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Multiplicative Functions and Small Divisors, II

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Let n be square-free and h a multiplicative function satisfying $0 \leqslant h(p) \leqslant 1/(k-1)$ on primes p, where $k \geqslant 2$. It is shown that

$$\sum_{d \mid n} h(d) \leqslant (2k + o(1)) \sum_{d \mid n, d \leqslant n^{1/k}} h(d), \qquad \text{for} \quad k = 2, 3, 4, \ldots,$$

where $o(1)$ is a quantity that tends to zero as $\sum_{p \mid n} 1 = \nu(n) \to \infty$. Such inequalities have applications to Probabilistic Number Theory.

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At the 1983 Western Number Theory Conference in Asilomar, one of us (K.A.) proposed as problem 407 the following:

CONJECTURE. (i) *Given $k \geqslant 2$, there exists $c_k > 0$ such that, for all multiplicative functions h satisfying $0 \leqslant h(p) \leqslant c_k$ on primes p,*

$$\sum_{d \mid n} h(d) \ll_k \sum_{d \mid n, d \leqslant n^{1/k}} h(d) \tag{1.1}$$

holds for all square-free n.

(ii) *In part* (i) *$c_k = 1/(k-1)$ is admissible for $k = 2, 3, \ldots$.*

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The purpose of this paper is to prove the stronger part of the conjecture, namely (ii). In the first paper under the same title [2], among other things we proved part (i) of the conjecture by establishing the following inequality more generally for sub-multiplicative functions $h \geqslant 0$ (these are functions satisfying $h(mn) \leqslant h(m)\, h(n)$, for $(m, n) = 1$).

THEOREM 1. *Let $h \geqslant 0$ be sub-multiplicative and satisfy*

$$0 \leqslant h(p) \leqslant c < \frac{1}{k-1}$$

for all primes p. Then for all square-free n we have

$$\sum_{d \mid n} h(d) \leqslant \left\{1 - \frac{kc}{1+c}\right\}^{-1} \sum_{d \mid n, d \leqslant n^{1/k}} h(d).$$

Clearly Theorem 1 settles Conjecture (i) with any $c_k < 1/(k-1)$. On the other hand, the conjecture is false with $c_k > 1/(k-1)$. For, let r be a large integer and $p_1, ..., p_r$ be distinct primes such that $p_1 \sim p_2 \sim p_3 \sim \cdots \sim p_r$. So a divisor d of n will satisfy $d \leqslant n^{1/k}$ if d has asymptotically fewer than r/k prime factors. Thus

$$\left\{\sum_{d \mid n} h(d)\right\}\left\{\sum_{d \mid n, d \leqslant n^{1/k}} h(d)\right\}^{-1} \sim (1+c)^r \left\{\sum_{l=0}^{r/k} \binom{r}{l} c^l\right\}^{-1}, \tag{1.2}$$

for the multiplicative function h satisfying $h(p) = c$ on primes p. The maximum value of $\binom{r}{l} c^l$ occurs when $l \sim rc/(1+c)$, as $r \to \infty$. So the quantities in (1.2) are unbounded if $c > 1/(k-1)$ and hence (ii) is best possible.

We had been aware of the validity of (ii) in the case $k = 2$ and one of us (K.A. [1]) applied this to Probabilistic Number Theory. Such applications motivated us to study the more general inequality (1.1).

We prove Conjecture (ii) in Section 3 by utilising a powerful result of Baranyai [3] on hypergraphs. Prior to proving Conjecture (ii) we establish in Section 2 a weaker version of (1.1) in the case $c_k = 1/(k-1)$, because its proof sheds some light on the scope of the method we had used earlier to prove Theorem 1.

Throughout, the latters p, q, with or without subscripts will denote primes and g, h will represent multiplicative functions. Implicit constants are absolute unless dependence is indicated by a subscript.

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THEOREM 2. *Let $k \geqslant 2$. If h satisfies $0 \leqslant h(p) \leqslant 1/(k-1)$, then we have*

$$\sum_{d \mid n} h(d) \leqslant \frac{k\nu(n)}{k-1} \sum_{d \mid n,\, d \leqslant n^{1/k}} h(d)$$

for all square-free n, where $\nu(n) = \sum_{p \mid n} 1$.

For Theorem 2 and for later use we establish

LEMMA 1. *Let n be square-free, $0 < \alpha < 1$. For fixed α and n, the quantity*

$$R_{\alpha,n}(h) = \left(\sum_{d \mid n,\, d \leqslant n^{\alpha}} h(d) \right) \Big/ \sum_{d \mid n} h(d)$$

decreases as h increases.

Proof. The lemma is trivial if $\nu(n) \leqslant 1$. So let $\nu(n) \geqslant 2$.
Define

$$\chi_\alpha(x) = \begin{cases} 1, & \text{if} \quad x \leqslant \alpha \\ 0, & \text{if} \quad x > \alpha. \end{cases}$$

Then

$$\begin{aligned} R_{\alpha,n}(h) &= \sum_{d \mid n} \chi_\alpha\left(\frac{\log d}{\log n}\right) \frac{h(d)}{\prod_{q \mid n}(1+h(q))} \\ &= \sum_{d \mid n/p} \left\{ \chi_\alpha\left(\frac{\log d}{\log n}\right) \frac{h(d)}{1+h(p)} \right. \\ &\qquad \left. + \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right) \frac{h(pd)}{1+h(p)} \right\} \frac{1}{\prod_{q \mid n,\, q \neq p}(1+h(q))} \\ &= \sum_{d \mid n/p} \left\{ \chi_\alpha\left(\frac{\log d}{\log n}\right) \left(1 - \frac{h(p)}{1+h(p)}\right) \right. \\ &\qquad \left. + \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right) \frac{h(p)}{1+h(p)} \right\} \frac{h(d)}{\prod_{q \mid n,\, q \neq p}(1+h(q))}, \end{aligned} \tag{2.1}$$

for some $p \mid n$. Note that

$$\chi_\alpha\left(\frac{\log d}{\log n}\right) \geqslant \chi_\alpha\left(\frac{\log p + \log d}{\log n}\right)$$

and so (2.1) implies that $R_{\alpha,n}(h)$ decreases by increasing $h(p)$ and not changing the values $h(q)$, for $q \neq p$. But then, by increasing the values $h(q)$ over the primes q in succession, we see that Lemma 1 is true.

In view of Lemma 1 it suffices to prove Theorem 2 in the case $h(p) = 1/(k-1)$ for all p. We shall now discuss somewhat more generally than what is required for Theorem 2, since this will reveal both the scope and limitations of the approach.

Let $F(\alpha, c, n)$ denote $R_{\alpha,n}(h)$ in the case where $h(p) = c$, for all p. To get a lower bound for $F(\alpha, c, n)$ we could attempt to bound $\chi_\alpha(x)$ from below. Here $x = \log d/\log n$. It is natural to minorize $\chi_\alpha(x)$ by a polynomial in x. The best linear polynomial which minorizes $\chi_\alpha(x)$ is

$$y = 1 - \frac{x}{\alpha},$$

which is the straight line obtained by joining $(0, 1)$ with $(\alpha, 0)$ in the (x, y) plane and, in fact, using this, Theorem 1 was proved in [2].

Next, we experiment with a polynomial of degree 2. Let t satisfy

$$-\alpha^{-2} \leqslant t \leqslant \alpha^{-1}. \tag{2.2}$$

Then

$$f(x) = tx^2 - \left(\alpha t + \frac{1}{\alpha}\right) x + 1 \tag{2.3}$$

minorizes $\chi_\alpha(x)$. Therefore

$$F(\alpha, c, n) \geqslant \frac{1}{H(n)} \sum_{d \mid n} \left\{ \frac{t \log^2 d \, h(d)}{\log^2 n} - \left(\alpha t + \frac{1}{\alpha}\right) \frac{\log d}{\log n} h(d) + h(d) \right\}, \tag{2.4}$$

where

$$H(n) = \sum_{d \mid n} h(d).$$

Note that

$$\begin{aligned} \frac{1}{\log n} \sum_{d \mid n} h(d) \log d &= \sum_{d \mid n} \frac{h(d)}{\log n} \sum_{p \mid d} \log p \\ &= \sum_{p \mid n} \frac{\log p}{\log n} \sum_{d \mid n/p} h(pd) \\ &= \frac{H(n)}{\log n} \sum_{p \mid n} \frac{h(p) \log p}{1 + h(p)} = \frac{cH(n)}{1 + c}. \end{aligned} \tag{2.5}$$

Similarly

$$\begin{aligned}
&\frac{1}{\log^2 n}\sum_{d\mid n} h(d)\log^2 d\\
&\quad=\frac{1}{\log^2 n}\sum_{d\mid n} h(d)\left(\sum_{p\mid d}\log p\right)^2\\
&\quad=\sum_{\substack{p,q\mid n\\ p\neq q}}\frac{\log p\log q}{\log^2 n}\sum_{d\mid n/pq} h(pqd)+\sum_{p\mid n}\frac{\log^2 p}{\log^2 n}\sum_{d\mid n/p} h(pd)\\
&\quad=\frac{H(n)}{\log^2 n}\sum_{p,q\mid n}\frac{\log p\log q\, h(pq)}{(1+h(p))(1+h(q))}\\
&\qquad+\frac{H(n)}{\log^2 n}\sum_{p\mid n}\log^2 p\left\{\frac{h(p)}{1+h(p)}-\frac{h^2(p)}{(1+h(p))^2}\right\}\\
&\quad=H(n)\left\{\left(\frac{c}{1+c}\right)^2+\frac{c}{(1+c)^2\log^2 n}\sum_{p\mid n}\log^2 p\right\}. \qquad (2.6)
\end{aligned}$$

So (2.3)–(2.6) yield

$$F(\alpha, c, n)\geqslant f\left(\frac{c}{1+c}\right)^2+\frac{tc}{(1+c)^2\log^2 n}\sum_{p\mid n}\log^2 p. \tag{2.7}$$

Note that

$$1=\sum_{p\mid n}\frac{\log p}{\log n}\leqslant v(n)^{1/2}\left(\sum_{p\mid n}\frac{\log^2 p}{\log^2 n}\right)^{1/2}$$

by the Cauchy–Schwartz inequality and so

$$\frac{1}{\log^2 n}\sum_{p\mid n}\log^2 p\geqslant\frac{1}{v(n)}. \tag{2.8}$$

Hence (2.8) and (2.7) combine to give

$$F(\alpha, c, n)\geqslant f\left(\frac{c}{1+c}\right)+\frac{tc}{(1+c)^2 v(n)}. \tag{2.9}$$

Obviously we want t as large as possible in (2.9). In Theorem 2, $\alpha=1/k$ and so, as permitted by (2.2), we take $t=k$. Also $c=1/(k-1)$. With these values of t and α, we find that

$$f\left(\frac{c}{1+c}\right)=0.$$

That is, the best quadratic polynomial passes through $(\alpha, 0)$. Thus the lower bound we get is

$$F\left(\frac{1}{k}, \frac{1}{k-1}, n\right) \geqslant \frac{(k-1)}{k\nu(n)},$$

which proves Theorem 2.

Theoretically, bounds for $F(\alpha, c, n)$ should get better by increasing the degree of the minorizing polynomial. But, from a practical point this would involve expressions of the form

$$\frac{1}{\log^m n} \sum_{p \mid n} \log^m p, \qquad m = 1, 2, 3, \ldots$$

which would give weaker lower bounds as m increases. However, it might be worthwhile to pursue this approach by taking into account the cancellation among the higher moments.

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THEOREM 3. *Let* $0 \leqslant h(p) \leqslant 1/(k-1)$ *for all* p. *Then, for* $k = 2, 3, 4, \ldots$,

$$\sum_{d \mid n} h(d) \leqslant (2k + o(1)) \sum_{d \mid n, d \leqslant n^{1/k}} h(d),$$

where $o(1)$ *tends to zero as* $\nu(n) \to \infty$. *In particular Conjecture* (ii) *is true.*

We will deduce Theorem 3 from the following result which is a special case of a deep theorem of Baranyai on hypergraphs [3, p. 93].

LEMMA 2. *Let* S *be a set of* km *elements. Then the* $\binom{km}{m}$ *subsets of* S, *comprised of* m *elements each, can be grouped* k *at a time, such that the* k *subsets (each of size* m) *in every such group generate a partition of* S.

Proof of Theorem 3. In view of Lemma 1, we may assume that $h(p) = 1/(k-1)$ in Theorem 3.

Let $\nu(n) = km + l$, $0 \leqslant l \leqslant k-1$, and $n = p_1 p_2 \cdots p_{\nu(n)}$. For some $j < m$ consider a particular divisor of n having $k(m-j)$ prime factors—say $N = p_1 \cdots p_{k(m-j)}$. Then, according to Lemma 2, the divisors of N having eactly $m - j$ prime factors can be grouped k at a time such that in every such group the divisors d_j are pairwise relatively prime and $d_1 \cdots d_k = N$. So there will be at least one divisor among the d_i that is $\leqslant N^{1/k}$ and in particular this divisor is $\leqslant n^{1/k}$ as well. Thus there are at least

$$\frac{1}{k}\binom{k(m-j)}{m-j}$$

divisors of N that are $\leqslant n^{1/k}$ which have exactly $(m-j)$ prime factors.

The number of ways of choosing such divisors N of n is

$$\binom{km+l}{k(m-j)}.$$

However, a divisor d of n with $v(d)=m-j$ could occur as a divisor of several such N. The maximum frequency of occurrence of such d will be

$$\binom{km+l-m+j}{(k-1)(m-j)}.$$

This is because $v(N)=k(m-j)$ and so given d, we have freedom in choosing the remaining $(k-1)(m-j)$ prime factors of N, and these primes are to be chosen from among the remaining $km+l-(m-j)$ prime factors of n. Thus we are guaranteed that there are at least

$$\frac{1}{k}\frac{\binom{k(m-j)}{m-j}\binom{km+l}{k(m-j)}}{\binom{km+l-m+j}{(k-1)(m-j)}}, \tag{3.1}$$

divisors of n which are $\leqslant n^{1/k}$. It turns out that the expression in (3.1) is equal to

$$\frac{1}{k}\binom{km+l}{m-j} \tag{3.2}$$

and this is a miraculous coincidence!

From (3.1) and (3.2) we see that

$$\sum_{d\mid n,\, d\leqslant n^{1/k}} h(d) \geqslant \frac{1}{k}\sum_{j=0}^{m}\binom{km+l}{m-j}\left(\frac{1}{k-1}\right)^{m-j}. \tag{3.3}$$

It is a well-known fact concerning the Binomial distribution that

$$\lim_{r\to\infty}\frac{1}{(1+c)^r}\sum_{l=0}^{[rc/(1+c)]}\binom{r}{l}c^l=\frac{1}{2}, \tag{3.4}$$

where [ ] is the greatest integer function. With $r = km + l$, $c = 1/(k-1)$, we have $[rc/(1+c)] = m$. Thus from (3.3) and (3.4) we deduce that

$$\sum_{d \mid n, d \leqslant n^{1/k}} h(d) \geqslant \frac{1}{(2k + o(1))} \left(1 + \frac{1}{k-1}\right)^{\nu(n)},$$

which is Theorem 3.

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While using Baranyai's result to construct groups of divisors satisfying $d_1 \cdots d_k = N$, we noted that one out of every k such divisors has to be $\leqslant N^{1/k}$. However, we should expect about half of such divisors to be $\leqslant n^{1/k}$. This suggests that $(2k + o(1))$ in Theorem 3 could perhaps be replaced by $4 + o(1)$. In particular we feel that the implicit constant in Theorem 3 will be absolute.

The use of hypergraphs restricted us in Section 3 to consider only integer values $k \geqslant 2$. This was sufficient for Conjecture (ii). But in view of Theorems 1 and 2 which hold for all real $k \geqslant 2$ we feel that Conjecture (ii) will hold as stated for all real $k \geqslant 2$ as well. Although the method of Section 2 did not give a proof of Conjecture (ii) but supplied only a partial result, still that approach was valid for all real $k \geqslant 2$. It might to worthwhile to see if the methods of Sections 2 and 3 could be combined to tackle some of these questions.

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