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ON THE NUMBER OF PARTITIONS OF n WITHOUT A GIVEN SUBSUM (I)

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1. Introduction

Let us denote by $p(n)$ the number of unrestricted partitions of n, by $r(n, m)$ the number of partitions of n whose parts are at least m, and by $R(n, a)$ the number of partitions of n:

$$n = n_1 + \cdots + n_t,$$

whose subsums $n_{i_1} + \cdots + n_{i_j}$ are all different from a. Furthermore, if $\mathscr{A} = \{a_1, \ldots, a_k\}$, we denote by $r(n, \mathscr{A})$ the number of partitions of n with no parts belonging to $\mathscr{A}$.

Let us consider now partitions of n for which each part is allowed to occur at most once. In that case the above notations will be changed for $q(n)$, $\rho(n, m)$, $Q(n, a)$, $\rho(n, \mathscr{A})$.

Clearly we have:

$$r(n, m) = r(n, \{1, 2, \ldots, m-1\})$$

$$R(n, a) \geq r(n, a+1) \tag{1}$$

$$R(n, a) \geq r(n, \{1, 2, \ldots, \lfloor a/2 \rfloor, a\}) \tag{2}$$

where $\lfloor x/2 \rfloor$ denotes the integral part of x.

In [4], the following estimation is given for $R(n, a)$: when a is fixed, and n tends to infinity,

$$R(n, a) \sim p(n) \left(\frac{\pi}{\sqrt{6n}} \right)^{\psi(a)} u(a) \tag{3}$$

where $\psi(a) = \lfloor a/2 + 1 \rfloor$, and $u(a) \in \mathbb{N}$. The value of $u(a)$ is computed for $1 \leq a \leq 20$, and it does not seem easy to get a simple formula for $u(a)$. The results are $u(1) = 1$, $u(2) = 4$, $u(3) = 3$, $u(4) = 16$, and u is increasing from $a = 3$ to

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$a = 20$. Moreover, J. Dixmier gives the following inequalities:

$$\text{for a even} \quad (\lfloor a/3 \rfloor - 1)!\, a^{a/6+3} \leq u(a) \leq 2^{a/2} a!/(a/2-1)! \tag{4}$$

$$\text{for a odd} \quad (\lfloor a/3 \rfloor - 1)!\, a^{a/6+2} \leq u(a) \leq 2^{a/2} a!/(\lfloor a/2 \rfloor)! \tag{5}$$

It follows from (3), the definition of ψ, and the behaviour of u, that for n large enough,

$$R(n, 1) > R(n, 2) > R(n, 3) > R(n, 4) \tag{6}$$

and for $a = 2b$, $2 \leq b \leq 9$,

$$R(n, 2b+2) < R(n, 2b) < R(n, 2b+1) < R(n, 2b-1).$$

At the end of the paper, a table of $R(n, a)$ is given. It has been calculated by J. Dixmier, H. Epstein and O.E. Lanford, using the induction formula.

$$f(n, p, \mathscr{A}) = \sum_{i \leq p} f(n-i, i, \mathscr{A} \cup \mathscr{A} - i).$$

Here, $f(n, p, \mathscr{A})$ denotes the number of partitions of n in parts $\leq p$ such that no subsum belongs to $\mathscr{A}$, and $\mathscr{A} - i = \{a - i;\, a \in \mathscr{A},\, a - i > 0\}$. It has been independently calculated by F. Morain and J.P. Massias. They have used computer algebra systems MAPLE and MACSYMA to compute polynomials mentioned by Diximier (cf. [4], 4.3 and 4.10). Unfortunately these polynomials are of degree $((a+1)(a+2)/2) - 2$, and it is not easy to deal with them for large values of a.

As observed in [4], $R(n, 2) < R(n, 3)$ for $10 \leq n \leq 106$, which contradicts (6). But (6) is true only for n large enough.

The aim of this paper is to study $R(n, a)$ for a depending on n, and smaller than $\lambda_0 \sqrt{n}$, where λ_0 is a small positive constant. The tools for that are an estimation for $r(n, \mathscr{A})$ (cf. Lemma 2 below), and inequalities involving $R(n, a)$, extending (1) and (2). We shall prove the following result.

Theorem 1. *There exists $\lambda_0 > 0$, such that uniformly for $1 \leq a \leq \lambda_0 \sqrt{n}$, we have, when n goes to infinity,*

(i) $$\log\left(\frac{R(n, a)}{p(n)}\right) \leq \left(\psi(a) \log \frac{\pi a}{\sqrt{6n}}\right) + O(1/\sqrt{n})$$

(ii) $$\log\left(\frac{R(n, a)}{p(n)}\right) \geq \psi(a) \log \frac{\pi a}{\sqrt{6n}} - \gamma_a a + O(a^2/\sqrt{n})$$

where $\gamma_a = \frac{1}{2}$ if a is odd, and, if a is even,

$$\gamma_a = \tfrac{1}{2} + \log 3 - \tfrac{7}{6} \log 2 + c\frac{\log a}{a} = +0.79 \cdots + c\frac{\log a}{a} \text{ where } c \text{ is a fixed constant.}$$

Let us observe that, when a goes to infinity, (4) or (5) gives

$$\left(-\frac{1+\log 3}{3}+o(1)\right)a \leqslant \log u(a) - \tfrac{1}{2}a \log a \leqslant (\log 2 - \tfrac{1}{2} + o(1))a \tag{7}$$

while (3), (i) and (ii) yield that

$$-\gamma_a a + o(a) \leqslant \log u(a) - \tfrac{1}{2}a \log a \leqslant o(a) \tag{8}$$

which is better than (7) except for the lower bound when a is even.

We intend to treat the case $\lambda_0\sqrt{n} \leqslant a$ in an other paper, by a different method, which will give also an estimation for $Q(n, a)$. For this quantity, we here give only a lower bound.

Theorem 2. *There exists $\lambda_1 > 0$, such that, uniformly for $1 \leqslant a \leqslant \lambda_1\sqrt{n}$, we have:*

$$\log\left(\frac{Q(n,a)}{q(n)}\right) \geqslant -\frac{a}{6}\log\frac{16}{3} - \log 3 + O\left(\frac{a^2}{\sqrt{n}}\right).$$

We thank very much J. Dixmier for several interesting remarks.

2. Preliminary Results

Let us first recall the definition of the mth Bessell polynomial $y_m(x)$; (cf. [9]):

$$\begin{aligned} &y_0(x) = 1 \\ &y_m(x) = (1+mx)y_{m-1}(x) + x^2 y'_{m-1}(x). \end{aligned} \tag{9}$$

From that definition, it is easy to see that, if we set

$$F(x) = (\exp(\sqrt{x}))/\sqrt{x},$$

then we have (cf. [5], Lemma 1)

$$F^{(m)}(x) = \frac{\exp\sqrt{x}}{2^m x^{(m+1)/2}}\, y_m\left(-\frac{1}{\sqrt{x}}\right). \tag{10}$$

Furthermore, it follows from (9) that $y_m(0) = 1$.

Lemma 1. *For m odd, the function $x \to y_m(x)$ is increasing for $x \in]-\infty, +\infty[$, and its zero α_m satisfies*

$$-\frac{\gamma}{m} < \alpha_m < -\frac{\gamma}{m+1} \tag{11}$$

where γ is a constant satisfying $1.5 < \gamma < 1.51$.

For m even, $y_m(x)$ is decreasing on $]-\infty, \alpha'_m[$ and increasing on $]\alpha'_m, +\infty[$, and

$$-\frac{\gamma}{m+0.77} < \alpha'_m < -\frac{\gamma}{m+1.61}. \tag{12}$$

Proof. This is proved in [1] and [2]. $\square$

Lemma 2. *Let us define P_m by*

$$P_m(x) = \sum_{k=0}^{m} \frac{(-1)^k (2k)!}{2^{2k}(k!)^2} \left(\prod_{j=-k+1}^{k} (m+j) \right) x^k.$$

Then we have

$$y_m(x) y_m(-x) = P_m(x^2).$$

Proof. It is known that $y_m(x)$ satisfies the differential equation (cf. [9], p. 7)

$$x^2 y'' + 2(x+1)y' - m(m+1)y = 0. \tag{13}$$

If we set $w(x) = y_m(-x)$, it satisfies

$$x^2 w'' + 2(x-1)w' - m(m+1)w = 0. \tag{14}$$

Now we set $Y = yw$. It is known that Y satisfies a linear differential equation, and with some calculation this equation writes:

$$x^4 Y''' + 6x^3 Y'' - ((4m(m+1) - 6)x^2 + 4)Y' - 4m(m+1)xY = 0. \tag{15}$$

We can easily check that Y satisfies (15), by calculating Y'' and Y''' in terms of yw, $y'w$, yw', $y'w'$ by (13) and (14).

Now, we are looking for polynomial solutions of (15). It turns out that these solutions are of the form $cP_m(x^2)$, and considering $x = 0$ yields Lemma 2. □

We are very pleased to thank A. Salinier for this proof of Lemma 2. This result is somewhat curious. We would expect that, in the product $y_m(x)y_m(-x)$ the coefficient of x^{2k} is a polynomial of degree $4k$ in m. Indeed, it follows from (9), cf. [9], p. 13, that

$$y_m(x) = 1 + \sum_{k=1}^{m} a_k^{(m)} x^k$$

with

$$a_k^{(m)} = \frac{1}{2^k k!} \prod_{j=-k+1}^{k} (m+j). \tag{16}$$

Lemma 3. *For x such that $0 \leqslant m \leqslant 1/\sqrt{2}$, we have*

$$\begin{aligned} y_m(-x) &\geqslant \left(1 - \frac{m(m+1)}{2} x^2\right) \exp\left(-\frac{m(m+1)}{2} x\right) \\ &= \exp\left(-\frac{m(m+1)}{2}(x + O(x^2))\right). \end{aligned}$$

Proof. From the obvious inequality $(m-i+1)(m+i) \leqslant (m+1)m$, (16) implies

$$a_k^{(m)} \leqslant \left(\frac{m(m+1)}{2}\right)^k \frac{1}{k!}$$

which gives

$$y_m(x) \leq \exp\left(\frac{m(m+1)}{2}x\right). \tag{17}$$

Now, let us write the polynomial P_m, defined in Lemma 2, in the form

$$P_m(x) = \sum_{k=0}^{m} d_k x^k.$$

Then we have

$$\left|\frac{d_{k+1}}{d_k}\right| = \frac{(m+k+1)(m-k)(2k+1)}{(2k+2)} \leq 2m^2.$$

Thus, the absolute value of the general term of $y_m(x)y_m(-x)$, that is $|d_k x^{2k}|$ is decreasing, and, as it alternates in sign, for $mx \leq 1/\sqrt{2}$, we have

$$1 - \frac{m(m+1)}{2}x^2 \leq y_m(x)y_m(-x) = P_m(x^2) \leq 1,$$

which, with (17) completes the proof of Lemma 3. □

More accurate estimations have been obtained by M. Chellali, using Agarwal's integral representation

$$y_m(x) = \frac{1}{n!}\int_0^\infty t^n\left(1 + \frac{tx}{2}\right)^n \mathrm{e}^{-t}\,\mathrm{d}t$$

and the saddle point method (cf. [3]).

Proposition. *There exists $\lambda_2 > 0$ such that, if $\mathcal{A} = \{a_1, \ldots, a_k\}$ satisfies $s = a_1 + a_2 + \cdots + a_k \leq \lambda_2 n$, then, when n tends to infinity, we have*

(i) $$r(n, \mathcal{A}) \leq \left(\prod_{i=1}^{k} a_i\right) p(n) \left(\frac{\pi}{\sqrt{6n}}\right)^k \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

(ii) $$r(n, \mathcal{A}) \geq \left(\prod_{i=1}^{k} a_i\right) p(n) \left(\frac{\pi}{\sqrt{6n}}\right)^k \exp(O(s/\sqrt{n})).$$

Proof. It is very similar to the proof of Theorem 1 of [5]. First we introduce the operator $D^{(m)}$. Let $f: \mathbb{R} \to \mathbb{R}$, $m \geq 1$, $u_1, \ldots, u_m$ be positive. We set

$$\begin{aligned} D^{(1)}(u_1; f, x) &= f(x) - f(x - u_1) \\ D^{(m)}(u_1, \ldots, u_m; f, x) &= D^{(m-1)}(u_1, \ldots, u_{m-1}; f, x) \\ &\quad - D^{(m-1)}(u_1, \ldots, u_{m-1}; f, x - u_m). \end{aligned}$$

From the generating functions, we observe

$$r(n, \{a_1, \ldots, a_k\}) = D^{(k)}(a_1, \ldots, a_k; p, n). \tag{18}$$

Now, with $F(x)=(\exp(\sqrt{x}))/\sqrt{x}$, the classical result of Hardy and Ramanujan can be written (cf. [5]) as follows:

$$p(n)=\frac{C^3}{2\pi\sqrt{2}}F'(C^2(n-1/24))+f_1(n) \tag{19}$$

with $C=\pi\sqrt{2/3}$, and

$$|f_1(n)|\leqslant\frac{0.11}{n}\exp\Big(\frac{C\sqrt{n}}{2}\Big).$$

Furthermore, using Lemma 3 of [5], (18) and (19) give

$$r(n,\{a_1,\ldots,a_k\})=\Big(\prod_{i=1}^{k}a_i\Big)\frac{C^{2k+3}}{2\pi\sqrt{2}}F^{(k+1)}(C^2(\xi-1/24))+O\Big(\frac{2^k}{n}\exp\Big(\frac{C\sqrt{n}}{2}\Big)\Big) \tag{20}$$

with

$$n-s\leqslant\xi\leqslant n. \tag{21}$$

We now use (10) to estimate $F^{(k+1)}(C^2(\xi-1/24))$. The function $x\to\dfrac{\exp(\sqrt{x})}{x^{(k+2)/2}}$ is increasing for $x\geqslant(k+2)^2$. As $s=a_1+\cdots+a_k\geqslant k(k+1)/2\geqslant k^2/2$, which implies $k\leqslant\sqrt{2s}$, for λ_2 small enough, from (21) we have

$$\frac{\exp(C\sqrt{n-s-1})}{(n-s-1)^{(k+2)/2}}\leqslant\frac{\exp(C\sqrt{\xi-1/24})}{(\xi-1/24)^{(k+2)/2}}\leqslant\frac{\exp(C\sqrt{n})}{n^{(k+2)/2}}. \tag{22}$$

Now let us turn to the proof of (i). By (22), the main term of (20), is at most

$$\Big(\prod_{i=1}^{k}a_i\Big)\frac{C^{2k+3}}{(2\pi\sqrt{2})2^{k+1}C^{k+2}}\frac{\exp(C\sqrt{n})}{n^{(k+2)/2}}y_{k+1}\Big(\frac{-1}{C\sqrt{\xi-1/24}}\Big).$$

For λ_2 small enough, we have

$$\frac{-1}{C\sqrt{\xi-1/24}}\geqslant\frac{\gamma}{k+3}$$

and thus Lemma 1 gives

$$y_{k+1}\Big(\frac{-1}{C\sqrt{\xi-1/23}}\Big)\leqslant 1. \tag{23}$$

Using the estimation

$$p(n)=\frac{\exp(C\sqrt{n})}{4\sqrt{3}}(1+O(1/\sqrt{n})),$$

the main term of (20) is at most

$$\Big(\prod_{i=1}^{k}a_i\Big)p(n)\Big(\frac{C}{2\sqrt{n}}\Big)^k\Big(1+O\Big(\frac{1}{\sqrt{n}}\Big)\Big). \tag{24}$$

To complete the proof of (i), it remains to check that the error term of (20) is included in the above error term. First, (24) is

$$\gg k!\,\frac{\exp(C\sqrt{n})}{n}\left(\frac{C}{2\sqrt{n}}\right)^k \gg \frac{\exp(C\sqrt{n})}{n}\left(\frac{Ck}{2\mathrm{e}\sqrt{n}}\right)^k$$

by Stirling's formula. So it is enough to show that

$$\left(\frac{4\mathrm{e}\sqrt{n}}{Ck}\right)^k \leqslant \exp\left(\frac{C}{2}\sqrt{n}\right).$$

But the left hand side of the above inequality is an increasing function on k, for $k \leqslant \frac{4}{C}\sqrt{n}$, and we know that $k \leqslant \sqrt{2s} \leqslant \sqrt{2\lambda_2 n}$. To conclude, we observe that for λ_2 small enough, we have:

$$\left(\frac{4\mathrm{e}}{C\sqrt{2\lambda_2}}\right)^{\sqrt{2\lambda_2 n}} \leqslant \exp\left(\frac{C}{2}\sqrt{n}\right).$$

In order to prove (ii), first we apply Lemma 3, to obtain

$$y_{k+1}\left(\frac{-1}{C\sqrt{\xi-1/24}}\right) \geqslant \exp\left(-\frac{(k+1)(k+2)}{2C\sqrt{n-s-1}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\right)$$

which with (23) yields

$$y_{k+1}\left(\frac{-1}{C\sqrt{\xi-1/24}}\right) = O(s/\sqrt{n}). \tag{25}$$

Then we observe that

$$\exp(C\sqrt{n-s-1}) = \exp(C\sqrt{n} + O(s/\sqrt{n}))$$

and

$$\begin{aligned}(n-s-1)^{(k+2)/2} &= \exp\left(\frac{k+2}{2}(\log n + O(s/n))\right)\\ &= n^{(k+2)/2}\exp(O(s/\sqrt{n}))\end{aligned}$$

since $k = O(\sqrt{n})$.

Furthermore, by (10), (22) and (25), the main term of (20) is at least

$$\left(\prod_{i=1}^{k} a_i\right)\left(\frac{C}{2}\right)^{k+1}\frac{\exp(C\sqrt{n})}{2\pi\sqrt{2}n^{(k+2)/2}}\exp(O(s/\sqrt{n})).$$

The end of the proof of (ii) goes in the same way as for (i). $\square$

Remark. A similar proposition is given in [7] in the case of restricted partitions. A more general estimation is given by J. Herzog (cf. [10] and [11]), using a Tauberian theorem.

3. The upper bound in Theorem 1

First let us say that a partition $n = n_1 + \cdots + n_t$ of n *represents* a if there is a subsum $n_{i_1}, \ldots, n_{i_j}$, $1 \leq i_1 < \cdots < i_j \leq t$ which is equal to a. Thus $R(n, a)$ counts the number of partitions of n which do not represent a.

Clearly if $b < a$, b and $a - b$ cannot be together parts of a partition which does not represent a.

Let us suppose first that a is odd. From the above remark, we deduce that for all integers, i, with $1 \leq i \leq \lfloor a/2 \rfloor$, at most one of i and $a - i$ can be a part, and thus

$$R(n, a) \leq \sum_{\varepsilon_1, \ldots, \varepsilon_{\lfloor a/2 \rfloor}} r\Big(n, \Big(\bigcup_{i=1}^{\lfloor a/2 \rfloor} \{i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}\}\Big) \cup \{a\}\Big) \tag{26}$$

where in the summation $\varepsilon_i \in \{0, 1\}$.

Now we apply our proposition, with $k = \psi(a)$, and $s \leq \sum_{a/2 < j \leq a} j$, and we obtain that

$$R(n, a) \leq p(n)\Big(\frac{\pi}{\sqrt{6n}}\Big)^{\psi(a)}\Big(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\Big) \sum_{\varepsilon_1, \ldots, \varepsilon_{\lfloor a/2 \rfloor}} a \prod_{i=1}^{\lfloor a/2 \rfloor} i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}.$$

But this summation is exactly

$$a \prod_{i=1}^{\lfloor a/2 \rfloor} (i + (a - i)) = a^{\psi(a)}$$

which proves (i) for a odd. When a is even, the part $a/2$ can occur but only once. Thus we have

$$\begin{aligned} R(n, a) \leq & \sum_{\varepsilon_1, \ldots, \varepsilon_{\lfloor a/2 \rfloor}} r\Big(n, \Big(\bigcup_{i=1}^{\lfloor a/2 \rfloor} \{i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}\}\Big) \cup \{a\}\Big) \\ & + \sum_{\varepsilon_i, \ldots, \varepsilon_{\lfloor a/2 \rfloor}} r\Big(n - \frac{a}{2}, \Big(\bigcup_{i=1}^{\lfloor a/2 \rfloor} \{i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}\}\Big) \cup \{a\}\Big) \end{aligned} \tag{27}$$

where the first summation counts partitions without any part equal to $a/2$, and the second counts partitions with one part equal to $a/2$.

For the second sum we obtain the upper bound

$$\begin{aligned} & p(n - a/2)\Big(\frac{\pi a}{\sqrt{6(n - a/2)}}\Big)^{\psi(a)}\Big(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\Big) \\ & \qquad \leq \Big[\frac{\exp(C\sqrt{n - a/2})}{4\sqrt{3}(n - a/2)^{\psi(a)/2+1}}\Big]\Big(\frac{\pi a}{\sqrt{6}}\Big)^{\psi(a)}(1 + O(1/\sqrt{n})). \end{aligned}$$

But, as already observed, the function $x \to \exp\sqrt{x}/x^k$ is increasing for $x \geq k^2$, and for λ_0 small enough, the expression between brackets is smaller than

$$\frac{\exp(C\sqrt{n})}{4\sqrt{3}n^{\psi(a)/2+1}} = p(n)n^{-\psi(a)/2}(1 + O(1/\sqrt{n})).$$

So, the second sum in (27) is not bigger than the first one, which was already estimated when a is odd. This completes the proof of (i). $\square$

4. The lower bound in Theorem 1

Let us suppose first that a is odd. Then

$$R(n, a) \geqslant r(n, \{1, 3, 5, \ldots, a\})$$

and, observing that $\psi(a) = (a+1)/2$, by the Proposition we have

$$R(n, a) \geqslant \frac{(2\psi(a))!}{2^{\psi(a)}\psi(a)!} p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a)} \exp(O(a^2/\sqrt{n})).$$

By Stirling's formula, $(2u)!/2^u u! \geqslant u^u 2^u \mathrm{e}^{-u}$, and since $\psi(a) \geqslant a/2$, we obtain (ii).

Let us suppose now that a is even. In fact, the following reasoning works also for a odd, but it gives a worse estimation than the preceding one. For real numbers x and y, let us denote the set of integers belonging to the real interval $]x, y[$ by $]x \cdots y[$. We set

$$\mathcal{A} = [1 \cdots a/3] \cup [a/2 \cdots 2a/3] \cup \{a\}.$$

Then, it is not difficult to see that

$$R(n, a) \geqslant r(n, \mathcal{A})$$

(which is slightly bettter than (2)), and considering the three possible cases $a \equiv 0, 2, 4 \bmod 6$, that card $\mathcal{A} = \psi(a)$. By the proposition, we get

$$R(n, a) \geqslant a \frac{\lfloor a/3 \rfloor! \, \lfloor (2a-1)/3 \rfloor!}{\lfloor a/2 - 1 \rfloor!} p(n)\left(\frac{\pi}{\sqrt{6n}}\right)^{\psi(a)} \exp(O(a^2/\sqrt{n})).$$

Using Stirling's formula in the form

$$[u + O(1)]! = u^u \mathrm{e}^{-u} \exp(O(\log u))$$

we obtain (ii) with an effectively computable constant c. $\square$

5. Proof of Theorem 2

We consider now only partitions without any repetition, and we look at a subset of $[1 \cdots (a-1)]$, say $\mathcal{A}$ with the following property:

no element $j \in \left[1 \cdots \frac{a}{3}\right]$ belongs to $\mathcal{A}$

for each $j \in \left]\frac{a}{3} \cdots \frac{a}{2}\right[$, there are 3 possibilities:

$j \in \mathcal{A}$ and $a - j \notin \mathcal{A}$, $j \notin \mathcal{A}$ and $a - j \in \mathcal{A}$, $j \notin \mathcal{A}$ and $a - j \notin \mathcal{A}$

for each $j \in \left[\frac{2a}{3} \cdots a - 1\right]$, there are 2 possibilities, $j \in \mathcal{A}$ or $j \notin \mathcal{A}$.

For any such $\mathscr{A}$, we have:

$$\operatorname{Card} \mathscr{A} \leqslant c_1 a^2.$$

How many such $\mathscr{A}$'s are there? As

$$\operatorname{Card}\left(\left]\frac{a}{3} \cdots \frac{a}{2}\right[\right) \geqslant \frac{a}{6} - 1 \quad \text{and} \quad \operatorname{Card}\left(\left[\frac{2a}{3} \cdots (a-1)\right]\right) \geqslant \frac{a}{3},$$

there are more than

$$3^{(a/6)-1} 2^{a/3}$$

such sets $\mathscr{A}$. Further, to build a partition of n, we choose such a set $\mathscr{A}$, and we complete by a partition of n-Card $\mathscr{A}$, without any part smaller than $a+1$. Thus, since $\rho(n, m)$ is non-decreasing in n (cf. [8]),

$$Q(n, a) \geqslant 3^{a/6-1} 2^{a/3} \rho(n - c_1 a^2, a+1).$$

Using Theorem 1 of [8], which gives $\rho(n, m) \geqslant q(n)/2^{m-1}$, and the classical estimation

$$q(n) \sim \frac{1}{4(3n^3)^{\frac{1}{4}}} \exp(\pi\sqrt{n/3}),$$

Table of $R(n, a)$

n	p(n)	a =	1	2	3	4	5	6	7	8	9	10	11	12
1	1		0											
2	2		1											
3	3		1											
4	5		2	2										
5	7		2	2										
6	11		4	3	5									
7	15		4	4	4									
8	22		7	5	7	8								
9	30		8	7	7	8								
10	42		12	9	12	9	17							
11	56		14	11	12	12	13							
12	77		21	15	19	15	21	24						
13	101		24	19	20	19	21	22						
14	135		34	23	30	24	30	25	46					
15	176		41	30	32	30	32	30	36					
16	231		55	38	46	35	50	36	50	64				
17	297		66	46	51	45	49	44	51	54				
18	385		88	58	70	55	72	50	73	63	107			
19	490		105	72	78	65	77	67	69	76	81			
20	627		137	88	105	81	103	80	103	81	112	147		
21	792		165	109	119	98	112	95	104	101	105	126		
22	1002		210	133	156	116	154	111	151	119	149	134	242	
23	1255		253	161	177	143	163	133	158	134	147	161	173	
24	1575		320	198	228	170	218	158	214	157	198	180	239	302
25	1958		383	240	262	202	241	187	219	209	195	200	236	250

Table of $R(n, a)$ (continued).

n	p(n)	a =	1	2	3	4	5	6	7	8	9	10
			11	12	13	14	15	16	17	18	19	20
26	2436		478	288	332	244	307	219	301	234	274	237
			303	276	488							
27	3010		574	349	381	291	343	261	309	279	268	273
			295	304	361							
28	3718		708	421	476	343	440	308	403	329	397	305
			414	338	457	629						
29	4565		847	503	550	410	483	362	438	371	395	374
			385	385	448	492						
30	5604		1039	604	680	485	611	420	565	435	534	402
			533	423	584	550	922					
31	6842		1238	722	785	571	688	494	597	519	540	513
			529	494	549	624	662					
32	8349		1507	859	961	677	846	581	772	579	710	618
			697	542	755	663	857	1172				
33	10143		1794	1024	1111	798	954	676	831	677	730	696
			691	623	706	750	801	930				
34	12310		2167	1216	1349	932	1177	783	1043	800	943	779
			1024	707	938	837	˙1050	995	1745			
35	14883		2573	1439	1560	1100	1318	916	1139	909	973	926
			983	825	933	902	963	1120	1223			
36	17977		3094	1706	1880	1287	1608	1063	1408	1058	1246	1046
			1302	890	1226	1043	1282	1190	1577	2108		
37	21637		3660	2014	2175	1503	1821	1235	1534	1239	1295	1181
			1355	1143	1160	1216	1217	1291	1496	1650		
38	26015		4378	2371	2603	1761	2189	1423	1905	1395	1662	1372
			1705	1309	1627	1272	1637	1477	1859	1773	3104	
39	31185		5170	2794	3008	2052	2483	1652	2079	1617	1735	1548
			1728	1502	1529	1472	1497	1592	1764	1909	2173	
40	37338		6153	3285	3581	2384	2980	1911	2525	1877	2179	1762
			2253	1684	2177	1723	2010	1704	2351	2096	2706	3737

which implies

$$\frac{q(n - c_1 a^2)}{q(n)} = \exp(O(a^2/\sqrt{n})),$$

we obtain Theorem 2. $\square$

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