

ENUMERATION OF INTERSECTING FAMILIES

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It is shown that the logarithm to the base 2 of the number of maximal intersecting families on m elements is asymptotically equal to $\binom{m-1}{n-1}$ where $n = \lfloor m \rfloor$.

1. Introduction

For a natural number m , let $\mathbf{m} = \{1, 2, \dots, m\}$. An intersecting family on \mathbf{m} is a set \mathcal{A} of sets such that $\bigcup \mathcal{A} \subseteq \mathbf{m}$ and any two members of \mathcal{A} have non-empty intersection. We let \mathcal{I}_m be the set of all maximal intersecting families on \mathbf{m} . We are concerned with estimating $|\mathcal{I}_m|$.

In Section 2 we obtain a lower bound by elementary counting methods. In Section 3 we obtain an upper bound using a result of Kleitman and Markowsky on the number of monotone Boolean functions.

Notice that if in the definition of intersecting families, the requirement that any two members of \mathcal{A} have non-empty intersection is raised to any three members, the problem becomes trivial. Indeed, by [1, Remark 7.5] any maximal intersecting family would be an ultrafilter; that is it would consist of all subsets of \mathbf{m} containing some singleton.

2. A lower bound and statistical remarks

We observe that an intersecting family \mathcal{A} on \mathbf{m} is maximal if and only if for every $A \subseteq \mathbf{m}$, either $A \in \mathcal{A}$ or $\mathbf{m} \setminus A \in \mathcal{A}$. Observe also that if \mathcal{A} is an intersecting family on \mathbf{m} , $A \in \mathcal{A}$, and $A \subseteq B \subseteq \mathbf{m}$, then $B \in \mathcal{A}$.

2.1. Definition. A subset \mathcal{B} of $\mathcal{P}(\mathbf{m})$ is a free choice family on \mathbf{m} if and only if whenever $\mathcal{C} \subseteq \mathcal{B}$, $\mathcal{C} \cup \{\mathbf{m} \setminus B : B \in \mathcal{B} \setminus \mathcal{C}\}$ is an intersecting family.

We denote by $[A]^k$ the set of k -element subsets of A .

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2.2. Lemma. Let $n = \lfloor \frac{1}{2}m \rfloor$ and let $\mathcal{B} = \{A \in [m]^n : 1 \in A\}$. Then \mathcal{B} is a free choice family on m with largest possible cardinality.

Proof. Trivially any two members of \mathcal{B} have non-empty intersection. Distinct members of $\{m \setminus B : B \in \mathcal{B}\}$ must meet because of their size. Likewise, if $B, C \in \mathcal{B}$ and $B \cap (m \setminus C) = \emptyset$, then $B = C$. Thus \mathcal{B} is a free choice family on m with cardinality $\binom{m-1}{n-1}$.

By Theorem 1 of [1], if \mathcal{C} is an intersecting family on m , each $A \in \mathcal{C}$ has $|A| \leq n$, and whenever A and B are distinct members of \mathcal{C} neither $A \subseteq B$ nor $B \subseteq A$, then $|\mathcal{C}| \leq \binom{m-1}{n-1}$. Given a free choice family \mathcal{D} on m , let $\mathcal{C} = \{A \in \mathcal{D} : |A| \leq n\} \cup \{m \setminus A : A \in \mathcal{D} \text{ and } |A| > n\}$. Then \mathcal{C} satisfies the conditions above, so \mathcal{C} (and hence \mathcal{D}) has at most $\binom{m-1}{n-1}$ elements. \square

Lemma 2.2 yields immediately a lower bound of $2^{\binom{m-1}{n-1}}$ for $|\mathcal{F}_m|$. As we shall see this is an asymptotically correct value in the exponent. However we do manage to raise the lower bound somewhat by considering free choices which remain given a particular choice from \mathcal{B} .

2.3. Theorem. Let $n = \lfloor \frac{1}{2}m \rfloor$.

- (a) If $m = 2n$, then $|\mathcal{F}_m| \geq 2^{\binom{m-1}{n-1} + \binom{m-1}{n} 2^{2n-1}}$.
 (b) If $m = 2n + 1$, then $|\mathcal{F}_m| \geq 2^{\binom{m-1}{n-1} + \binom{m-1}{n} 2^{2n}}$.

Proof. Let $\mathcal{B} = \{A \in [m]^n : 1 \in A\}$. Given $\mathcal{F} \subseteq \mathcal{B}$, let $\mathcal{C}(\mathcal{F}) = \mathcal{F} \cup \{m \setminus B : B \in \mathcal{B} \setminus \mathcal{F}\}$ (so that $\mathcal{C}(\mathcal{F})$ is the choice induced by \mathcal{F}). If $\mathcal{F} \subseteq \mathcal{B}$ and $m = 2n$, let

$$\mathcal{D}(\mathcal{F}) = \{A \in [m]^{n+1} : \text{for all } B \in \mathcal{C}(\mathcal{F}), B \setminus A \neq \emptyset\}.$$

If $\mathcal{F} \subseteq \mathcal{B}$ and $m = 2n + 1$, let

$$\mathcal{D}(\mathcal{F}) = \{A \in [m]^{n+1} : \{1, 2\} \subseteq A \text{ and for all } B \in \mathcal{C}(\mathcal{F}), B \setminus A \neq \emptyset\}.$$

For any $\mathcal{F} \subseteq \mathcal{B}$, let $d(\mathcal{F}) = |\mathcal{D}(\mathcal{F})|$. We claim that

(*) If $\mathcal{F} \subseteq \mathcal{B}$ and $\mathcal{G} \subseteq \mathcal{D}(\mathcal{F})$, then $\mathcal{C}(\mathcal{F}) \cup \mathcal{G} \cup \{m \setminus A : A \in \mathcal{D}(\mathcal{F}) \setminus \mathcal{G}\}$ is an intersecting family.

To see (*) note that $\mathcal{D}(\mathcal{F})$ was defined so that whenever $B \in \mathcal{C}(\mathcal{F})$ and $A \in \mathcal{D}(\mathcal{F})$, both $B \cap A$ and $B \cap (m \setminus A)$ are non-empty. Also if $A, B \in \mathcal{D}(\mathcal{F})$ and $A \neq B$, then $B \cap A$ and $B \cap (m \setminus A)$ are non-empty by virtue of their sizes. (If one had $B \cap (m \setminus A) = \emptyset$ one would have $B = A$.) Consequently we need only show that if $A, B \in \mathcal{D}(\mathcal{F})$ and $A \neq B$, then $(m \setminus A) \cap (m \setminus B) \neq \emptyset$. If $m = 2n + 1$, then $\{1, 2\} \subseteq A \cap B$ and hence $|A \cup B| \leq 2n$ so we can assume $m = 2n$. Suppose $(m \setminus A) \cap (m \setminus B) = \emptyset$. Then $A \cup B = m$ so $|A \cap B| = 2$. Pick $x, y \in m$ such that $A \cap B = \{x, y\}$. Then either $\{x\} \cup (A \setminus B)$ or $\{y\} \cup (B \setminus A)$ is in $\mathcal{C}(\mathcal{F})$ and we may assume the former. Then since $\{x\} \cup (A \setminus B) \subseteq A$ we have $A \notin \mathcal{D}(\mathcal{F})$, a contradiction.

Since (*) holds, we have $|\mathcal{F}_m| \geq \sum_{\mathcal{F} \subseteq \mathcal{B}} 2^{d(\mathcal{F})}$.

Let $G = \{(\mathcal{F}, A) : \mathcal{F} \subseteq \mathcal{B} \text{ and } A \in \mathcal{D}(\mathcal{F})\}$. We count G in two ways. On the one hand $|G| = \sum_{\mathcal{F} \subseteq \mathcal{B}} d(\mathcal{F})$. Given $A \in [\mathbf{m}]^{n+1}$ (with $\{1, 2\} \subseteq A$ if $m = 2n + 1$) and $\mathcal{F} \subseteq \mathcal{B}$, we have $A \in \mathcal{D}(\mathcal{F})$ if and only if no subset of A is in $\mathcal{C}(\mathcal{F})$. Assume now $m = 2n$ and $A \in [\mathbf{m}]^{n+1}$. There are $n + 1$ n -element subsets of A and $\binom{m-1}{n-1}$ elements of \mathcal{B} so $|\{\mathcal{F} \subseteq \mathcal{B} : A \in \mathcal{D}(\mathcal{F})\}| = 2^{\binom{n-1}{n-1} - n - 1}$. Since $|[\mathbf{m}]^{n+1}| = \binom{m}{n+1} = \binom{m}{n-1}$ we have

$$|G| = \binom{m}{n-1} \cdot 2^{\binom{n-1}{n-1} - n - 1}.$$

Now assume $m = 2n + 1$ and $A \in [\mathbf{m}]^{n+1}$ with $\{1, 2\} \subseteq A$. Any subset B of A which is in $\mathcal{C}(\mathcal{F})$ must in fact be in \mathcal{F} and hence must have $1 \in B$. There are n such n -element subsets so $|\{\mathcal{F} \subseteq \mathcal{B} : A \in \mathcal{D}(\mathcal{F})\}| = 2^{\binom{m-1}{n-1} - n}$. Since $|\{A \in [\mathbf{m}]^{n+1} : \{1, 2\} \subseteq A\}| = \binom{m-1}{n-1}$ we have

$$|G| = \binom{m-1}{n-1} \cdot 2^{\binom{m-1}{n-1} - n}.$$

Let $\bar{d} = (\sum_{\mathcal{F} \subseteq \mathcal{B}} d(\mathcal{F})) / |\mathcal{P}(\mathcal{B})|$. (Thus \bar{d} is the mean value of the $d(\mathcal{F})$'s.) We have then

$$|\mathcal{I}_m| \geq \sum_{\mathcal{F} \subseteq \mathcal{B}} 2^{d(\mathcal{F})} \geq \sum_{\mathcal{F} \subseteq \mathcal{B}} 2^{\bar{d}} = 2^{\binom{m-1}{n-1} + \bar{d}}.$$

Inserting the value for \bar{d} obtained by our double counting of G we have the desired result. \square

We now restrict our attention to the simpler case when $m = 2n$ and discuss the distribution of $\{d(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{B}\}$. We obtained above the value $\binom{m-1}{n-1} / 2^{n+1}$ for the mean by counting twice the set $\{(\mathcal{F}, A) : \mathcal{F} \subseteq \mathcal{B} \text{ and } A \in \mathcal{D}(\mathcal{F})\}$, we can also compute the variance by counting twice the set

$$\{(\mathcal{F}, A, B) : \mathcal{F} \subseteq \mathcal{B}, A \in \mathcal{D}(\mathcal{F}), \text{ and } B \in \mathcal{D}(\mathcal{F})\}.$$

(In this computation we consider separately pairs (A, B) where $A = B$, $|A \cap B| = n$, and $3 \leq |A \cap B| \leq n - 1$. As we saw earlier if $|A \cap B| = 2$, then for no $\mathcal{F} \subseteq \mathcal{B}$ do we have $A \in \mathcal{D}(\mathcal{F})$ and $B \in \mathcal{D}(\mathcal{F})$.) This computation yields the result (for $n \geq 3$)

$$\sum_{\mathcal{F} \subseteq \mathcal{B}} (d(\mathcal{F}) - \bar{d})^2 = \binom{m}{n-1} 2^{\binom{n-1}{n-1} - n - 1} (1 + (n^2 - n - 4) 2^{-n-2})$$

so that the variance is $\binom{m}{n-1} / 2^{n+1} \cdot (1 + o(1))$.

3. An upper bound

Let \mathcal{I}_m be the set of antichains in $\mathcal{P}(\mathbf{m})$. (A set $\mathcal{A} \subseteq \mathcal{P}(\mathbf{m})$ is an anti-chain provided that whenever $A, B \in \mathcal{A}$ with $A \subseteq B$ one has $A = B$.) It was shown in [3], improving an earlier result [2], that there is a constant c such that $|\mathcal{I}_m| < 2^{(1+c) \binom{m}{n} / 2}$ where $n = \lfloor \frac{1}{2} m \rfloor$. We show in this section that $|\mathcal{I}_m| \leq |\mathcal{I}_{m-1}|$.

3.1. Definition. Define a function g on \mathcal{F}_m by $g(\mathcal{A}) = \{A \subseteq m-1 : A = B \cap m-1 \text{ for some } B \in \mathcal{A} \text{ and there does not exist } C \in \mathcal{A} \text{ such that } C \cap m-1 \subseteq A\}$.

3.2. Lemma. Let $\mathcal{A} \in \mathcal{F}_m$ and let $A \subseteq m-1$.

(a) $A \cup \{m\} \in \mathcal{A}$ if and only if there exists $B \in g(\mathcal{A})$ such that $B \subseteq A$.

(b) $A \in \mathcal{A}$ if and only if there exists $B \in g(\mathcal{A})$ such that $B \subseteq A$ and there does not exist $C \in g(\mathcal{A})$ such that $C \cap A = \emptyset$.

Proof. (a) Assume $A \cup \{m\} \in \mathcal{A}$. Pick $D \in \mathcal{A}$ such that $D \subseteq A \cup \{m\}$ and $|D \cap m-1|$ is minimal among all such members of \mathcal{A} . Let $B = D \cap m-1$. Then $B \in g(\mathcal{A})$ and $B \subseteq A$.

Now assume we have $B \in g(\mathcal{A})$ such that $B \subseteq A$. Pick $D \in \mathcal{A}$ such that $D \cap m-1 = B$. Then $D \subseteq A \cup \{m\}$ so $A \cup \{m\} \in \mathcal{A}$.

(b) Assume $A \in \mathcal{A}$. Then $A \cup \{m\} \in \mathcal{A}$ so by (a) we have some $B \in g(\mathcal{A})$ such that $B \subseteq A$. Suppose we have $C \in g(\mathcal{A})$ such that $C \cap A = \emptyset$. Pick $D \in \mathcal{A}$ such that $D \cap m-1 = C$. Then $D \cap A = \emptyset$, a contradiction.

Finally assume we have some $B \in g(\mathcal{A})$ such that $B \subseteq A$ and have no $C \in g(\mathcal{A})$ such that $C \cap A = \emptyset$. By (a) we have $A \cup \{m\} \in \mathcal{A}$. Suppose that $A \notin \mathcal{A}$ so that $m \setminus A \in \mathcal{A}$. Again by (a) pick $C \in g(\mathcal{A})$ such that $C \subseteq m \setminus A$. Then $C \cap A = \emptyset$, a contradiction. \square

3.3. Theorem. The function g is one-to-one and takes \mathcal{F}_m to \mathcal{F}_{m-1} .

Proof. By Lemma 3.2, \mathcal{A} is completely determined by $g(\mathcal{A})$ so g is one-to-one. Let $\mathcal{A} \in \mathcal{F}_m$. To see that $g(\mathcal{A}) \in \mathcal{F}_{m-1}$ suppose instead we have $B, C \in g(\mathcal{A})$ with $C \subseteq B$. Pick $D, F \in \mathcal{A}$ such that $D \cap m-1 = B$ and $E \cap m-1 = C$. Then $E \cap m-1 \subseteq B$ so $B \notin g(\mathcal{A})$. \square

3.4. Corollary. Let $n = \lfloor \frac{1}{2}m \rfloor$. There is a constant c such that:

(a) If $m = 2n$, then

$$|\mathcal{F}_m| \leq 2^{(1+c \log m/m) \binom{m-1}{n}},$$

(b) If $m = 2n+1$, then

$$|\mathcal{F}_m| \leq 2^{(1+c \log m/m) \binom{m-1}{n}}.$$

Proof. By Theorem 3.3, $|\mathcal{F}_m| \leq |\mathcal{F}_{m-1}|$ so the theorem of Kleitman and Markowsky cited above applies. \square

3.5. Corollary. $\log_2 |\mathcal{F}_m|$ is asymptotically equal to $\binom{m-1}{n}$ where $n = \lfloor \frac{1}{2}m \rfloor$.

Proof. Theorem 2.3 and Corollary 3.4. \square

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A graph G is k -regular if it has k neighbours for each vertex. The set of all k -regular graphs with n vertices is denoted by $\mathcal{R}(n, k)$. Every vertex-transitive graph is k -regular for some k . In fact, by a well-known theorem of Frucht, every finite group G is isomorphic to the automorphism group of some k -regular graph. In this paper we study the number of k -regular graphs with n vertices and m edges, for $k \geq 2$ and $m \geq kn/2$. We show that the number of k -regular graphs with n vertices and m edges is asymptotically equal to the number of k -regular graphs with n vertices and m edges, for $k \geq 2$ and $m \geq kn/2$. The case $k=1$ is treated separately, and the case $k=0$ is also considered.



1. Introduction

For general concepts see either the monograph [1] or the survey [2]. Let D be a graph. $V(D)$ and $E(D)$ (or ED) will denote the set of vertices and the set of edges, respectively. Given any subset S of $V(D)$, instead of $\{v_1, \dots, v_n\}$ let S_1, \dots, S_r be subsets of $V(D)$. The arc span_D of D will be called an (S_1, S_2) -arc whenever $v_1 \in S_1$ and $v_2 \in S_2$. A directed (S_1, S_2) -path is any v_1, v_2, \dots, v_r with $v_1 \in S_1$ and $v_i \in S_i$, $i=2, \dots, r$. Let $\mathcal{P}(S_1, S_2)$ be the set of all (S_1, S_2) -paths of D . The length of a path P is denoted by $|P|$.

Definition 1. N is k -regular if $V(D)$ is independent if $|E(D \cap N)| = k$.

A subset N of $V(D)$ is an independent set of vertices such that for each $v \in V(D) \setminus N$ there exists a kN -arc to v .

The concept of k -regular was introduced by the author and Alon [3] in the context of Game Theory. This also proved that k -regular graphs through the existence lemma. The problem of the number of k -regular graphs has been studied by several authors, in particular by Bender [4–6], Neumann-Lara [7] and, recently, by Deza and Mészáros [8, 9]. As well known result of