Invariant Random Subgroups and their Applications

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Lemma (A-Glasner-Virag)

Every IRS of Γ arises as the stabilizer for a p.m.p. action of Γ.

General aspects of invariant random subgroups

Invariant random subgroups:

- Tend to behave like normal subgroups, rather than arbitrary subgroups

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- Tends to carry over spectral information (spectral measure, L^2 Betti numbers, Plancherel measure, etc)

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- This convergence notion is equivalent to local sampling convergence: from a typical place in M_n, and looking at a bounded distance, we wont be able to distinguish M_n and $\widetilde{M}$.
- Typically, whatever is continuous for normal chains, is expected to be continuous for this convergence notion.

The Lück Approximation Theorem for IRS's

b_k: k-th Betti number over $\mathbb{Q}$; β_k^2: k-th L^2 Betti number.

Theorem (Lück Approximation)

Let M be a finite complex and let $H_n \leq \pi_1(M)$ be finite index subgroups such that $\mu_{H_n} \to \mu_1$. Then for all k we have

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Lück: normal chains, Farber: approximating chains, Bergeron-Gaboriau: any chain. Proof: weak and pointwise convergence of spectral measure. Gaboriau: L^2 Betti numbers of a p.m.p. action only depend on its IRS.

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- Minimal number of generators (converges on chains, but does the limit depend on the chain?). Fixed Price Problem of Gaboriau.

Sofic groups and IRS

A countable group Γ is *sofic* if it admits a sequence of maps $\phi_n : \Gamma \to \mathrm{Sym}(n_k)$ such that for every finite subset $S \subseteq \Gamma$, ϕ_n restricted to S behaves like an injective group homomorphism with ratio of error tending to 0 (Gromov, Weiss).

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Lemma

Let $\Gamma = F/N$ where F is a free group. Then Γ is sofic if and only if there exist subgroups $H_n \leq F$ of finite index such that

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Generalized sofic question (Aldous-Lyons): is every IRS in a free group the weak limit of finite index IRS's? Also open.

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Let G_n be finite d-regular graphs with $|G_n| \to \infty$. If $\lim \lambda_{G_n}$ is supported on $[-\rho(T_d), \rho(T_d)]$ then

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If Serre's condition holds, then $\gamma_k = 0$ for all k and $\lim_n \lambda_{G_n} = \lambda_{T_d}$.

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Does not work for rank 1 in general ($\mathrm{SL}_2(R)$).

Works for semisimple Lie groups. New proofs and extensions are in the works.

Other classifications and boundaries

Vershik: Classification of IRS's for $\mathrm{FSym}(\mathbb{N})$

Bowen + Grigorchuk + Kravchenko: Zoos and shape of the simplex of IRS's for large groups, analysis of IRS's that are invariant under automorphisms (lamplighter group, $\mathrm{Aut}(F_n)$).

[7Samurai] Let K be any discrete subgroup in G and let H be a nontrivial IRS in K. Then the limit set of H equals the limit set of K a.s. In particular, any IRS in G has full limit set.

[Cannizzo-Kaimanovich] Let H be a stationary random subgroup of a free group F. Then the action of H on the boundary of F is conservative a.s.

[Glasner-Weiss] Topological version of IRS: Uniformly Recurrent Subgroup (minimal subshift of Sub_Γ).


Tsachik Gelander
Jean Raimbault
Ian Biringer
Nik Nikolov
Miklos Abert
Nicolas Bergeron
Iddo Samet

Big higher rank locally symmetric spaces are also fat

For a Lie group G let $X = G/K$ be its symmetric space.
If Y is connected, complete, locally-X, then $Y = \Gamma \backslash X$ where $\Gamma \leq G$ is discrete. Let

$$(Y)_{<r} = \{x \in Y \mid \text{injrad}(x) < r\}$$

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Theorem (7Samurai)

Let G be a higher rank simple Lie group with symmetric space X. Let $\Gamma_n \leq G$ be lattices and let $X_n = \Gamma_n\backslash X$ with $\mathrm{vol}(X_n) \to \infty$. Then for all $r > 0$ we have

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When Γ is a fixed arithmetic lattice and $\Gamma_n \leq \Gamma$ is a sequence of congruence subgroups, we have explicit bounds on the size of the thin part and the typical injrad.

The IRS behind

Theorem (7Samurai)

Let G be a higher rank simple Lie group and let $\Gamma_n \leq G$ be lattices with $\mathrm{vol}(X_n) \to \infty$. Then we have $\lim_{n\to\infty} \mu_{\Gamma_n} = \mu_1$.

$m(\pi, \Gamma)$: multiplicity of $\pi \in \widehat{G}$ in $L^2(\Gamma \backslash G)$. $d(\pi)$: multiplicity in $L^2(G)$.

Theorem (7Samurai Limit Multiplicity)

Let (Γ_n) be a uniformly discrete sequence of lattices in G such that $\lim_{n\to\infty} \mu_{\Gamma_n} = \mu_1$. Then for all $\pi \in \widehat{G}$, we have

$$\frac{m(\pi, \Gamma_n)}{vol(\Gamma_n \backslash G)} \to d(\pi).$$

Also implies weak convergence of Plancherel measures. For chains, these are due to DeGeorge-Wallach and Delorme. Lots of deep papers. For the non-uniform case, recent work of Finis, Lapid and Müller.

Character rigidity

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Theorem (Peterson-Thom)

No nontrivial characters (and hence IRS's) for $SL_n(K)$ where K is an infinite field or the localization of an order in a number field.

Much more on semisimple lattices: Creutz, Creutz-Peterson.

Open problems: rank 1 simple Lie groups

In rank 1, not every sequence of lattices approximate G.

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- Question [Weinberger] Assume G has finitely many non-conjugate lattices below any given volume. Do random lattices converge to μ_1?

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- Discrete version [A] Let Γ be a f.g. residually finite group. Let H_n be a uniform random subgroup of Γ of index $\leq n$. Does $\mu_{H_n} \to \mu_1$ a.s.?

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- What is a character for a Lie (or locally compact) group? Ideally should be induced from lattices and should be connected to IRS's.

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Covering towers (chains) admit a stronger limit: graphing, profinite action, foliated space with transversal measure. Let the *rank* of a measured groupoid be the infimum of measures of its generating subsets.

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- [A-Szegedy] The normalized entropy

$$h(A, \Gamma, H) = H(A_H(k\text{-i.i.d.})) / |\Gamma : H|$$

is continuous in IRS convergence. Would imply Lück Approx. mod p.

The end

THANK YOU!!!