A Collection of Open Problems

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One of the most satisfying aspects of the Third International Conference on Combinatorial Mathematics was the continual exchange of open problems that occurred. These problems were presented during the course of lectures, during the discussion periods following lectures, and during the special problem session that was a major activity of one evening.

The fourteen questions posed below were asked at that special session. For the benefit of the reader, extra material has been appended to the questions in the form of "Remarks" in those cases when solutions are known, and by "References" to provide some background information. The addresses of the posers of the problems are included so that the readers of this volume can obtain additional current information from them or perhaps provide more information to them.

The number of labeled graphs not containing a specific subgraph.

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Let G be a graph. Denote by $T(n; G)$ (T for Turán) the largest integer for which there is a graph on n vertices and $T(n; G)$ edges that does not contain G as a subgraph. The number $f(n; G)$ of labeled graphs on n vertices that do not contain G as a subgraph clearly satisfies

$$f(n; G) > 2^{T(n; G)}.$$

Also, it is known that

$$f(n; G) < 2^{(1+o(1))T(n; G)}. \tag{1}$$

For $G = K(r)$, (1) was proved by Erdös, Kleitman, and Rothschild [2], and in a stronger form for $G = K(3)$. In a forthcoming paper, Colaitis, Prömel, and Rothschild proved (1) in a stronger form for $G = K(r)$. Recently, Erdös, Frankl, and Rödl [1] proved (1) for all G. Kleitman and Winston [3] proved

$$f(n; C_4) < 2^{cn^{3/2}}.$$

As far as I know

$$f(n; C_4) = 2^{(1/2+o(1))n^{3/2}}$$

is still open.

REFERENCES

1. ERDÖS, P., P. FRANKL & V. RÖDL. The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. Graphs Comb. In press.
2. ERDÖS, P., D. J. KLEITMAN & B. ROTHSCHILD. 1976. Asymptotic enumeration of K_n-free graphs. Coll. Intern. sulla Teorie Comb. Roma 1973, Tome II, Atti dei Con. Lincei No. 1 Acad. Naz. Lincei, pp. 19–27.
3. KLEITMAN, D. J. & K. J. WINSTON. 1980. The asymptotic number of lattices. Ann. Discrete Math. **6:** 243–249.

The number of triangles that must contain some edge of a graph.

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Let $G(n; cn^2)$ be a graph on n vertices and cn^2 edges. Assume that every edge of our graph is contained in at least one triangle. Denote by $f(n; c)$ the largest integer for which at least one edge is contained in at least $f(n; c)$ triangles. Estimate $f(n; c)$ as well as possible both from above and below.

Noga Alon proved $f(n; c) < \alpha_c n^{1/2}$, and Szemerédi proved by his regularity lemma that, for every $c > 0, f(n; c) \to \infty$ as $n \to \infty$.

One could also ask for the largest e_n for which there is a $G(n; e_n)$ with the property that every edge is contained in exactly one triangle. This question is identical with a question of Brown, Sós, and Erdös [1], and was successfully attacked by Ruzsa and Szemerédi [2], who proved

$$cnr_3(n) < e_n < o(n^2)$$

where $r_3(n)$ is the largest integer t such that there is a sequence of integers $a_1 < a_2 < \cdots < a_t \leq n$ that does not contain an arithmetic progression of three terms.

REFERENCES

1. BROWN, W. G., P. ERDÖS & V. T. SÓS. 1973. Some extremal problems on r-graphs. *In* New Directions in the Theory of Graphs (Proc. Third Ann Arbor Conference on Graph Theory, Univ. Michigan, Ann Arbor, Michigan, 1971). Academic Press, New York: 53–63.
2. RUZSA, I. & E. SZEMERÉDI. 1973. On the existence of triangulated spheres in 3-graphs and related problems. Periodica Math. Hungar. **3/4:** 221–228. (See also Keszthely meeting 1975.)

Complete prime subsets of consecutive integers.

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Find the smallest integer $k > 1$ and an integer n such that $gcd(n, n + k) = 1$ and $gcd(n + i, n(n + k)) > 1$ for each i, $1 \leq i < k$.

If $gcd(n, n + k) > 1$, the smallest k is 16. Every integer from 2184 to 2200 has a prime factor in common with 2184 or 2200 [1].

REFERENCE

1. ERDÖS, P. & J. L. SELFRIDGE. 1971. Complete prime subsets of consecutive integers. *In* Proceedings of the Manitoba Conference on Numerical Mathematics, pp. 1–14. MR49 #2597.

An extremal set coloring problem.

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Consider a family of 4-element subsets of a set S that has the property that any 2-coloring of S forces at least one of the 4-sets to be monochromatic. Let $m(4)$ denote the minimal cardinality that such a family may have.

It is easy to show that $15 \leq m(4) \leq 27$. Paul Seymour [1] has exhibited a family of twenty-three 4-element subsets of an 11-element set, thus showing that $m(4) \leq 23$. Aizley and Selfridge [2] have announced that $m(4) \geq 19$.

Can you improve the lower or the upper bound?

REFERENCES

1. SEYMOUR, P. D. 1974. A note on a combinatorial problem of Erdös and Hajnal. J. London Math. Soc. **8**(2): 681–682. MR50 #1894.
2. AIZLEY, P. & J. L. SELFRIDGE. 1977. A lower bound for $m(4)$. Abstract 747-05-18. Notices Am. Math. Soc. **24**: A-452.

Orientably simple graphs.

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Every graph that embeds in the torus T can be embedded in the surface with three crosscaps. Many can also be drawn in the Klein bottle K, that is, the sphere with two crosscaps. A graph G that embeds in T but not K we call *orientably simple*. The problem is to characterize orientably simple graphs, not necessarily by excluded minors. A number of examples are known, for example, K_7, but virtually no theory exists.

An extremal clique covering problem.

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Does almost every graph have a collection of cliques such that each vertex appears in not too many cliques ($\leq f(n)$ per vertex) and almost all edges are covered (at most $f^2(n)$ uncovered)? If cn cliques per vertex are allowed, we can cover all edges. Can one find cliques with each vertex in $o(n)$ cliques such that $o(n^2)$ edges are uncovered? The best possible result would be $f(n) = O(n/\log n)$. This follows from results on interval number, and would imply that the interval number of the random graph is $O(n/\log n)$.

REFERENCE

1. ERDÖS, P. & D. WEST. A note on the interval number of a graph. Discrete Math. In press.

The maximum distance between triangulations of an n-gon.

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Consider triangulations of a fixed n-gon. Each diagonal cuts a quadrilateral into two triangles. Call two triangulations *adjacent* if one is obtained from the other by changing a single diagonal to cut its quadrilateral the other way. Let $f(n)$ be the maximum distance between two triangulations. It is easy to show $f(n) \leq 2n - 10$ (for $n \geq 8$), which is conjectured optimal. Sleator and Tarjan showed $f(n) \geq 1.75n$ constructively, and Thurston showed $f(n) > 2n - \log n$ nonconstructively.

Threshold graph covering and embedding problems.

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One can ask similar questions to those about embeddings of or coverings by cliques, using threshold graphs (as defined by Chvátal and Hammer) instead of cliques. For example, I have a family of graphs such that the nth graph can be edge-covered by $2n$ threshold graphs, but requires $3n$ threshold graphs to edge-partition it. Is this best possible?

A graph with $2n$ nodes and $n^2 + 1$ edges must contain a threshold graph with $n(1 + \frac{1}{12})$ edges (joint result with Paul Erdös). We think it need not contain a threshold graph with $n(1 + \frac{3}{8})$ edges. What is the correct multiple of n?

A conjectured ratio of central and bicentral unlabeled n-vertex trees.

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CONJECTURE: Let $c(n)$ and $b(n)$ be the numbers of central and bicentral unlabeled n-vertex trees, respectively. Then $c(n)/b(n) \to 1$ as $n \to \infty$.

Labeled and unlabeled trees have been counted by number of vertices and diameter, and tables of numbers appear in the *IBM Journal of Research and Development* 4 (1960), p. 476, and *Mathematics of Computation* 25 (1971), p. 632. Since a tree is central if and only if its diameter is even, $c(n)$ and $b(n)$ can be calculated from the tables. The ratio $c(n)/b(n)$ appears to tend to 1 (with an error of less than 1 percent for $n = 20$), whence the above conjecture. The corresponding conjecture for labeled trees has been proved by G. Szekeres, but the unlabeled case is apparently still open.

Interval k-colorings of m-partitions of guaranteed minimum size.

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An *interval* k*-coloring* is a coloring of the unit interval I by k colors, such that each color forms a measurable set. An m*-partition* of the coloring is a partition of I into m disjoint subsets $A_1, A_2, \ldots, A_m$, each a union of intervals, and each capturing precisely $1/m$ of the measure of each color. The *size* of the partition is the number of cuts that form all these intervals. Let $c(m, k)$ be the minimum number c (possibly ∞) such that every interval k-coloring has an m-partition of size at most c.

CONJECTURE: $c(m, k) = (m-1)k$.

This conjecture is true for $m = 2^j$ and every k, and for $k \leq 2$ and every m (see [1], [2]). Also, easily $c(m, k) \geq (m-1)k$ for every m, k. The smallest unknown case is $m = k = 3$. (We do not even know if $c(3, 3)$ is finite.)

REMARK: The conjecture has now been proved by Alon.

REFERENCES

1. ALON, N. & D. B. WEST. The Borsuk-Ulam theorem and bisection of necklaces. In press.
2. GOLDBERG, C. H. & D. B. WEST. 1985. Bisection of circle colorings. SIAM J. Alg. Discrete Math. **6**: 93–106.

Diameters of Cayley graphs.

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Let G be a group and $T \subset G$ a set of generators. The *Cayley graph* $\Gamma = \Gamma(G, T)$ has $V(\Gamma) = G$ for its vertex set and $E(\Gamma) = \{[g, gt]: g \in G, t \in T\}$ for its edge set. Prove (or disprove) that there exists a constant c such that for every n and for every set T of generators of the symmetric group S_n, the diameter of $\Gamma(S_n, T)$ is less than n^c.

The longest simple path with increasing labels in a labeled graph.

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Given a labeling $\lambda: E(K_n) \to \{1, 2, \ldots, \binom{n}{2}\}$ of the edges of K_n with distinct integers, denote by $I(\lambda)$ the length of a longest simple path with increasing labels. Determine $f(n) = \min_\lambda I(\lambda)$.

It is known that

$$cn^{1/2} \leq f(n) \leq \frac{n}{2}$$

Is $f(n) = o(n)$? Is $f(n) > n^{(1/2)+\varepsilon}$ for some $\varepsilon > 0$?

Fast determination of graph diameters.

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Find a fast algorithm to determine the diameter of a graph.

Using matrix multiplication one can determine the diameter in time $O(n^{2.496})$ [1]. On the other hand, the best known lower bound is $O(n \log n)$.

REFERENCE

1. CHUNG, F. R. K. 1984. Diameters of Communications Networks 1984, AMS Short Course Lecture Notes.

The existence of spanning bipartite subgraphs of specified connectivity.

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Is it true that every $(2k-1)$-connected graph has a k-connected spanning bipartite subgraph?

REMARK: This is true for $k = 1$ (trivial) and $k = 2$ (easy). It is also very easy to prove the assertion obtained by replacing connectivity by edge-connectivity.