an } (1, r'') \text{ graph } \bar{H}(e) \text{ with root } x \text{ and such that } g(e') = x \text{ for } e' \in \bar{H}(e), \text{ and } |\bar{f}(e)| \geq c'n \text{ for } \bar{f}(e) = \cap \{\bar{f}(e') \in \bar{H}(e)\}\}$$

$|\bar{\bar{E}}| \geq tn$ holds. Let now $e \in \bar{\bar{E}}$. Clearly, there is an $(r'+1, r'')$-graph H contained in $\cup \{\bar{H}(e'): e' \in \bar{H}(e)\}$ such that e is a special edge of H and

$$|\cap \{f(e'): e' \in H\}| \geq |\cap \{\bar{f}(e'): e' \in \bar{H}(e)\}| \geq c'n.$$

3. The generalization of Turán's theorem for weight functions taking values $0, \frac{1}{2}, 1$

In this $W=\langle V, w\rangle$ will denote a multigraph i.e. V is a set, and $w: [V]^2 \to \{0, \frac{1}{2}, 1\}$.

In [2] these objects were called multigraphs, for $e \in [V]^2$, $w(e)=0$ means that e is not an edge, $w(e)=\frac{1}{2}$ means that e is a simple edge $w(e)=1$ means that e is a double edge. Our notation suits our present purposes more, and our lemma is unfortunately not covered by the numerous interesting results proved there.

Definition 3.1. Given $W=\langle V, w\rangle$, we define two graphs $G_{1/2}^W=\langle V, E_{1/2}^W\rangle$ and $G_1^W=\langle V, E_1^W\rangle$ by setting

$$E_{1/2}^w = \{e \in [V]^2: w(e) \geq \tfrac{1}{2}, \quad E_1^w = \{e \in [V]^2: w(e) = 1\}.$$

Put $e(w) = \sum_{e \in [V]^2} w(e)$ (i.e. "the number of edges" of W), and $d_w(x) = \sum_{y \in V} w(\{x, y\})$ ("the degree of $x \in V$ in w").

Now the following definition of a "complete l-subgraph of w" is not entirely natural but seems to work well, and suites our final aim.

Definition 3.2. Let $W=\langle V, w\rangle$ be given. The pair (X, Y) $X \subset Y \subset V$ is a complete l-subgraph of w iff $[X]^2 \subset E_1^w$, $[Y]^2 \subset E_{1/2}^w$ and $|X|+|Y| \geq l$.

Lemma 3.3. *(The generalization of Turán's theorem.) Assume $l \geq 3$, $|V|=n$ and $W=\langle V, w\rangle$ does not contain a complete l-subgraph (X, Y). Then $e(w) \leq a_l n^2(1+o(1))$.*

Remark. Note that we will prove a much better result since we can determine an extreme cinfiguration for sufficiently large n. It is also obvious that generalizations for weight functions say taking values $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ are immediate but we consider this out of the scope of this paper.

Proof. The idea of Zykov's proof of Turán's theorem, the so-called symmetrization, works. See [13]. We will only outline the proof.

Assume $W=\langle V, w\rangle$ does not contain a complete l subgraph (X, Y), and $e(w)$ is maximal. Choose two vertices $x\neq y$, $\{x, y\}\notin E^w_{1/2}$. Assume $d_w(x)\geqq d_w(y)$. Define w' so that $w(e)=w'(e)$ for $y\notin e$ and $w'(\{z, y\})=w(\{z, x\})$ for $z\in V$. Then $e(w')$ is still maximal and w' does not contain a complete l-subgraph. Indeed, if (X, Y) were a complete l subgraph of w', only one of x, y could occur in Y, and if it was y, we could change it to x. By repeated applications of this operation we can assume that there is a w with maximal $e(w)$ and such that $\{x, y\}\notin E^w_{1/2}$ is an equivalence relation. Denoting the equivalence classes by $A_0, \dots, A_{m-1}$ we know that for $i<j<m$ either $w(x, y)=\frac{1}{2}$ for all $x\in A_i$, $y\in A_j$ or $w(x, y)=1$ for all $x\in A_i$, $y\in A_j$. Moreover $m<l-1$, since any one element subset of a $[Y]^2\subset E^w_{1/2}$ can be chosen as X. We now know that G^w_1 does not contain a complete $l-m$ subgraph. Choose now $x\in A_i$, $y\in A_j$ $i\neq j<m$, $w(x, y)=\frac{1}{2}$. Assume $d_w(x)\geqq d_w(y)$. Define w' as follows: $w(e)=w'(e)$ if $e\cap A_j=\emptyset$, and for $e=\{z, y\}$, $y\in A_j$ let $w'(e)=w(\{x, y\})$. Then $e(w')\geqq e(w)$. Now to see that W' does not contain a complete l-graph (X, Y) it is sufficient to see that $G^{w'}_1$ does not contain a complete $(l-m)$-graph. If it did, then this complete $(l-m)$-graph could contain at most one element from each A_i: $i<m$ and it could only meet one of the equivalence classes A_i, A_j. If it met A_j, we could change the common element with A_j to an element of A_i obtaining a complete $(l-m)$-graph in G^w_1 as well.

By repeated application of the above operation we can obtain a w with maximal $e(w)$, not containing a complete l-graph (X, Y) and such that $\{x, y\}\notin E^w_1$ is an equivalence relation, and denoting the equivalence classes of this relation by $B_0, \dots, B_{m'-1}$, the A-partition is a refinement of the B-partition.

From now on we assume that n is sufficiently large. Next we can assume that each B_i contains at most two A_j-s, since an easy computation shows that if B_i contains more than two A_j's then splitting B_i to two almost equal parts $B_{i,0}$, B_{i1}, and defining for $x, y\in B_i$ $w'(x, y)=1$ iff $x\in B_{i,0}$, $y\in B_{i,1}$ and $w(x, y)=0$ otherwise, $e(w')\geqq e(w)+o(1)$ but w' still does not contain a complete l-graph (x, y).

Next, if two B's say B_i and B_j, $i\neq j$ contain two A_k-s, then an easy computation shows that one does not decrease w more than by $o(1)$ by first equalizing the size of B_i, B_j and then the A_k's contained in their union. After that an easy computation shows that we are better off by splitting $B_i\cup B_j$ into three equal parts choosing them as new B'-s and making them A_k-s well. It follos that there is a basically maximal configuration either of the form $V=B_0\cup\dots\cup B_{m'-1}$, $B_0=A_0\cup A_1$, $A_{i+1}=B_i$ for $2\leqq i<m=m'+1$, $m+m'<l$ or of the form

$$V = B_0\cup\dots\cup B_{m'},\quad m = m',\quad A_i = B_i\ \ for\ \ i < m = [\tfrac{l}{2}]$$

In case l is odd, only the second possibility can occur and $||B_i|-|B_j||\leqq 1$ for $i, j<m'$. In case l is even, the first possibility occurs and computation shows that one gets the "largest" $e(w)$ in case w is regular, i.e. the sizes of the sets are determined in such a way that $d_w(x)\simeq d_w(y)$ for $x, y\in V$. The reader can easily check that this gives the numbers a_l for $l\geqq 3$. ∎

4. Restatement of the regularity lemma. Proof of Theorem 1

Definition 4.1. Let $G=\langle V, E\rangle$ be a graph

(i) For $A, B\subset V$ set $e_G(A, B)=e(A, B)=|E\cap[A, B]^{1,1}|$ and $d_G(A, B)=d(A, B)=\frac{e(A, B)}{|A|\cdot|B|}$ provided $A, B\neq\emptyset$.

(ii) For $\varepsilon>0$, $A, B\neq 0$, $A, B\subset V$ the pair (A, B) is said to be *ε-regular* if for all $X\subset A$, $Y\subset B$ with $|X|\geqq\varepsilon|A|$, $|Y|\geqq\varepsilon|B|$

$$|d(X, Y)-d(A, B)|<\varepsilon \quad \text{holds.}$$

(iii) A disjoint partition of V, $V=\bigcup_{i\leqq m} C_i$, is said to be an *equitable partition* of length $m+1$ if $|C_1|=|C_i|$ for $1\leqq i\leqq m$, C_0 is the exceptional class of the partition.

(iv) An equitable partition $V=\bigcup_{i\leqq m} C_i$ is said to be *ε-regular* if $|C_0|\leqq\varepsilon n$ and (C_i, C_j) is ε-regular for all but εm^2 pairs (i, j) $1\leqq i<j\leqq m$.

Szemerédi's regularity lemma [12]. For every $\varepsilon_0>0$ and m_0 there are $n_0=n_0(\varepsilon_0, m_0)$ and $m_1=m_1(\varepsilon_0, m_0)$ such that for every graph $G=\langle V, E\rangle$ with $|V|=n>n_0(\varepsilon_0, m_0)$ there is an ε_0-regular partition $V=\bigcup_{i\leqq m} C_i$ of V with

$$m_0<m<m_1(\varepsilon_0, m_0).$$

Now to prove our Theorem 1 we will prove its "finite" form, i.e.

Theorem 2. *Assume $l\geqq 3$, $H=(V_H, E_H)\in\mathrm{Arb}(l)$ and $\varepsilon>0$. Then there exist $\delta>0$ and n_0 such that for a graph $G=\langle V, E\rangle$ with $|V|=n>n_0$, $|E|\geqq(a_l+\varepsilon)n^2$ and $\alpha(G)\leqq\delta n$, H is isomorphic to a subgraph of G. (Here $\alpha(G)$ as usual denotes the size of the largest independent set of G.)*

Proof. Let $|V_H|=r$. We now describe the order in which we are going to choose the parameters featuring in the lemmas. In the proof we are going to apply Lemma 2.4 repeatedly $\leqq l$ times with $p\leqq l$ and $r', r''=r$ starting with $c_1=\varepsilon/2$ and then applying the lemma for sets of size $c'n$ $c'=c_2$ and so on. This gives us numbers $c_1, \ldots, c_l>0$, and numbers $s_1, \ldots, s_l$ (from Lemma 2.4). We than fix ε_0 of the regularity lemma so that ε_0 is smaller than $c_l\cdot\varepsilon/4$. We then choose m_0 of the regularity lemma so large that Lemma (3.3) should apply for graphs having $(a_l+\varepsilon/4)m^2$ edges and $m\geqq m_0$ vertices, and so that $1/m_0<\varepsilon/4$. The regularity lemma yields us numbers n_1 and m_1. We will assume that $n>n_1$ and we will choose δ so small, that even a subset $A\subset V$ of size $\geqq(1/m_1)\varepsilon_0 n$ must contain sn edges where $s\geqq\max\{s_i\colon i\leqq l\}$.

Let $V_H=\bigcup_{i\leqq k} V_i^H$, $k=[l/2]$ be a partition establishing that $H\in\mathrm{Arb}(l)$. First we apply the regularity lemma and obtain an ε_0-regular partition of length $m+1$ $(m_0<m<m_1)$ $V=\bigcup_{i\leqq m} C_i$ of the set V.

We now define a weight function $w: \{1, \dots, m\} \to \{0, 1/2, 1\}$ as follows: For $1 \leqq i < j \leqq m$

$$w(\{i,j\}) = \begin{cases} 0 & \text{if } d(C_i, C_j) < \frac{\varepsilon}{2} \text{ or } (C_i, C_j) \text{ is not } \varepsilon_0\text{-regular} \\ \frac{1}{2} & \text{if } \frac{\varepsilon}{2} \leqq d(C_i, C_j) < \frac{1}{2} + \frac{\varepsilon_0}{2} \\ 1 & \text{if } d(C_i, C_j) > \frac{1}{2} + \frac{\varepsilon}{2} \end{cases} \left.\right\} \text{and } (C_i, C_j) \text{ is } \varepsilon_0\text{-regular}$$

Clearly, $e(w) \geqq \sum_{1 \leqq i < j \leqq m} d(C_i, C_j) - \frac{\varepsilon}{4} m^2 - \varepsilon_0 m^2 \geqq \left(a_l + \frac{\varepsilon}{4}\right) m^2$ because $1/m_0$ is small. Hence we can apply Lemma 3.3. We get subsets $X \subset Y \subset \{1, \dots, m\}$ such that the pair (X, Y) is a complete l-graph for the weight function w.

Let $\{i_0, \dots, i_{u-1}\} = Y \setminus X$, $\{i_u, \dots, i_{u+v-1}\} = X$ $u + 2v = l$.

To continue we need more notation. For $x \in V$, $B \subset V$, $V_G(x, B) = \{y \in B: \{x, y\} \in E\}$ and $v_G(x, B) = |V_G(x, B)|$. $v_G(x, B)$ is the degree of the vertex x for B.

For $0 \neq X \subset V$ let $\Pi_G(X, B) = \bigcap_{x \in X} V_G(x, B)$.

Assume now for a minute that $u \neq 0$. The pairs (C_{i_0}, C_{i_j}) $0 < j < u + v$ are all ε_0-regular, by the definition of w, and because of $[Y]^2 \subset E^w_{1/2}$. Hence $v_G(x, C_{i_j}) \geqq \left|\frac{\varepsilon}{2} - \varepsilon_0\right| \cdot |C_{i_j}|$ for all $1 \leqq j < u + v$, for a positive portion of $x \in C_{i_0}$.

Applying Lemma 2.1 we can arrange that there is a subset $A_0 \subset C_{i_0}$, $|A_0| = r$ such that $|\Pi_G(A_0, C_{i_j})| \geqq c_1 \dfrac{n(1-\varepsilon_0)}{m_1}$ for $1 \leqq j < u + v$, where c_1 is still larger than ε_0. Repeating this procedure u times we get sets $A_0 \subset C_{i_0}, \dots, A_{u-1} \subset C_{i_{u-1}}$, $|A_0| = \dots = |A_{u-1}| = r$ and such that the sets $\Pi_G(\bigcup_{i<u} A_i, C_{i_j}) = C'_j$ have size $\geqq c_u \dfrac{n(1-\varepsilon_0)}{m_1}$ for $u \leqq j < u + v$, and c_u is still larger than ε_0.

Now we turn our attention to the set C''_u. Now because of $[X]^2 \subset E^w_1$, for all j, $u < j < u + v$, $v_G(x, C'_j) \geqq \left(\frac{1}{2} + \frac{\varepsilon - \varepsilon_0}{2}\right) |C'_j| \geqq \left(\frac{1}{2} + \frac{\varepsilon}{4}\right) |C'_j|$, for $x \in C^*_u$ where $C^*_u \subset C''_u$ is still large enough to contain s edges as required by Lemma 2.4. Now for $x, y \in C^*_u$, $\{x, y\} \in E$, $u < j < u + v$ let $f_j(\{x, y\}) = V_G(x, C'_j) \cap V_G(y, C'_j) = \Pi_G(\{x, y\}, C'_j)$. Then $f_j(\{x, y\}) \geqq \frac{\varepsilon}{4} |C'_j|$. Now, by Lemma 2.4 we can choose an (r, r)-graph $H_u \subset E$, $\bigcup H_u \underset{def}{=} A_u \subset C^*_u \subset C_{i_u}$ in such a way that for each $u < j < u + v$

$$\left|\bigcap_{E \in H_u} \Pi_G(e, C_{i_j})\right| \geqq c_{u+1} \frac{n(1-\varepsilon_0)}{m_1}.$$

Since $A_u = \bigcup H_u$ this implies

$$|\Pi_G(A_u, C_{i_j})| \geqq c_{u+1} \frac{n(1-\varepsilon_0)}{m_1}.$$

Clearly, we can continue this procedure, and define the sets $A_j \subset C_{ij}$ for $j < u + v$

in such a way that for $j<j'<u+v$ $[A_j, A_{j'}]^{1,1}\subset E$, and that for $u\leqq j<u+v$, $G(A_j)$ contains an (r, r)-graph. Let $B_v=A_{2v}\cup A_{2v+1}$ for $v<[u/2]=k$

$$B_{(u/2)+j} = A_{u+j} \quad \textit{for} \quad j<v, \quad \textit{and}$$

$$B_k = 0 \quad \textit{for} \quad l \quad \textit{even and}$$

$$B_k = A_{(u/2)+1} \quad \textit{for} \quad l \quad \textit{odd.}$$

Now for $v<k$, the graphs $G(B_v)$ contain a tree isomorphic to $H(V_v^H)$. For $v<[u/2]$ this is true because $G(B_v)$ contains a $K_{r,r}$ and for $[u/2]<v<k$ this is true by (2.3) because $G(B_v)$ contains an (r, r)-graph. ∎

5. The counterexamples

5.1. There exists a sequence $G_{1,n}=\langle \mathbf{n}, E_{1,n}\rangle$ of graphs, such that $|E_{1,n}|=o(n^2)$, $\alpha(G_n)=o(n)$ and the girth of $G_{1,n}$ tends to infinity. See [3].

5.2. There exists a sequence $G_{2,n}=\langle \mathbf{n}, E_{2,n}\rangle$ of graphs satisfying the following conditions:

$$\frac{|E_{2,n}|}{n^2}\to\frac{1}{8}, \quad K_4 \not\subset G_{2,n}, \quad \alpha(G_{2,n})=o(n)$$

Note that $G_{2,n}$ can be chosen so that $n=A_{2,n}\cup B_{2,n}$ and all but $o(n^2)$ edges of $G_{2,n}$ are in $[A_{2,n}, B_{2,n}]^{1,1}$. See [1].

As it is pointed out in the paper of Bollobás and Erdős, it is not known if $G_{2,n}$ can be chosen so that $|E_{2,n}|\geqq n^2/8$ holds for all n. This leaves a corresponding open problem for all even l, $l\geqq 4$. It is also not known if $G_{2,n}$ can be chosen so that $G_{2,n}(A_{2,n})$, $G_{2,n}(B_{2,n})$ have large girth as well. If this was the case, our argument on p. 20 would yield that $c_{RT}(H)=c_l$ for some l for all graphs H.

5.3. Let $k\geqq 1$. There exists a sequence $G^k_{3,n}=\langle \mathbf{n}, E^k_{3,n}\rangle$ of graphs such that, $K_{2k+1}\not\subset G^k_{3,n}$, $|E^k_{3,n}|\geqq\frac{1}{2}\left(1-\frac{1}{k}\right)n^2=a_{2k+1}n^2$, and $\alpha(G^k_{3,n})=o(n)$. See [4].

For the convenience of the reader we describe a proof. We assume that n is divisible by k. In the other cases we can argue similarly. Let $\mathbf{n}=\bigcup_{i<k} A_i$, $|A_i|=n/k$ for $i<k$. Let $\tilde{G}_i=\langle A_i, \tilde{E}_i^n\rangle$, $i<k$ be isomorphic to $G_{1,n/k}$ defined in 5.1.

Clearly $E^k_{3,n}=\bigcup_{i<j<k}[A_i, A_j]^{1,1}\cup\left(\bigcup_{i<k}\tilde{E}_i\right)$ satisfies the requirements of 5.3.

5.4. Let $k\geqq 2$. There exists a sequence $G^k_{4,n}=\langle \mathbf{n}, E^k_{4,n}\rangle$ of graphs satisfying the following conditions

$$K_{2k}\not\subset G^k_{4,n}, \quad \frac{|E^k_{4,n}|}{n^2}\to\frac{1}{2}\left(\frac{3k-5}{3k-2}\right)=a_{2k} \quad \text{and}$$

$$\alpha(G^k_{4,n})=o(n) \quad \text{for} \quad n\to\infty.$$

Proof. We assume for the sake of simplicity that n is divisible by $3k-2$. Choose the pairwise disjoint sets A_i: $i<k-1$ in such a way that $\mathbf{n}=\bigcup_{i<k-1} A_i$, $|A_0|=\frac{4n}{3k-2}$, and $|A_i|=\frac{3n}{3k-2}$ for $0<i<k-1$. Let $\tilde{G}_0=\langle A_0, \tilde{E}_0\rangle$ be isomorphic to the graph $G_{2,4n/3k-2}$ defined in 5.2 and for $0<i<k-1$ let $\tilde{G}_i=\langle A_i, \tilde{E}_i\rangle$ be isomorphic to $G_{1,3n/3k-2}$ defined in 5.1.

Put $E^k_{4,n}=\bigcup_{i<k-1}\tilde{E}_i\cup\bigcup_{i<k<j-1}[A_i, A_j]^{1,1}$. A computation shows that, by 5.2.

$$\frac{|E^k_{4,n}|}{n^2}\to a_{2k} \quad \textit{if} \quad n\to\infty.$$

By 5.1 and 5.2, $\alpha(G^k_{4,n})=o(n)$. Finally if $X\subset\mathbf{n}$ and $[X]^2\subset E^k_{4,n}$ then $|X\cap A^n_0|\leqq 3$ and $|X\cap A^n_i|\leqq 2$ for $0<i<k-1$, hence $|X|\leqq 3+2(k-2)=2k-1$. ∎

5.3 and 5.4 conclude the proof of (1.8). We are now in a position to prove our claim (1.14).

Assume that for some graph $G=\langle V, E\rangle$ $c_{RT}(G)<a_l$ for some odd $l>3$. Then, by 5.3, for all sufficiently large n, $G\subset G^{[l/2]}_{3,n}$ holds. But, because of, by 5.1, the girth of $G_{1,n}$ tends to the infinity, $G\subset G^{[l/2]}_{3,n}$ implies that the arboricity of G is at most $[l/2]$. This in turn implies that $G\in\mathrm{Arb}\,(l-1)$ and then, by Theorem 1, $c_{R,T}(G)\leqq a_{l-1}$ holds as well. ∎

6. Miscellaneous remarks

First we would like to mention that using the methods of this paper we can prove a more general statement.

6.1. *Assume* $k_1, \ldots, k_r\geqq 3$. *There is a constant* $a_{k_1,\ldots,k_r}$ *such that*

$$R(n;\ k_1, \ldots, k_r;\ o(n)) = a_{k_1,\ldots,k_r} n^2(1+o(1)).$$

Moreover the numbers $a_{k_1,\ldots,k_r}$ *can be obtained as Ramsey numbers of multiple graphs* $W=\langle v, w\rangle$.

We preserve the formulation of a more precise statement, and the proof of it for later publication. This will be done in a joint work with M. Simonovits, to whom we would like to express our thanks for his helpful comments concerning the work published in this paper as well.

To make the rather vague statement 6.1 a little more comprehensible we write down a special case of it we obtained earlier.

Definition 6.2. Let $r\geqq 2$ and let $R^*_2(3, \ldots, 3^r)$ be the largest integer for which there is a 2-partition $[\mathbf{n}]^2=\bigcup_{i<r} E_i$ of length r of $\mathbf{n}$ such that the graphs $G_i=\langle\mathbf{n}, E_i\rangle$

do not contain K_3, and is such that for all $j<n$ there is an $i<r$ such that G does not contain an edge adjacent to j. Clearly

$$R_2(\underbrace{3, \ldots, 3}_{r-1}) \leqq R_2^*(\underbrace{3, \ldots, 3}_{r}) \leqq R_2(\underbrace{3, \ldots, 3}_{r}).$$

Now our methods give

Theorem 3. $RT(n, \underbrace{3, \ldots, 3}_{r}, o(n)) = \frac{1}{2}\left(1-\frac{1}{R_2^*(\underbrace{3, \ldots, 3}_{r})}\right) n^2(1+o(1)).$

The proof is based on a simple application of the regularity lemma, along the lines described in the proof of Theorem 2. Since $R_2^*(3, 3, 3) = R_2(3, 3) = 5$ we get

Corollary. $RT(n; 3, 3, 3, o(n)) = \frac{2}{5} n^2(1+o(1))$.

This was explicitly stated as a problem.

The first author stated several times the following problem: Is it true that $R_2^*(\underbrace{3, \ldots, 3}_{r}) = R_2(\underbrace{3, \ldots, 3}_{r-1})$ for all $r \geqq 3$? Finally Fan Chung proved (oral communication) that this is not the case. The constructions are quite involved.

Finally we mention another type of problems. For a graph $G=\langle V, E\rangle$ let $\alpha_r(G)$ be the size of the largest subset $A \subset V$ for which $G(A)$ does not contain a complete K_r graph. Clearly $\alpha(G) = \alpha_2(G)$.

Let $RT(n; k; l|r) = \max\{e : \exists G = \langle V, E\rangle \ (|V| = n \wedge |E| = e \wedge K_k \not\subset G \wedge \alpha_r(G) < l)\}$, provided the set after the max sign is nonempty. We are again interested in $RT((n, k, o(n)|r)$. Again, as in the original problem, one can with a special argument generalize (1.4) and show that for $k \geqq 1$

$$RT(n, 3k+1, o(n)|3) = \frac{1}{2}\left(1-\frac{1}{k}\right) n^2(1+o(1)). \tag{6.3}$$

Now one would conjecture that an application of the regularity lemma and an appropriate generalization of Turán's theorem for weight functions taking values $\{0, 1/3, 2/3, 1\}$ should yield the answer in cases $3k+2$, $3k+3$ as well. However, this is not the case and though we have partial results there remain simple unsolved problems. Here is the simplest unsolved case:

We can prove that $R(n, 5, o(n)|3) \leqq 1/12\, n^2(1+o(1))$. Is this best possible? To show this an analogue of the Bollobás—Erdős graph (2) would be needed which we think will be extremely hard to find. At the moment we can not even disprove $RT(n, 6, o(n)|3) = o(n^2)$.

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