

A RAMSEY PROBLEM OF HARARY ON GRAPHS WITH PRESCRIBED SIZE

P. ERDÖS

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

R.J. FAUDREE, C.C. ROUSSEAU and R.H. SCHELP

Memphis State University, Memphis, TN 38152, U.S.A.

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Lower bounds on the Ramsey number $r(G, H)$, as a function of the size of the graphs G and H , are determined. In particular, if H is a graph with n lines, lower bounds for $r(H) = r(H, H)$ and $r(K_m, H)$ are calculated in terms of n in the first case and m and n in the second case. For $m = 3$ an upper bound is also determined. These results partially answer a question raised by Harary about the relationship between Ramsey numbers and the size of graphs.

1. Introduction

Let G and H be finite graphs without loops or multiple edges. If for any 2-coloring of the lines of a complete graph K_n , there is a copy of G in the first color, red, or a copy of H in the second color, blue, we will say $K_n \rightarrow (G, H)$. The Ramsey number $r(G, H)$ is the smallest positive integer n such that $K_n \rightarrow (G, H)$.

The Ramsey number $r(G, H)$ has been calculated for many pairs of graphs. However, in most cases the Ramsey number is expressed in terms of the order (number of points) of the graph. Harary posed the following natural question in 1980 at a meeting at Kent State University: What is the relationship between $r(G, H)$ and the size (number of lines) of the graphs G and H ?

A partial answer to this general question is contained in the following results. In the statements which follow and throughout the remainder of the paper, $p(G)$ will denote the order and $q(G)$ the size of the graph G . Notation not specifically mentioned will follow [4].

2. Theorems

This section develops our results and the next section supplies their proofs.

The first result gives bounds on the minimum value of the diagonal Ramsey number for all graphs of a fixed size n . The result is sharp except for the value of the constant.

Theorem 1. For any integer $n > 1$, there is an $\varepsilon < 1$ such that

$$\frac{n}{2 \log n} < \min_{q(G)=n} r(G) < \frac{(8 + \varepsilon)n}{2 \log n}.$$

The next result, a non-diagonal Ramsey number, involves a complete graph. In this case the bounds are not sharp, even when the complete graph is a triangle.

Theorem 2. Let $m \geq 3$ be fixed. There exist constants C_1 and C_2 such that for all sufficiently large n

$$C_1 \cdot n^{m/(m+2)} < \min\{r(K_m, G) : q(G) = n\} < C_2 \cdot n^{(m-1)/m}.$$

The immediate special case when the complete graph in Theorem 2 is a triangle is of special interest, and leads to one of the specific problems posed.

Corollary 3. There exist constants C_1 and C_2 such that for all sufficiently large n ,

$$C_1 \cdot n^{\frac{3}{2}} < \min\{r(K_3, G) : q(G) = n\} < C_2 \cdot n^{\frac{3}{2}}.$$

Another specific question, which is a special case of the general question posed earlier, involves an upper bound for the Ramsey number $r(K_3, G)$, where G is any graph of size n . The previous corollary gives a lower bound for this Ramsey number. Harary made the following conjecture about the upper bound.

Conjecture. For any graph G of size n and without isolates,

$$r(K_3, G) \leq 2n + 1.$$

This bound is the best possible, since $r(K_3, T_{n+1}) = 2n + 1$ for any tree with n lines (see [2]). Also, it is trivial to show that $r(K_3, nK_2) = 2n + 1$, and it is well known that $r(K_3, K_n) < 2\binom{n}{2} + 1$ (see [5]). Therefore the conjecture is confirmed in the extreme cases of the most sparse and dense connected graphs, and the most disconnected graph. We were unable to verify the conjecture, but the following result is proved.

Theorem 4. For any graph G of size n without isolated points,

$$r(K_3, G) \leq \left\lceil \frac{8}{3}n \right\rceil.$$

A fixed graph G is said to be *size linear* if for all n there is a constant C such that $r(G, H) \leq Cn$ for all graphs H of size n . Thus, Theorem 4 implies that K_3 is size linear. Not all graphs are size linear. For example, K_4 is not, since there is a constant $D > 0$ such that $r(K_4, K_n) > D(n/\ln n)^{\frac{3}{2}}$ for all n sufficiently large (see [5]). The following result implies that graphs which are too dense cannot be size linear.

Theorem 5. *Let G be a fixed graph with $p(G) = m \geq 3$ and $q(G) = q$. There exists a positive constant C such that for n sufficiently large*

$$r(G, K_n) > C(n/\log n)^{(q-1)/(m-2)}.$$

Corollary 6. *If $p(G) \geq 3$ and $q(G) \geq 2p(G) - 2$, then G is not size linear.*

Additional results about size linear graphs can be found in [3]. In particular, there is an example of a size linear graph G with $p(G) = p$ and $q(G) = 2p - 3$ for any integer $p \geq 2$. Thus the result stated in Corollary 6 is the best possible. Note that this does not imply that any connected graph of order p and size at most $2p - 3$ must be size linear. In fact a K_4 with a long suspended path attached to one of its points is not size linear, but this graph has approximately the same number of points and lines.

3. Proofs

We now indicate the proofs of the theorems.

Proof of Theorem 1. The proof of the lower bound uses a very simple counting argument. Let $N = n/(\log n)$ and let G be an arbitrary graph with $p(G) = p$ and $q(G) = n$. We claim that $K_N \not\rightarrow (G, G)$. If $p > N$ there is nothing to prove. The number of 2-colorings of K_N which contain a monochromatic G is not more than

$$2 \binom{N}{p} p! 2^{\binom{N}{2} - n}.$$

It follows that if $2N^p < 2^n$, there is a 2-coloring of the lines which avoids G in each color. Since

$$1 + p^2 \log N \leq 1 + N^2 \log N \leq 1 + \frac{n}{2 \log n} (2 \log n - \log^2 \log n) < n$$

for all $n > 2$, the result follows.

The upper bound uses a simple example. Let x be the unique positive root of the equation $x^2 \cdot 4^x = n$. Note that $x = \frac{1}{2} (2 \log n) (1 - o(1))$. Set $m = \lceil x \rceil + 1$ and let s be the smallest integer such that $s \binom{m}{2} \geq n$. Let G_1, \dots, G_s each be graphs of order m such that $q(G_1) + \dots + q(G_s) = n$. Let G be the point disjoint union of these graphs. Clearly $r(G) \leq r(sK_m) \leq (2s - 2)m + r(K_m)$. By our choice of s , $(2s - 2)m < 4n/(m - 1) = 8(1 + o(1))n/(2 \log n)$. Thus, $r(G)$ is bounded above as stated, which completes the proof. \square

The example used to verify the upper bound in Theorem 1 is disconnected. However, a connected example can be constructed by replacing the complete

graphs with complete bipartite graphs and connecting a new point to each of the points of the bipartite graphs.

Theorem 2 is closely related to a result appearing in [1]. Before starting the proof of Theorem 2, we will state the related result from [1] along with the appropriate definition. All of the graphs considered during this discussion will be connected.

For fixed positive integers m and n ,

$$g(m, n) = \max\{q(G) : p(G) = n \text{ and } r(K_m, G) = (m-1)(n-1) + 1\}.$$

Theorem A ([1]). For $m \geq 3$, there exist a positive $\varepsilon < 1$ and positive constants C and D such that for n sufficiently large

$$Cn^{m/(m-1)} < g(m, n) < Dn^{(m+2)/m}(\log n)^\varepsilon.$$

Proof of Theorem 2. The upper bound in Theorem 2 is a direct consequence of the lower bound of Theorem A. From the proposition there is a graph G with $p(G) = n$, $q(G) > Cn^{m/(m-1)}$ and such that $r(K_m, G) = (m-1)(n-1) + 1$. Thus for appropriate constants C' and C_2 , $q(G) = q$, $p(G) < C'q^{(m-1)/m}$ and $r(K_m, G) = (m-1)(n-1) + 1 < C_2q^{(m-1)/m}$. It follows immediately that

$$\min_{q(G)=q} r(K_m, G) < C_2q^{(m-1)/m}.$$

The lower bound of Theorem 2 does not follow directly from the upper bound of Theorem A. However, the same proof technique used in the proof of Theorem A can be used to verify the lower bound of Theorem 2. This same proof technique, which utilizes the lemma of Lovász (Lemma B, which is stated later), is applied later to prove Theorem 5. Therefore we omit the details of the proof of the lower bound. \square

Upper bounds for Ramsey numbers as a function of the size of graphs appear to be difficult to obtain. A possible reason for this is that upper bounds may involve Ramsey numbers for complete graphs. The following proof considers the case when one of the graphs is a triangle.

Proof of Theorem 4. The proof will be induction on n . The result is trivial for $n = 1$. We can assume G is connected since $r(K_3, G_1 \cup G_2) \leq r(K_3, G_1) + r(K_3, G_2)$ for point disjoint graphs G_1 and G_2 . Let $N = \lceil \frac{8}{3}n \rceil$ and assume that K_N is 2-colored with no red K_3 or blue G . By the induction assumption any graph with at most $n-1$ lines is contained in the blue subgraph of K_N . Note that no point of K_N can have red degree as large as $p(G)$, because a red line in the neighborhood of this point would imply a red K_3 and no red line would imply a blue copy of G .

Let v be a point of minimal degree δ in G and let M be the neighborhood of v in G . The graph $G' = G - v$ can be assumed to be a blue subgraph of K_N . Let X

be the points of K_N not in G' . Three cases depending on the minimal degree of a vertex will be considered.

Case 1. $\delta \geq 3$.

Since $\delta \cdot p(G) \leq 2n$, $p(G) \leq 2n/\delta$. Each point in X must be adjacent in red to at least one point in M . Thus some point in M has red degree at least $(\frac{8}{3}n - \frac{2}{3}n)/\delta \geq 2n/\delta \geq p(G)$, a contradiction.

Case 2. $\delta = 1$

If u is the point in M , then u is adjacent in red to each point of X . Since X has at least $\frac{8}{3}n - n \geq \frac{5}{3}n \geq p(G)$ points, this gives a contradiction.

Case 3. $\delta = 2$

First consider the case when G has a suspended path (interior point on the path have degree two in G) with five points. Decrease the length of this path by one to obtain a graph G'' . Let u_1, u_2, u_3, u_4 be the consecutive points of the shortened suspended path P in G'' .

The graph G'' , which has at most n points, can be assumed to be in the blue subgraph of K_N . Let Y be the points of K_N not in G' . Since Y has at least $\frac{5}{3}n$ points, there are y_1 and y_2 in Y which have a red line between them. Each y_i cannot be adjacent in blue to two consecutive points of P and no vertex of P is adjacent in red to both y_1 and y_2 . Therefore, with no loss of generality, we can assume that $y_1u_1, y_1u_3, y_2u_2, y_2u_4$ are precisely the blue lines between $\{y_1, y_2\}$ and P . If u_1u_3 is blue, there is a blue copy of G , and if u_1u_3 is red there is a red K_3 .

We can now assume that G has no suspended path with more than four points. Let H be the graph (possibly a multigraph) obtained from G by shrinking each of the suspended paths to a line. Thus for some $s \geq 0$, H has $p(G) - s$ points and $n - s$ lines. Since each point of H has degree at least three, $3(p(G) - s) \leq 2(n - s)$ and

$$p(G) \leq \frac{1}{3}(2n + s). \tag{1}$$

Since there is no suspended path with five points,

$$p(G) \leq (p(G) - s) + 2(n - s), \text{ and } s \leq \frac{2}{3}n. \tag{2}$$

From (1) and (2) it follows that $p(G) \leq \frac{8}{9}n$.

As before, each point of X has at least one red adjacency in M . Thus, some point of M is adjacent in red to at least one half of the vertices of X . Hence there is a point of red degree at least

$$\frac{1}{2}(\frac{8}{3}n - \frac{8}{9}n) \geq \frac{8}{9}n \geq p(G).$$

This gives a contradiction which completes the proof of this case and of Theorem 4. \square

In the proof of Theorem 5, $[N]^k$ will denote the set of all k -element subsets of $\{1, 2, \dots, N\}$. Any 2-coloring of the lines $[N]^2$ of the complete graph with points $[N]$ will be denoted by (R, B) . Thus R is the red subgraph and B is the blue subgraph. If $S \subset [N]$, then the red subgraph (blue subgraph) induced by $R(B)$ will be denoted by $\langle S \rangle_R$ ($\langle S \rangle_B$).

Proof of Theorem 5. The proof uses the Lovász–Spencer method (see [5]). For an appropriately large N , we will demonstrate the existence of a 2-coloring (R, B) of $[N]^2$ such that $R \not\supset G$ and $B \not\supset K_n$. Randomly two-color $[N]^2$, each edge being red with independent probability p . For each $S \subset [N]^m$ let A_S denote the event: $\langle S \rangle_R \supset G$. Similarly for each $T \subset [N]^n$ let B_T denote the event: $\langle T \rangle_B \supset K_n$. The fundamental result to be used here is

Lemma B (Lovász [5]). *Let C_1, C_2, \dots, C_n be events with probabilities $P(C_i)$, $i = 1, 2, \dots, n$. Suppose there exist corresponding positive numbers x_1, x_2, \dots, x_n such that $x_i \cdot P(C_i) < 1$ and*

$$\log x_i > \sum_{j \neq i} x_j P(C_j), \quad i = 1, 2, \dots, n,$$

where the sum is taken over all $j \neq i$ such that C_i and C_j are dependent. Then

$$P(\bigcap \bar{C}_i) > 0.$$

To implement Lovász' Lemma in the setting previously described, we make the following simplification. For each $C_i = A_S$, let $x_i = a$, and for each $C_i = B_T$, let $x_i = b$. For a fixed A_S , let N_{AA} denote the number of $S' \neq S$ such that A_S and $A_{S'}$ are dependent. Similarly, define N_{AB} to be the number of T such that A_S and B_T are dependent. In exactly the same way, define N_{BA} and N_{BB} . Letting A and B denote typical A_S and B_T respectively, note that the desired conclusion follows if there exist positive numbers a and b such that $aP(A) < 1$, $bP(B) < 1$,

$$\log a > N_{AA}aP(A) + N_{BB}bP(B), \quad (3)$$

and

$$\log b > N_{BA}aP(A) + N_{BB}bP(B). \quad (4)$$

Note that A_S and B_T are dependent only if $|S \cap T| \geq 2$. A similar observation holds for the pairs $(A_S, A_{S'})$ and $(B_T, B_{T'})$.

For the purpose of this calculation, it suffices to use the following bounds:

$$N_{AA} \leq \binom{m}{2} \binom{N-2}{m-2} = O(N^{m-2}),$$

$$N_{AB}N_{BB} \leq \binom{N}{n} < N^n,$$

$$N_{BA} \leq \binom{n}{2} \binom{N-2}{m-2} = O(n^2N^{m-2}),$$

$$P(A) \geq m! p^q \quad \text{and} \quad P(B) = (1-p)^{\binom{q}{2}}.$$

Let $s = (m-2)/(q-1)$ and set

$$p = C_1 \cdot N^{-s}, \quad n = C_2 \cdot N^s \cdot \log N, \\ a = C_3 > 1 \quad \text{and} \quad b = \exp(C_4 \cdot N^s \cdot (\log N)^2),$$

where C_1 – C_4 are positive constants. Then $\log a > 0$,

$$N_{AA} \cdot a \cdot P(A) = O(N^{m-2} N^{-sq}) = o(1),$$

and

$$N_{AB} \cdot b \cdot P(B) < \exp((C_2 + C_4 - C_1 C_2^2/2) N^s (\log N)^2) = o(1),$$

if $\frac{1}{2} C_1 C_2^2 > C_2 + C_4$. Similarly, both sides of (4) are of order

$$c N^s (\log N)^2,$$

for an appropriate constant c . The constants C_1 – C_4 may be chosen so that (4) holds. Thus there is a 2-coloring of $[N]^2$ with no red G and no blue K_m , where $n = C_2 N^s \log N$. Solving for N in terms of n , we get the stated result. This completes the proof of Theorem 5. \square

There are numerous interesting problems that remain unsolved. Verification of the conjecture about the upper bound on the Ramey number $r(K_3, G)$ for any graph G of a fixed size n would be of interest. The determination of all graphs which are size linear is probably very difficult. However, even partial solutions to this problem are worth some effort.

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