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GENERATION OF ALTERNATING GROUPS BY PAIRS OF CONJUGATES

L. B. BEASLEY (Logan), J. L. BRENNER (Palo Alto), P. ERDŐS (Budapest), M. SZALAY (Budapest) and A. G. WILLIAMSON (Haywards Heath)

Abstract

Considering the conjugacy classes of the alternating group of degree n, those classes that contain a pair of generators are in the majority. In fact, the proportion of such classes is $1 - \varepsilon(n)$, and $\varepsilon(n) \to 0$ as $n \to \infty$.

1. Introduction

In this article, we obtain the following result for the alternating groups $\mathsf{Alt}\,(n)$:

The proportion of conjugacy classes in $\mathsf{Alt}\,(n)$ *that contain a pair of generators approaches* 1 *as* $n \to \infty$.

In Section 2 we give a quick proof of a weaker form of this asymptotic result. In the weaker form, "$n \to \infty$" is replaced by the condition "n increases through some set Σ_0 that has density 1 in the set $\mathbf{Z}$ of all integers". The argument uses results of Erdős, Lehner, Cameron, Neumann and Teague ([9], [5]) together with a combinatorial construction.

The strong form of the theorem is proved in Section **3**.

Some results from number theory needed in the proofs are established in Sections 2 and 3.

The number of classes involved in the construction in Section **3** is large enough to constitute an overwhelming majority in the set of all classes (as we show). It is reasonable to suppose that many more classes contain a pair of generators. This is indeed true; the proof would be too intricate to include,

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Akadémiai Kiadó, Budapest
D. Reidel, Dordrecht

since no uniform arguments seem to be available. On the other hand, although the discussion in Section **3** is not limpid in its simplicity, the argument does use a single technique.

2. A preliminary result

We first recall a theorem, due to P. Erdős and J. Lehner, concerning the number of summands in a partition.

2.01. THEOREM [9]. *Denote by $p(n)$ the number of unrestricted partitions of a positive integer n and by $p_k(n)$ the number of partitions of n which have at most k summands. If*

$$k = C^{-1}n^{1/2}\log n + xn^{1/2}\,,$$

then

$$\lim \frac{p_k(n)}{p(n)} = \exp\left[-2C^{-1}\exp - (1/2)Cx\right]$$

as $n \to \infty$. Here $C = \pi\,(2/3)^{1/2}$.

2.02. COROLLARY. *If $p^{(l)}(n)$ denotes the number of partitions of n such that the average size of a summand is at least l and $1 \leq l \leq n^{1/2}/\log n$, then*

$$\lim \frac{p^{(l)}(n)}{p(n)} = 1$$

as $n \to \infty$.

PROOF. 2.01 yields that, for almost all partitions of n (i.e. with the exception of $o(p(n))$ partitions of n, as $n \to \infty$), the number of summands is

$$\bigl(1 + o(1)\bigr)C^{-1}n^{1/2}\log n\,,$$

consequently the average size of a summand is

$$\bigl(1 + o(1)\bigr)\frac{Cn^{1/2}}{\log n} > \frac{n^{1/2}}{\log n}$$

in almost all partitions of n.

We also recall a result of P. J. Cameron, P. M. Neumann, and D. N. Teague:

2.03. THEOREM [5]. *The number of integers n that can be the degree of a primitive group contained properly in $\mathrm{Alt}\,(n)$ is vanishingly small. More precisely* *if $T(n_0)$ represents the number of such values of $n \leq n_0$, then* $\lim T(n_0)/n_0 = 0$ *as* $n_0 \to \infty$.

We emphasize that Theorem 2.03 heavily relies on the classification of finite simple groups.

For the symmetric groups $\mathsf{Sym}\,(n)$ we assert:

2.04. Theorem. *Let C be a class in $\mathsf{Sym}\,(n)$ of type*

$$T = 1^{e(1)}2^{e(2)}3^{e(3)} \ldots .$$

If T is not the type of an involution, and if the relation

$$\sum_{j \geq 1} e(j) \leq \frac{n}{2}$$

holds, then C contains a pair of elements that generate a primitive group.

Proof. Let

$$a = (1, 2, \ldots, k_1)(k_1 + 1, \ldots, k_2) \ldots (k_{r-1} + 1, \ldots, k_r)(k_r + 1) \ldots (n)$$

be a member of C, where the cycles are of decreasing length. By assumption, a is not an involution, hence k_1 is at least 3. Take an involution

$$t = (1, 2)(k_1, k_1 + 1) \ldots (k_{r-1}, k_{r-1} + 1)(i_1, k_r + 1) \ldots (i_{n-k_r}, n),$$

where $i_1, \ldots, i_{n-k_r} \leq k_r$ are such that the transpositions in t are pairwise disjoint. Such a t exists, since the i's can be chosen from $k_r - 2r$ elements and by assumption we have $2(r + n - k_r) \leq n$, so $n - k_r \leq k_r - 2r$. Now take $b = tat \in C$. The group generated by a and b is easily seen to be transitive. It contains $ab = (at)^2$. We have

$$a(1, 2)(k_1, k_1 + 1) \ldots (k_{r-1}, k_{r-1} + 1) =$$
$$= (1)(2, \ldots, k_1 - 1, k_1 + 1, \ldots, k_2 - 1, k_2 + 1, \ldots, k_{r-1} - 1, k_{r-1} + 1, \ldots$$
$$\ldots, k_r, k_{r-1}, k_{r-2}, \ldots, k_2, k_1)\,,$$

and by induction on $n - k_r$ one can easily check that at fixes 1 and permutes all other letters cyclically. To see that $\langle a, b\rangle$ is primitive, suppose if possible that Φ is a nontrivial set of imprimitivity that contains 1. Since $ab = (at)^2$ fixes 1, and since Φ contains elements in another cycle of ab, it follows that

$$|\Phi| \geq 1 + \frac{n-1}{2} > \frac{n}{2}\,,$$

which cannot be true.

Now we need information concerning the classes in $\mathsf{Alt}\,(n)$.

2.05. **Lemma** [7]. *The number of classes in* $\mathsf{Alt}\,(n)$ $(n > 1)$ *is equal to* $\alpha_n + \beta_n$, *where* α_n *is the number of partitions of* n *in which the number of even parts is even, and* β_n *is the number of partitions of* n *into unequal odd parts.*

Remark. If the orbits in a permutation P have odd and unequal length, the permutation $(12)P(12)$ has the same type as P, but is not conjugate to P inside $\mathsf{Alt}\,(n)$. [$P^{(12)}$ *is* conjugate to P in $\mathsf{Sym}\,(n)$.] This explains the term β_n.

Except in the above case, two permutations in $\mathsf{Alt}\,(n)$ are conjugate if they have the same type. (See [7].)

2.06. **Lemma**. *Let* η_n *be the proportion of classes*

$$T = 1^{e(1)}2^{e(2)}3^{e(3)} \dots$$

in $\mathsf{Alt}\,(n)$ $[\mathsf{Sym}\,(n)]$ $(n = \Sigma j \cdot e(j))$ *such that*

$$e(1) > \sum_{j>1} (j-2)e(j) .$$

(Thus $1 - \eta_n$ *is the proportion of classes in* $\mathsf{Alt}\,(n)$ $[\mathsf{Sym}\,(n)]$ *such that* $\sum_{j\geq 1} e(j) \leq$ $\leq n/2$.) *Then* $\eta_n \to 0$ *as* $n \to \infty$.

Remark. η_n is not the same for $\mathsf{Alt}\,(n)$ and for $\mathsf{Sym}\,(n)$.

Proof. The classes with $e(1) \leq \Sigma \dots$ are precisely the classes in which the average size of the orbits is ≥ 2. Apply Corollary 2.02, Lemma 2.05, and $\alpha_n + \beta_n \sim (1/2)p(n)$. (See [7].)

As a consequence of all the above results, none of which required the expenditure of great effort on our part, we come to the following conclusion (the number of involutory classes in $\mathsf{Alt}\,(n)$ is also negligibly small: $[n/4]$):

2.07. **Theorem**. *Let* $1 - \varepsilon_n$ *be the proportion of classes in* $\mathsf{Alt}\,(n)$ *that contain a pair of (conjugate) elements that generate* $\mathsf{Alt}\,(n)$. *Then as* $n \to \infty$ *through a certain set* Σ_0 *of integers that has density* 1 *in the set of all positive integers, the relation* $\lim \varepsilon_n = 0$ *holds.*

It will take considerably more effort to strengthen this last theorem to one in which Σ_0 is replaced by $\mathbf{Z}$, the set of all integers.

We conclude this section with an argument that shows how the stronger conclusion (with $\mathbf{Z}$ in place of Σ_0) follows.

In Section 3 we shall show that almost all partitions of n have a summand that is >1 and relatively prime to the other summands (see Theorem 3.04).

To see how this fact could be used we recall a theorem of Williamson.

2.08. THEOREM [16]. *If a primitive permutation group G of degree n contains a t-cycle (a permutation of type $1^{n-t}t^1$, $t > 1$) then G contains* $\mathsf{Alt}\,(n)$ *unless $t > (n-t)\,!$.*

If t is a summand in a partition τ of n, and if $t > 1$ is prime to the other summands, then any permutation of type τ generates a t-cycle. If, in addition, such a value t satisfies $t > (n-t)\,!$, then t is exponentially close to n. (If $n = 1000$, then $t \geq 994$.) Thus the number of summands in the partition τ is extremely small: $o(\log n)$. Such partitions are (asymptotically) in the minority, by the theorem of Erdős and Lehner, see 2.02. Thus 3.04 will yield 3.05.

3. The main theorem, and some lemmas from Number Theory

3.01. LEMMA [7]. *The number $a(n)$ of conjugacy classes in* $\mathsf{Alt}\,(n)$ *satisfies*

$$\lim \frac{a(n)}{p(n)} = 1/2$$

as $n \to \infty$.

We remind the reader of the asymptotic formula of Hardy and Ramanujan (see [1]), according to which

$$p(n) \sim 4^{-1}3^{-1/2}n^{-1}\exp\left(\pi(2/3)^{1/2}n^{1/2}\right).$$

This gives at once the following

3.02. LEMMA. *For $j = o(n^{1/2})$, we have*

$$\lim \frac{p(n-j)}{p(n)} = 1$$

as $n \to \infty$.

3.03. DEFINITION. A partition of n *has a prime part* if (at least) one of its summands is >1 and is relatively prime to the other summands. The symbol $p^{(0)}(n)$ denotes the number of partitions of n that have a prime part.

3.04. THEOREM. *Almost all partitions of n have a prime part, that is*

$$\lim \frac{p^{(0)}(n)}{p(n)} = 1 \qquad (n \to \infty).$$

PROOF. The proof relies on a result of Erdős and Turán. See the italicized theorem on page 6 of [11].

$$\text{Set} \quad m = [n^{1/5}],$$

$$l_n = \sum_{k=1}^{m} k = \frac{1}{2} m \, (m+1).$$

Then (from 3.02) $p(n - l_n) \sim p(n)$, so that almost all partitions of n contain every one of the summands $1, 2, \ldots, m$. Using the conjugate partition (in the dot diagram) this means that almost all partitions $n = \Sigma a_i$, $a_1 \geq a_2 \geq \geq \ldots \geq a_t$ have the property that

$$a_1 > a_2 > \ldots > a_m > a_{m+1} \geq \ldots \geq a_t .$$

Now we refer to some other known results. The asymptotic estimate

$$a_1 \sim \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n$$

for almost all partitions appears in [9]. Also, by [14] for almost all partitions,

$$a_m \sim \frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{\sqrt{6n}}{\pi n^{1/5}} < (1 - \delta) \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n$$

for any δ near 0 (and sufficiently large n).

Now let $\varrho(P)$ denote the largest prime factor of the period (order) $o(P)$ of the permutation P, i.e. $o(P) = \operatorname{lcm} [a_1, a_2, \ldots]$. (By [11],

$$\varrho(P) \sim \sqrt{6n} \, \frac{\log n}{2\pi}$$

for almost all partitions.) Then $\varrho(P)$ divides some one of $a_1, a_2, \ldots$, say

$$\varrho(P) | a_{i_0} .$$

The asymptotic result on a_m shows that $i_0 < m = [n^{1/5}]$; therefore the prime $\varrho(P) = a_{i_0}$ occurs just once and it is relatively prime to the other summands (in the case of almost all partitions).

3.05. THEOREM. *Let* $1 - \varepsilon_n$ *be the proportion of classes in* $\mathsf{Alt}\,(n)$ *that contain a pair of generators. Then* $\varepsilon_n \to 0$ *as* $n \to \infty$. *(In other words,* ~ 100 *percent of all classes contain a pair of generators.)*

PROOF. The truth of this assertion follows from 2.04, 3.04 with a small amount of additional argument. By Lemma 2.06, Theorem 2.04 says that almost all classes contain a pair that generate a primitive group; and from 3.04 it is clear that a single cycle is almost always contained in this group.

Let t be the size of this cycle. We recall Theorem 2.08 (of Williamson): *If a primitive permutation group G of degree n contains a t-cycle, then G contains* $\mathsf{Alt}\,(n)$ *unless* $t > (n-t)\,!$. The exceptional case $t > (n-t)\,!$ is the only sticking point to completion of the proof.

Now if $t > (n-t)\,!$, then the permutation P has type $a_1 \geq a_2 \geq \ldots$, with $a_1 = t$. (Here t is the prime a_{i_0} mentioned in the proof of 3.04.) In the extreme case $a_2 = \ldots = 1$, P has only $n - t + 1$ orbits; in any other case (with $t > (n-t)\,!$), P has even fewer orbits. Note that $n - t$ must be a very small number here; certainly if $\delta > 0$, is given, there is an n_0 so large that if $t > (n-t)\,!$, $n > n_0$, then $n - t < \delta \log n$. This last assertion follows from any (weak) form of Stirling's formula. One of the results of [9] is that a partition with so few summands is rare. Theorem 3.05 is proved.

We note that when t is a prime $\leq n-3$, the conclusion also follows from a theorem of Jordan (see [15], Theorem 13.9). For some partitions, Williamson's theorem goes further than Jordan's so we give some complements to our main theorem 3.05.

3.06. THEOREM. *Let P be a permutation of arbitrary type in* $\mathsf{Alt}\,(n)$. *Then it is true with probability* $1 - \varepsilon_n'$ *that an involution T exists such that* $\langle P, T\rangle \supseteq$ $\supseteq \mathsf{Alt}\,(n)$; *moreover* $\varepsilon_n' \to 0$ *as* $n \to \infty$.

In other words, almost all type can generate $\mathsf{Alt}\,(n)$ or $\mathsf{Sym}\,(n)$ with the help of a mate of period 2. (This explains work of G. A. Miller [13], H. R. Brahana [4]. See also M. D. E. Conder [6].)

PROOF. The construction of 2.04 used only involutions; and $\langle P, TPT\rangle$ is contained in $\langle P, T\rangle$.

3.07. THEOREM. *The proportion of permutations P in* $\mathsf{Alt}\,(n)$ *such that for some involution T (depending on P) the relation* $\langle P, T\rangle \supseteq \mathsf{Alt}\,(n)$ *holds is* $1 - \varepsilon_n^*$, *and* $\varepsilon_n^* \to 0$ *as* $n \to \infty$.

PROOF. For an analogue of 2.06 we can use V. L. Gončarov's theorem [12] (cf. [10]) that, for almost all permutations of degree n (i.e. with the exception of $o(n\,!)$ permutations), the total number of cycles in the canonical decomposition is $\sim \log n$. Next, instead of 3.04, we can apply J. D. Dixon's result

([8], Lemma 3) (cf. [2], [3]) that almost all permutations of degree n have, for a suitable *prime* $q \in [\log^2 n, n-3]$, exactly one cycle of length q and all other cycles of length relatively prime to q.

We remark that we do have a proof of the main theorem using only combinatorial arguments. In this proof we bypass theorem 3.04. The alternative proof is straightforward, but lengthy.

We can prove that if C is a class in $\mathit{Sym}\,(n)$, $n > 6$, C is not an involutory class, and if C has no fixed points, then C contains a pair of elements that generate $\mathit{Alt}\,(n)$ or $\mathit{Sym}\,(n)$. We do not go into details since the number of classes in $\mathit{Sym}\,(n)$ with $e(1) = 0$ is only

$$p(n) - p(n-1) \sim \frac{\pi}{\sqrt{6n}}\, p(n)\,.$$

Let $1 - \varepsilon_n''$ be the proportion of classes in $\mathit{Sym}\,(n)$ that contain a pair that generates $\mathit{Alt}\,(n)$ or $\mathit{Sym}\,(n)$. The proof of Theorem 3.05 yields that $\varepsilon_n'' \to 0$ as $n \to \infty$.

The rate at which $\varepsilon_n'' \to 0$ (as an infinitesimal in n) remains to be investigated. It is probable that there exists a positive constant c such that, for sufficiently large n,

$$\varepsilon_n'' \leq \exp\left(-\frac{cn^{1/2}}{\log n}\right)$$

owing to the *prime* "prime parts" close to $n^{1/2}$. We prove only the following *lower* estimate.

3.08. Lemma. *The infinitesimal ε_n'' does not approach* 0 *any faster than*

$$\left(1 + O(n^{-1/2}\log^2 n)\right) n^{1/2} 12^{1/2} \pi^{-1} \exp\left(-(2 - 2^{1/2})\,\frac{n^{1/2}\pi}{6^{1/2}}\right)$$

$$\left(\sim 3^{1/2}\pi^{-1} n^{1/2}\,\frac{p(n/2)}{p(n)}\right).$$

Proof. First we establish the claim that if a class has more than $(n + 1)/2$ orbits, the class cannot contain a pair of generators of a transitive group. Suppose that the type is $1^{e(1)}2^{e(2)}\ldots$. Then any element of the class can be written as a product of $\Sigma_{j\geq 2}(j-1)e(j)$ transpositions. If we assume that the number of orbits

$$k = \Sigma_{j\geq 1}\, e(j) > \frac{n+1}{2}$$

then the number of transpositions in the factorization of a member of the class is $n - k < (n - 1)/2$, hence the subgroup generated by two members of the class is contained in a subgroup generated by $2(n - k) < n - 1$ transpositions. As it is well-known, at least $n - 1$ transpositions are needed to generate a transitive group, thus no two members of our class can generate a transitive group.

Now set $p_k(n)$ = number of partitions of n with at most k parts. Then $p_k(n)$ = number of unrestricted partitions of n into summands not exceeding k, according to the dot diagram for each such partition. We need the theorem of Hardy and Ramanujan (see [1]), according to which

$$p(n) = \left(1 + O(n^{-1/2})\right)\left(4^{-1}n^{-1}3^{-1/2}\exp\left(\pi(2/3)^{1/2}n^{1/2}\right)\right). \tag{3.09}$$

Set $C = \pi(2/3)^{1/2}$. Then, for $k \geq n/2$,

$$p(n) - p_k(n) = \sum_{k<j\leq n} p_j(n - j) = \sum_{k<j\leq n} p(n-j). \tag{3.10}$$

From (3.09), it can be seen that if $t = O(n^{1/2}\log n)$, then

$$\frac{p(n-t)}{p(n)} = \left(1 + O\left(n^{-1/2}\log^2 n\right)\right)\exp\left(\frac{-Ctn^{-1/2}}{2}\right). \tag{3.11}$$

Now use the value $k = (n + 1)/2$, and set

$$j = [k] + 1 = \frac{n}{2} + O(1),$$

and take

$$i = \left[\frac{2\cdot 6^{1/2}}{\pi}(n - j)^{1/2}\log(n - j)\right]$$

(greatest integer). Using $p(n) \leq p(n + 1)$, (3.10), (3.11), it is seen that the conclusion of the lemma follows from the analysis below:

$$\frac{p(n) - p_k(n)}{p(n)} = \frac{p(n-j)}{p(n)}\left\{\sum_{s=j}^{i+j}\frac{p(n-s)}{p(n-j)} + O(n)\,\frac{p(n-j-i)}{p(n-j)}\right\} =$$

$$= \frac{p(n-j)}{p(n)}\left\{1 + \sum_{t=1}^{i}\frac{p(n-j-t)}{p(n-j)} + O(n^{-1})\right\} =$$

$$= \frac{p(n-j)}{p(n)}\left\{1 + \left(1 + O(n^{-1/2}\log^2 n)\right)\sum_{t=1}^{\infty}\exp\left(-t\pi 6^{-1/2}(n-j)^{-1/2}\right) + O(n^{-1})\right\} =$$

$$= \left(1 + O(n^{-1/2} \log^2 n)\right) \frac{p(n-j)}{p(n)} \{1 - \exp\left(-\pi 6^{-1/2}(n-j)^{-1/2}\right)\}^{-1} =$$

$$= \left(1 + O(n^{-1/2} \log^2 n)\right) 6^{1/2}(n-j)^{1/2}\pi^{-1} \frac{p(n-j)}{p(n)}$$

because of **(3.09)**, this is equal to

$$\left(1 + O(n^{-1/2} \log^2 n)\right) 6^{1/2}(n-j)^{1/2}\pi^{-1} n(n-j)^{-1} \exp\left(-C(n^{1/2} - (n-j)^{1/2})\right) =$$
$$= \left(1 + O(n^{-1/2} \log^2 n)\right) 6^{1/2}\pi^{-1}(n/2)^{1/2} \cdot 2 \cdot \exp\left(-C(n^{1/2} - (n/2)^{1/2})\right) =$$
$$= \left(1 + O(n^{-1/2} \log^2 n)\right) n^{1/2} 12^{1/2} \pi^{-1} \exp\left(-(2 - 2^{1/2}) n^{1/2}\pi/6^{1/2}\right).$$

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UTAH STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
LOGAN, UT 84322
U.S.A.

10 PHILLIPS ROAD
PALO ALTO, CA 94303
U.S.A.

MTA MATEMATIKAI KUTATÓ INTÉZET
H−1364 BUDAPEST
P. O. BOX 127.
REÁLTANODA U. 13−15.
HUNGARY

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
ALGEBRA ÉS SZÁMELMÉLET TANSZÉK
H−1088 BUDAPEST
MÚZEUM KRT. 6−8.
HUNGARY

HAYWARDS HEATH COLLEGE
HAYWARDS HEATH
WEST SUSSEX
ENGLAND