Classical analytic number theory

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[1] I took some parts of these notes from Terry Tao's blog (written for his class Analytic Prime Number Theory a few years ago), and Antal Balog's lecture notes An introduction to analytic number theory (written for his class Analytic Number Theory at BSM). On some points, however, I might follow alternative explanation and structure, e.g. that of Montgomery's and Vaughan's Multiplicative Number Theory I. Classical Theory.

Contents

Chapter 1
Elementary theory

1.1 The Riemann ζ function on $s > 1$

Proposition 1.1.1. *Let $y < x$ be real numbers, and $f : [y, x] \to \mathbf{R}$ a monotone function. Then*

$$\sum_{n\in\mathbf{Z}:y\leqslant n\leqslant x} f(n) = \int_y^x f(t)dt + O(|f(x)|) + O(|f(y)|).$$

Proof. We can assume that $y, x \in \mathbf{Z}$. Also, we can assume that f is monotone increasing. Then

$$\int_y^x f(t)dt + f(y) \leqslant \int_y^x f(\lceil t\rceil)dt + f(y) \leqslant \sum_{n\in\mathbf{Z}:y\leqslant n\leqslant x} f(n)$$
$$\leqslant \int_y^x f(\lfloor t\rfloor)dt + f(x) \leqslant \int_y^x f(t)dt + f(x).$$

The proof is complete. □

Proposition 1.1.2. *Assume $f : \mathbf{N} \to \mathbf{C}$, $F : \mathbf{R}_+ \to \mathbf{C}$, $g : \mathbf{R}_+ \to \mathbf{R}_+$ such that $g(x) \to 0$ as $x \to \infty$. Then the following are equivalent.*

(i) For all $1 \leqslant y < x$, we have

$$\sum_{y\leqslant n<x} f(n) = F(x) - F(y) + O(g(x)) + O(g(y)).$$

(ii) There exists some $c \in \mathbf{C}$ such that for all $x \geqslant 1$,

$$\sum_{n<x} f(n) = c + F(x) + O(g(x)).$$

Proof. Assume (ii) holds. Then, for any $1 \leqslant y < x$,

$$\sum_{y\leqslant n<x} f(n) = \sum_{n<x} f(n) - \sum_{n<y} f(n) = c + F(x) + O(g(x)) - (c + F(y) + O(g(y)))$$
$$= F(x) - F(y) + O(g(x)) + O(g(y)),$$

hence (i) follows.

Now assume (i) holds. Then consider $h(x) = \sum_{n<x} f(n) - F(x)$. Then for any $1 \leqslant y < x$, we have $h(x) - h(y) = O(g(x)) + O(g(y))$, and this tends to 0, as $y, x \to \infty$, implying by Cauchy's criterion that $\lim_{x\to\infty} h(x) = c$ for some $c \in \mathbf{C}$, i.e.

$$\sum_{n<x} f(n) = c + F(x) + o(1), \qquad x \to \infty.$$

Now fix some $y \in \mathbf{R}_+$, and let $x \to \infty$. By (i), we obtain

$$\sum_{n<y} f(n) = \underbrace{\sum_{n<x} f(n) - F(x) + O(g(x))}_{=c+o(1)} + F(y) + O(g(y)) = c + F(y) + O(g(y)),$$

hence (ii) follows. □

Remark 1.1.3. The quantity c in (ii) is obviously unique, and writing $x = 1$, we see that $c = -F(1)+O(g(1))$. Also, if f and F are both real-valued, then $c \in \mathbf{R}$. We also note that if $\lim_{x\to\infty} F(x) = 0$, then $\lim_{x\to\infty} \sum_{n<x} f(x) = c$.

Definition 1.1.4 (Riemann ζ for $s > 1$)**.** For $s > 1$, we define $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

Proposition 1.1.5. *The series defining ζ is convergent for any $s > 1$. Also*

$$\zeta(s) = \frac{1}{s-1} + O(1), \qquad s > 1.$$

Proof. First, by Proposition 1.1.1, we have, for $1 \leqslant y < x$,

$$\sum_{y\leqslant n\leqslant x} n^{-s} = \int_y^x t^{-s} dt + O(y^{-s}) = \frac{y^{1-s}}{s-1} - \frac{x^{1-s}}{s-1} + O(y^{-s}).$$

Then Proposition 1.1.2 applies with $F(x) = x^{1-s}/(s-1)$, which tends to 0 as $x \to \infty$, giving a certain $c \in \mathbf{R}$ satisfying $\sum_{n<x} n^{-s} = c + O(x^{-s})$. As x tends to infinity, this gives $c = \zeta(s)$, and that the series in question is convergent.

The second statement follows from

$$\sum_{y\leqslant n\leqslant x} n^{-s} = \frac{y^{1-s}}{s-1} - \frac{x^{1-s}}{s-1} + O(y^{-s})$$

by setting $y = 1$, and $x \to \infty$. □

Remark 1.1.6. The same argument gives, for $s > 0$, that there exists some $c \in \mathbf{C}$ such that

$$\sum_{n\leqslant x} n^{-s} = c - \frac{x^{1-s}}{s-1} + O(x^{-s}).$$

In fact, this c equals $\zeta(s)$ for $s > 0$.

Proposition 1.1.7. *For any $x > 1$, we have*

$$\sum_{n\leqslant x} \frac{1}{n} = \log x + \gamma + O(x^{-1})$$

for some constant γ.

Proof. We combine again Proposition 1.1.1 and Proposition 1.1.2. First, for any $1 \leqslant y < x$,

$$\sum_{y\leqslant n\leqslant x} \frac{1}{n} = \int_y^x \frac{dt}{t} + O(y^{-1}) = \log x - \log y + O(y^{-1}).$$

Then Proposition 1.1.2 applies and gives that

$$\sum_{n\leqslant x} \frac{1}{n} = \gamma + \log x + O(x^{-1})$$

for some constant γ. □

Remark 1.1.8. We have that $\gamma \approx 0.577$.

Interlude on infinite products

Let $a_1, a_2, \ldots$ be an inifinite sequence of complex numbers, and we assume that none of them is -1.

Definition 1.1.9 (convergence and absolute convergence of infinite products)**.** We say that the infinite product

$$\prod_{n=1}^{\infty}(1+a_n)$$

is convergent and its value is c, if

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)=c,$$

and $c \in \mathbf{C}$ is nonzero. We say that it is absolutely convergent, if $\prod_{n=1}^{\infty}(1+|a_n|)$ is convergent.

Remark 1.1.10. One can easily see that when $a_n > -1$ for all $n \in \mathbf{N}$, then $\prod_{n=1}^{N}(1+a_n)$ converges to c if and only if $\sum_{n=1}^{N}\log(1+a_n)$ converges to $\log c$, i.e. $\prod_{n=1}^{\infty}(1+a_n) = c$ is equivalent to $\sum_{n=1}^{\infty}\log(1+a_n) = \log c$.

Proposition 1.1.11. *Assume $a_n \geqslant 0$. Then $\prod_{n=1}^{\infty}(1+a_n)$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent. (Note that in this case, convergence and absolute convergence are the same.)*

Proof. Clearly the sequence $\prod_{n=1}^{N}(1+a_n)$ is monotone increasing in N, in particular, it is separated from zero. Then the inequalities

$$\sum_{n=1}^{N} a_n \leqslant \prod_{n=1}^{N}(1+a_n) \leqslant e^{\sum_{n=1}^{N} a_n}$$

show that as $N\to\infty$, $\sum_{n=1}^{N} a_n \to \infty$ if and only if $\prod_{n=1}^{N}(1+a_n)\to\infty$. □

Proposition 1.1.12. *If $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent, then it is also convergent.*

Proof. Assume $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent. Then clearly only finitely many of the a_n's can be $-1/2$, and dropping the first few terms of the sequence $a_1, a_2, \ldots$ does not affect the convergence or absolute convergence. Therefore, we may assume that $a_n \neq -1/2$ for all $n \in \mathbf{N}$.

First we show that $\prod_{n=1}^{\infty}(1+a_n)$ tends to a finite limit. Let $P_N = \prod_{n=1}^{N}(1+a_n)$ and $Q_N = \prod_{n=1}^{N}(1+|a_n|)$. Then

$$|P_{N+1}-P_N| = |(1+a_1)\cdot\ldots\cdot(1+a_n)a_{n+1}| \leqslant (1+|a_1|)\cdot\ldots\cdot(1+|a_n|)|a_{n+1}| = Q_{N+1}-Q_N.$$

The series $\sum_{N=1}^{\infty}(Q_{N+1}-Q_N)$ is convergent by assumption, so is $\sum_{N=1}^{\infty}|P_{N+1}-P_N|$, and this means that P_N tends to a finite limit.

We still have to show that this finite limit cannot be 0. To this aim, observe that $1+a_n\to 1$, implying

$$\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$$

is absolutely convergent: indeed, dropping the first few terms,

$$\frac{|a_n|}{2} \leqslant \left|\frac{a_n}{1+a_n}\right| \leqslant 2|a_n|.$$

Then the product

$$\prod_{n=1}^{\infty}\left(1-\frac{a_n}{1+a_n}\right)$$

is absolutely convergent by Proposition 1.1.11, that is,

$$\prod_{n=1}^{N}\left(1-\frac{a_n}{1+a_n}\right) = P_N^{-1}$$

also tends to a finite limit. □

Proposition 1.1.13. *Let $D \subseteq \mathbf{C}$ be a domain, and $a_n(z) : D \to \mathbf{C}$ a sequence of functions such that $\sum_{n=1}^{\infty} |a_n(z)|$ converges uniformly to a bounded function on $D \to \mathbf{C}$. Then $\prod_{n=1}^{\infty}(1+a_n(z))$ also converges uniformly to a bounded function on $D \to \mathbf{C}$.*

Proof. Let M be an upper bound of $\sum_{n=1}^{\infty} |a_n(z)|$ on D. Then

$$\prod_{n=1}^{N}(1 + |a_n(z)|) \leqslant e^{\sum_{n=1}^{N} |a_n(z)|} \leqslant e^M.$$

Setting $Q_N(z) = \prod_{n=1}^{N}(1 + |a_n(z)|)$, this means that

$$Q_{N+1}(z) - Q_N(z) = (1 + |a_1(z)|) \cdot \ldots \cdot (1 + |a_n(z)|)|a_{n+1}(z)| \leqslant e^M |a_{n+1}(z)|.$$

If $P_N(z) = \prod_{n=1}^{N}(1 + |a_n(z)|)$, then, as in the proof of Proposition 1.1.12, $\sum_{N=1}^{\infty} |P_{N+1}(z) - P_N(z)|$ converges uniformly, since its dominating series $\sum_{N=1}^{\infty}(Q_{N+1}(z) - Q_N(z))$ converges uniformly (vanishing limit is excluded pointwise by Proposition 1.1.12). □

End of interlude

Proposition 1.1.14. *We have, for $s > 1$,*

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

and the infinite product is absolutely convergent. This is known as the Euler product of the ζ function.

Proof. First we prove that the infinite product is absolutely convergent for any $s > 1$. Clearly,

$$p^{-s} + p^{-2s} + \ldots = p^{-s}(1 - p^{-s})^{-1} \leqslant 2p^{-s},$$

and $\sum_p p^{-s} \leqslant \sum_{n=1}^{\infty} n^{-s} < \infty$. This gives absolute convergence of the infinite product.

Also, for any $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{N\to\infty} \sum_{\substack{n\in\mathbf{N} \\ \text{the largest prime divisor of } n \text{ is at most } N}} \frac{1}{n^s} = \lim_{N\to\infty} \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right),$$

by the fundamental theorem of arithmetic. The proof is complete. □

Definition 1.1.15 (von Mangoldt function)**.** Let

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n \text{ is a positive power of a prime } p, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.1.16. *For $s > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s \log n} = \log\zeta(s) = \log\frac{1}{s-1} + O(s-1).$$

Proof. Combining Proposition 1.1.5 and Proposition 1.1.14, we obtain

$$\prod_p \left(1 - p^{-s}\right)^{-1} = \zeta(s) = \frac{1}{s-1} + O(1).$$

Taking logarithms, we obtain (since $\zeta(s) > 1$)

$$-\sum_p \log\left(1 - p^{-s}\right) = \log\zeta(s) = \log\frac{1}{s-1} + O(s-1).$$

Applying the Taylor expansion of $\log(1 - x)$, we obtain

$$\sum_p \sum_{k=1}^{\infty} \frac{1}{p^{ks}} = \log\zeta(s) = \log\frac{1}{s-1} + O(s-1),$$

and the leftmost expression is clearly $\sum_{n=1}^{\infty} \Lambda(n)/(n^s \log n)$. □

1.2 Mertens' theorems

Theorem 1.2.1 (Chebyshev's theorem, upper bound)**.** *We have, for any* $x \geqslant 2$*,*

$$\sum_{n \leqslant x} \Lambda(n) \ll x.$$

Proof. First of all, instead of summing over prime powers, we can restrict to primes, since the number of other powers is $O(\sqrt{x} \log x)$, and each of them contributes $O(\log x)$.

By dyadic decomposition, it suffices to show that

$$\sum_{x/2 < p \leqslant x} \log p \ll x.$$

We can clearly assume that x is an even number. Then observe that

$$\prod_{x/2 < p \leqslant x} p \mid \binom{x}{x/2} \leqslant 2^x.$$

Taking the logarithm of both sides, we obtain the statement. □

Proposition 1.2.2. *We have* $L = \Lambda * 1$*, where* $*$ *is the convolution of number-theoretic functions,* L *is the logarithm function* $n \mapsto \log n$*, and* 1 *is the constant* 1 *function.*

Proof. Straight-forward calculation. □

Theorem 1.2.3 (Mertens' first theorem)**.** *We have, for any* $x \geqslant 2$*,*

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Proof. We have

$$\sum_{n \leqslant x} \log n = \sum_{n \leqslant x} \sum_{d|n} \Lambda(d) = \sum_{d \leqslant x} \Lambda(d) \left(\frac{x}{d} + O(1) \right).$$

Then applying Proposition 1.1.1 and Theorem 1.2.1, we obtain

$$x \sum_{d \leqslant x} \frac{\Lambda(d)}{d} = x \log x + O(x),$$

and the proof is complete. □

Corollary 1.2.4. *We have, for any* $x \geqslant 2$*,*

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

Proof. It is easy to see that

$$\sum_{p} \sum_{k=2}^{\infty} \frac{\log p}{p^k} \ll \sum_{n=1}^{\infty} n^{-3/2} < \infty,$$

and then Theorem 1.2.3 gives immediately the statement. □

Theorem 1.2.5 (Mertens' second theorem)**.** *There exists a constant* $c \in \mathbf{R}$ *such that for any* $x \geqslant 2$*,*

$$\sum_{p \leqslant x} \frac{1}{p} = \log \log x + c + o(1).$$

Proof. Let $2 \leqslant y < x$. Then for any $y \leqslant p \leqslant x$, we have

$$\frac{1}{p} = \frac{\log p}{p}\left(\frac{1}{\log x} + \int_y^x 1_{p\leqslant t}\frac{dt}{t\log^2 t}\right),$$

since $\frac{d}{dt}(\log t)^{-1} = -(t\log^2 t)^{-1}$. Then summing this over all the primes between y and x, we obtain

$$\sum_{y\leqslant p\leqslant x}\frac{1}{p} = \frac{1}{\log x}\sum_{y\leqslant p\leqslant x}\frac{\log p}{p} + \int_y^x \sum_{y\leqslant p\leqslant t}\frac{\log p}{p}\frac{dt}{t\log^2 t}.$$

Applying Corollary 1.2.4, we see

$$\sum_{y\leqslant p\leqslant x}\frac{\log p}{p} = \log x - \log y + O(1), \qquad \sum_{y\leqslant p\leqslant t}\frac{\log p}{p} = \log t - \log y + O(1).$$

Then using again $\frac{d}{dt}(\log t)^{-1} = -(t\log^2 t)^{-1}$ and also $\frac{d}{dt}\log\log t = (t\log t)^{-1}$, we obtain

$$\begin{aligned}\int_y^x \sum_{y\leqslant p\leqslant t}\frac{\log p}{p}\frac{dt}{t\log^2 t} &= \int_y^x (\log t - \log y + O(1))\frac{dt}{t\log^2 t}\\ &= \log\log x - \log\log y + \log y\left(\frac{1}{\log x} - \frac{1}{\log y}\right) + O\left(\frac{1}{\log y}\right).\end{aligned}$$

Then altogether

$$\sum_{y\leqslant p\leqslant x}\frac{1}{p} = \log\log x - \log\log y + O\left(\frac{1}{\log y}\right),$$

and Proposition 1.1.2 completes the proof. □

Corollary 1.2.6. *There exists a constant $c \in \mathbf{R}$ such that for any $x \geqslant 2$,*

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n\log n} = \log\log x + c + o(1).$$

Proof. The same way as Theorem 1.2.3 implies Corollary 1.2.4. □

Proposition 1.2.7. *Let $0 < a < b < \infty$. Then*

$$\sum_{x^a\leqslant p\leqslant x^b}\frac{1}{p} = \log\frac{b}{a} + o(1), \qquad x\to\infty.$$

Let f be a fixed compactly supported function from $\mathbf{R}_+$ to $\mathbf{C}$. Then

$$\sum_p \frac{1}{p} f\left(\frac{\log p}{\log x}\right) = \int_0^\infty f(t)\frac{dt}{t} + o(1), \qquad x\to\infty.$$

The same holds for $\Lambda(n)/(n\log n)$ in place of $1/p$.

Proof. The first statement follows from Theorem 1.2.5:

$$\sum_{x^a\leqslant p\leqslant x^b}\frac{1}{p} = \log(b\log x) + c + o(1) - \log(a\log x) - c - o(1) = \log b - \log a + o(1).$$

The second statement follows by observing that it simplifies to the first one for $f = 1_{[a,b]}$, and that every compactly supported function is approximated by characteristic ones (in L^∞-norm).

The proof is the same for $\Lambda(n)/(n\log n)$. □

Proposition 1.2.8. *We have*

$$\gamma = \int_0^\infty \frac{1_{[0,1]}(t) - e^{-t}}{t} dt.$$

Also, for $\varepsilon > 0$ small enough and N large enough, we have

$$\int_\varepsilon^N \frac{1_{[0,1]}(t) - e^{-t}}{t} dt = \gamma + O(\varepsilon) + O(1/N).$$

Proof. We have, by definition,

$$\gamma = \lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) - \log n \right).$$

Here,

$$\begin{aligned} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) - \log n &= \int_0^1 \frac{1-u^n}{1-u} du - \int_0^{1-\frac{1}{n}} \frac{1}{1-u} du \\ &= \int_0^1 \frac{1_{[1-1/n,1]} - u^n}{1-u} du \\ &= \int_0^n \frac{1_{[0,1]}(t) - \left(1 - \frac{t}{n}\right)^n}{t} dt. \end{aligned}$$

As $n \to \infty$, the integrand tends pointwise to $(1_{[0,1]}(t) - e^{-t})/t$, and it is dominated in absolute value by $O(e^{-t})$. Then the first statement follows by dominated convergence. The second statement follows by noting that the integral from 0 to ε is $O(\varepsilon)$, and from N to ∞ it is $O(1/N)$. □

Theorem 1.2.9 (Mertens' third theorem)**.** *We have*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p} \right) = \frac{e^{-\gamma} + o(1)}{\log x}, \qquad x \to \infty.$$

Proof. Recall that for any $s > 1$, we have

$$\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s \log n} = -\log(s-1) + O(s-1).$$

Writing $s = 1 + 1/\log x$, we obtain, as $x \to \infty$,

$$\sum_{n=1}^\infty \frac{\Lambda(n)}{n \log n} e^{-\log n/\log x} = \log\log x + o(1).$$

Then

$$\sum_{n=1}^\infty \frac{\Lambda(n)}{n \log n} \left(e^{-\log n/\log x} - 1_{[0,1]} \left(\frac{\log n}{\log x} \right) \right) = -c + o(1),$$

where c is the same constant as in Corollary 1.2.6.

On the other hand, consider the quantity, for any small $\varepsilon > 0$ fixed,

$$\sum_{n < x^\varepsilon} \frac{\Lambda(n)}{n \log n} \left(e^{-\log n/\log x} - 1_{[0,1]} \left(\frac{\log n}{\log x} \right) \right),$$

which is, combining Proposition 1.2.7 and Proposition 1.2.8 (see also its proof),

$$\int_0^\varepsilon \frac{(e^{-t} - 1_{[0,1]}(t))}{t} dt + o(1) = O(\varepsilon) + o(1),$$

and similarly, for any large N fixed,

$$\sum_{n > x^N} \frac{\Lambda(n)}{n \log n} \left(e^{-\log n/\log x} - 1_{[0,1]} \left(\frac{\log n}{\log x} \right) \right) = O(1/N) + o(1).$$

Then for some small $\varepsilon > 0$ and large N, we see

$$\sum_{x^\varepsilon \leqslant n \leqslant x^N} \frac{\Lambda(n)}{n \log n} \left(e^{-\log n / \log x} - 1_{[0,1]} \left(\frac{\log n}{\log x} \right) \right) = -c + O(\varepsilon) + O(1/N) + o(1).$$

By Proposition 1.2.7, this is the same as

$$\int_\varepsilon^N \frac{e^{-t} - 1_{[0,1]}}{t} dt + o(1),$$

which is $-\gamma + O(\varepsilon) + O(1/N) + o(1)$, therefore letting $\varepsilon \to 0$, $N \to \infty$, we see $c = \gamma$.

Then Theorem 1.2.6 gives

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n \log n} = \log \log x + \gamma + o(1).$$

The left-hand side can be rewritten to give

$$\sum_{p \leqslant x} \sum_{k=1}^{\infty} \frac{1}{kp^k} = \log \log x + \gamma + o(1).$$

By the Taylor expansion of $\log(1 - x)$, we obtain

$$-\sum_{p \leqslant x} \left(1 - \frac{1}{p} \right) = \log \log x + \gamma + o(1).$$

Taking exponentials, the proof is complete. □

1.3 Dirichlet characters

Definition 1.3.1 (Dirichlet character)**.** Given a modulus $q \in \mathbf{N}$, we say a function $\chi : \mathbf{Z} \to \mathbf{C}$ is a Dirichlet character, if

(a) for any $n \in \mathbf{Z}$, $\chi(n + q) = \chi(n)$;

(b) for any $n \in \mathbf{Z}$, $\chi(n) \neq 0$ if and only if $\gcd(n, q) = 1$;

(c) for any $m, n \in \mathbf{Z}$, $\chi(mn) = \chi(m)\chi(n)$.

Proposition 1.3.2 (properties of Dirichlet character)**.** *Dirichlet characters of modulo q have the following properties.*

(a) The function χ can be considered as a function $\mathbf{Z}_q \to \mathbf{C}$.

(b) The function χ is a group homomorphism from $\mathbf{Z}_q^\times$ to the multiplicative group of $\varphi(q)$th roots of unity.

Proof. (a) This is obvious from the fact that χ has period q.

(b) By multiplicativity $\chi(1) = \chi(1 \cdot 1) = \chi(1)\chi(1)$, which implies $\chi(1) = 1$, since $\chi(1) = 0$ is excluded by $\gcd(1, q) = 1$. The fact that χ is a group homomorphism is a simple consequence of multiplicativity. Assume $a \in \mathbf{Z}_q^\times$. Then by Euler-Fermat theorem, and by iterating the multiplicativity of χ,

$$(\chi(a))^{\varphi(q)} = \chi(a^{\varphi(q)}) = \chi(1) = 1,$$

which means that $\chi(a)$ is indeed a $\varphi(q)$th root of unity. □

Proposition 1.3.3 (the group of Dirichlet characters)**.** *The Dirichlet characters of modulo q form a group under pointwise multiplication. The unit element is the principal character*

$$\chi_0(a) = \begin{cases} 1 & \text{if } \gcd(a,q)=1, \\ 0 & \text{if } \gcd(a,q)=0; \end{cases}$$

and the inverse χ^{-1} of χ is the complex conjugate character

$$\chi^{-1}(a) = \overline{\chi(a)}.$$

The group Ξ of characters has $\varphi(q)$ elements, and it is isomorphic to the group $\mathbf{Z}_q^\times$.

Proof. The group properties (multiplication, unit element, inverse) follow easily. As for the structure of Ξ, recall that

$$\mathbf{Z}_q^\times = C_{q_1} \times \ldots \times C_{q_r}$$

for some cyclic groups $C_{q_1}, \ldots, C_{q_r}$ of order $q_1, \ldots, q_r$, respectively, satisfying $q_1 \cdot \ldots \cdot q_r = \varphi(q)$. Take some generators $c_1, \ldots, c_r$ of $C_{q_1}, \ldots, C_{q_r}$. Then a $\chi \in \Xi$ is determined by its values on $c_1, \ldots, c_r$, which must be q_1th, ..., q_rth roots of unity, respectively. From this, both the order and the structure of Ξ are clear. □

Remark 1.3.4. More generally, for any finite abelian group G, its dual group $\widehat{G}$ is isomorphic to G. Note on the other hand that this isomorphism is not canonical (in the sense of universal algebra), since it depends on the generators of the cyclic parts. The proof of the general statement is the same, starting out from the fundamental theorem of finite abelian groups.

Proposition 1.3.5. *For any $n \in \mathbf{Z}$,*

$$\frac{1}{\varphi(q)} \sum_{\chi\in\Xi} \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod q; \\ 0 & \text{if } n \not\equiv 1 \bmod q. \end{cases}$$

Proof. If $n \equiv 1 \bmod q$ or if $\gcd(n,q) \neq 1$, then the statement trivially holds. Assume that $\gcd(n,q)=1$ and $n \not\equiv 1 \bmod q$. When writing

$$\mathbf{Z}_q^\times = C_{q_1} \times \ldots \times C_{q_r},$$

and taking the generators $c_1, \ldots, c_r$, we have $n \equiv c_1^{n_1} \cdot \ldots \cdot c_r^{n_r} \bmod q$, where for each $1 \leqslant j \leqslant r$, $0 \leqslant n_j \leqslant q_j$, and for some $1 \leqslant i \leqslant r$ strictly $0 \leqslant n_i < q_i$. Then in that particular factor i, map c_i to a primitive q_ith root of unity, while at other factors $j \neq i$, map c_j's to 1. Obviously the resulting χ' satisfies $\chi'(n) \neq 1$. Then

$$\sum_{\chi\in\Xi} \chi(n) = \sum_{\chi\in\Xi} (\chi'\chi)(n) = \chi'(n) \sum_{\chi\in\Xi} \chi(n).$$

Here, since $\chi'(n) \neq 1$, $\sum_{\chi\in\Xi} \chi(n) = 0$, and the proof is complete. □

Corollary 1.3.6. *If $\gcd(a,q)=1$, then for any $n \in \mathbf{Z}$,*

$$\frac{1}{\varphi(q)} \sum_{\chi\in\Xi} \overline{\chi(a)}\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \bmod q; \\ 0 & \text{if } n \not\equiv a \bmod q. \end{cases}$$

Proof. Apply Proposition 1.3.5 to the residue class $a^{-1}n$. □

Proposition 1.3.7. *We have*

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Proof. If $\chi = \chi_0$, the statement is obvious. If $\chi \neq \chi_0$, then take $a \in \mathbf{Z}_q^\times$ such that $\chi(a) \neq 1$. When n runs through the residue classes modulo q, so does an, which implies

$$\sum_{n \bmod q} \chi(n) = \sum_{n \bmod q} \chi(an) = \chi(a) \sum_{n \bmod q} \chi(n).$$

Here, since $\chi(a) \neq 1$, $\sum_{n \bmod q} \chi(n) = 0$, and the proof is complete. □

1.4 The function $L(s,\chi)$ for $s>0$ and the value $L(1,\chi)$

Proposition 1.4.1. *Let $\chi \neq \chi_0$ be a character modulo q, and $s>0$. Then*

$$\sum_{y \leqslant n \leqslant x} \frac{\chi(n)}{n^s} = O_q\left(\frac{1}{y^s}\right).$$

There exists a complex number $L(1,\chi)$ satisfying

$$\sum_{n \leqslant x} \frac{\chi(n)}{n} = L(1,\chi) + O\left(\frac{1}{x}\right), \qquad \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = L(1,\chi).$$

Also,

$$\sum_{n \leqslant x} \frac{\chi(n)\log n}{n} = O_q(1).$$

Proof. Let $X(x) = \sum_{n \leqslant x} \chi(n)$. Then clearly $|X(x)| \leqslant q$, and by partial summation,

$$\sum_{y \leqslant n \leqslant x} \frac{\chi(n)}{n^s} = \sum_{y \leqslant n \leqslant x} \frac{1}{n^s}(X(n) - X(n-1)) = \sum_{y \leqslant n \leqslant x} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) X(n) + O_q\left(\frac{1}{y^s}\right) = O_q\left(\frac{1}{y^s}\right),$$

and the first statement is established. The second one then follows from Proposition 1.1.2 and setting $s=1$. The third statement is proven along the same calculation by writing $y=1$ and $\log n/n$ in place of $1/n^s$. □

Remark 1.4.2. Instead of weights $1/n$, $\log n/n$, one can use other sequences which are monotone decreasing from a certain point on. Of course, the resulting implied constant in $O_q(1)$ depends not only on q, but also on the sequence we have chosen.

Definition 1.4.3 (Dirichlet L-functions for $s>0$)**.** For a character χ modulo q, and $s>0$, we define $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$.

Then Proposition 1.1.5 easily gives that the series defining $L(s,\chi)$ is absolutely convergent for $s>1$ and conditionally convergent for $s>0$. Also, we have, similarly to the ζ function that

$$L(s,\chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}, \qquad \log L(s,\chi) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s \log n}, \qquad s>1,$$

where in the logarithm, we mean the principal branch (which is real for nonnegative numbers). We also note

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p}\right).$$

Proposition 1.4.4. *We have, for $\chi \neq \chi_0$,*

$$L(s,\chi) = L(1,\chi) + O_q(s-1).$$

Proof. First, $L(s,\chi)$ can be differentiated in s for $s>0$, and we see that

$$L'(s,\chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n^s},$$

since the partial sums converge locally uniformly on $(0,\infty)$ by applying Remark 1.4.2 to the sequence $\log n/n^s$ (which, for any s in a compact subset of $(0,\infty)$, is a monotone decreasing sequence from a certain point on), and the partial sums $\sum_{n \leqslant x} \chi(n)/n^s$ converge on $(0,\infty)$.

Now consider the function $L(u,\chi)$ on the interval $[1,s]$. By Newton-Leibniz,

$$L(s,\chi) - L(1,\chi) = \int_1^s L'(u,\chi)du, \qquad \left|\int_1^s L'(u,\chi)du\right| \leqslant (s-1) \max_{1 \leqslant u \leqslant s} |L'(u,\chi)|.$$

Since $L'(u,\chi) = O_q(1)$, the proof is complete. □

Theorem 1.4.5. *If $\chi \neq \chi_0$, then $L(1,\chi) \neq 0$.*

Proof. First we prove that $L(1,\chi)$ can be zero for only at most one character χ modulo q. To this aim, let Ξ be the set of all modulo q characters. Then Proposition 1.3.5 gives

$$\sum_{\chi\in\Xi} \log L(s,\chi) = \varphi(q) \sum_{n=1}^{\infty} \frac{\Lambda(n) 1_{n\equiv 1 \bmod q}}{n^s \log n}.$$

Here, the right-hand side is nonnegative, therefore $\prod_{\chi\in\Xi} L(s,\chi) \geqslant 1$. We know that

$$L(s,\chi_0) = O_q\left(\frac{1}{s-1}\right),$$

and

$$L(s,\chi) = L(1,\chi) + O_q(s-1), \qquad \chi \neq \chi_0.$$

These would immediately contradict, if there were two χ's with $L(1,\chi) = 0$.

Therefore, at most one $L(1,\chi)$ vanishes, in particular, it can only correspond to a real character χ (obviously $L(1,\overline{\chi}) = \overline{L(1,\chi)}$).

Let then χ be real, then its nonzero values are ± 1 (and not always $+1$, since $\chi \neq \chi_0$). Then, in particaular, χ is quadratic, i.e. if $\gcd(n,q) = 1$, then $\chi(n^2) = \chi(n)\chi(n) = 1$. We claim further that $1 * \chi$ is a nonnegative number-theoretic function. Indeed, it is multiplicative (since it is the convolution of multiplicative functions), therefore, it suffices to prove this for powers of primes. And indeed, if $n = p^k$, then

$$(1*\chi)(n) = 1 + \chi(p) + \ldots + \chi(p)^k,$$

which is obviously nonnegative, since $\chi(p^0), \chi(p^2), \ldots$ are all 1 for $p \nmid q$, while if $p \mid q$, the nonnegativity statement is obvious. It is also easy to check that $(1*\chi)(n)$ is at least 1 for all squares.

Consider then, for some $x \geqslant 2$,

$$\sum_{n\leqslant x} \frac{(1*\chi)(n)}{\sqrt{n}}.$$

First, each square n contributes at least $1/\sqrt{n}$, giving

$$\sum_{n\leqslant x} \frac{(1*\chi)(n)}{\sqrt{n}} \gg \log x.$$

On the other hand,

$$\sum_{n\leqslant x} \frac{(1*\chi)(n)}{\sqrt{n}} = \sum_{d\leqslant\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \sum_{m\leqslant x/d} \frac{1}{\sqrt{m}} + \sum_{m\leqslant\sqrt{x}} \frac{1}{\sqrt{m}} \sum_{\sqrt{x}<d\leqslant x/m} \frac{\chi(d)}{\sqrt{d}}.$$

In the second term, applying Proposition 1.4.1,

$$\sum_{m\leqslant\sqrt{x}} \frac{1}{\sqrt{m}} \sum_{\sqrt{x}<d\leqslant x/m} \frac{\chi(d)}{\sqrt{d}} = O_q\left(\sum_{m\leqslant\sqrt{x}} \frac{1}{m^{1/2}x^{1/4}}\right) = O_q(1).$$

In the first term, for some $c \in \mathbf{C}$, by Remark 1.1.6,

$$\sum_{m\leqslant x/d} \frac{1}{\sqrt{m}} = 2\frac{\sqrt{x}}{\sqrt{d}} + c + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right).$$

Hence

$$\sum_{d\leqslant\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \sum_{m\leqslant x/d} \frac{1}{\sqrt{m}} = \sum_{d\leqslant\sqrt{x}} \frac{\chi(d)}{\sqrt{d}} \left(2\frac{\sqrt{x}}{\sqrt{d}} + c + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right)\right) = 2\sqrt{x}L(1,\chi) + O_q(1),$$

by applying again Proposition 1.4.1.

Altogether,

$$\log x \ll \sqrt{x}L(1,\chi) + O_q(1),$$

which, by choosing x large enough, guarantees $L(1,\chi) \neq 0$. □

1.5 Dirichlet's theorem

Theorem 1.5.1. *Let $\chi \neq \chi_0$ be a character modulo q. Then, for any $x \geqslant 2$,*

$$\sum_{n \leqslant x} \frac{\Lambda(n)\chi(n)}{n} = O_q(1).$$

Proof. Using Proposition 1.2.2, one can easily see that

$$\frac{\chi(n)\log n}{n} = \frac{\chi(n)\sum_{d|n}\Lambda(d)}{n} = \sum_{d|n} \frac{\Lambda(d)\chi(d)}{d} \cdot \frac{\chi(n/d)}{n/d},$$

that is,

$$\sum_{n \leqslant x} \frac{\chi(n)\log n}{n} = \sum_{d \leqslant x} \frac{\Lambda(d)\chi(d)}{d} \sum_{m \leqslant x/d} \frac{\chi(m)}{m}.$$

The inner sum is, by Proposition 1.4.1, $L(1,\chi) + O_q(d/x)$, therefore, applying also

$$\sum_{n \leqslant x} \frac{\chi(n)\log n}{n} = O_q(1)$$

from Propostion 1.4.1, we obtain

$$L(1,\chi)\sum_{d \leqslant x} \frac{\Lambda(d)\chi(d)}{d} = O_q(1) + O_q\left(\sum_{d \leqslant x} \frac{\Lambda(d)}{x}\right) = O_q(1),$$

by Theorem 1.2.1. Now the proof is complete by Theorem 1.4.5. ☐

Now this immediately gives rise to the following.

Theorem 1.5.2. *Let $\gcd(a,q) = 1$. Then, for any $x \geqslant 2$,*

$$\sum_{n \leqslant x} \frac{\Lambda(n)1_{n \equiv a \bmod q}}{n} = \frac{1}{\varphi(q)}\log x + O_q(1).$$

Proof. Denote by Ξ the set of modulo q characters. By Corollary 1.3.6,

$$\sum_{n \leqslant x} \frac{\Lambda(n)1_{n \equiv a \bmod q}}{n} = \frac{1}{\varphi(q)} \sum_{\chi \in \Xi} \sum_{n \leqslant x} \frac{\Lambda(n)\chi(n)\overline{\chi(a)}}{n}.$$

The terms on the right-hand side are evaluated via Theorem 1.2.3 (by noting that only finitely many primes divide q) and Theorem 1.5.1. ☐

Now the same way as Theorem 1.2.3 implies Theorem 1.2.5, we have the analogue for arithmetic progressions.

Theorem 1.5.3. *Let $\gcd(a,q) = 1$. Then, for any $x \geqslant 2$,*

$$\sum_{p \leqslant x} \frac{1_{p \equiv a \bmod q}}{p} = \frac{1}{\varphi(q)}\log\log x + c_{a,q} + o(1).$$

Proof. The proof of Theorem 1.2.5 can be essentially repeated. This time we refer to Theorem 1.5.2 in place of Theorem 1.2.3. ☐

Finally we obtain what originally motivated Dirichlet.

Theorem 1.5.4. *Let $\gcd(a,q) = 1$. Then there are infinitely many prime numbers $p \equiv a \bmod q$.*

Proof. Immediate from Theorem 1.5.3. ☐

Chapter 2
The prime number theorem

2.1 Dirichlet series

Definition 2.1.1 (Dirichlet series)**.** Let $f : \mathbf{N} \to \mathbf{C}$ be a number-theoretic function. Then the associated Dirichlet series is

$$\mathcal{D}f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad s \in \mathbf{C}.$$

All our functions below satisfy

$$f(n) = O(n^{o(1)}),$$

hence we assume this from now on.

Proposition 2.1.2. *If*

$$f(n) = O(n^{o(1)}),$$

then the series defining $\mathcal{D}f(s)$ is absolutely convergent for $\Re s > 1$.

Proof. Let $\Re s = \sigma > 1$. Clearly

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \ll \sum_{n=1}^{\infty} \frac{1}{n^{(1+\sigma)/2}} = \zeta((1+\sigma)/2) < \infty,$$

and the proof is complete. □

It is easy to see that the partial sums

$$\sum_{n=1}^{N} \frac{f(n)}{n^s} = \sum_{n=1}^{N} f(n) e^{-s \log n}$$

converge locally uniformly to $\mathcal{D}f(s)$. Then the derivative can be computed termwise to obtain

$$\frac{d}{ds}\mathcal{D}f(s) = \sum_{n=1}^{\infty} (\log n) f(n) n^{-s} = \mathcal{D}(-Lf)(s), \qquad \Re s > 1.$$

Another simple observation is that if $f, g : \mathbf{N} \to \mathbf{C}$ are two such functions, then

$$\mathcal{D}f(s) \cdot \mathcal{D}g(s) = \mathcal{D}(f * g)(s), \qquad \Re s > 1,$$

where $*$ is the convolution of number-theoretic function. Indeed,

$$\mathcal{D}f(s)\mathcal{D}g(s) = \sum_{m=1}^{\infty} f(m) m^{-s} \sum_{k=1}^{\infty} g(k) k^{-s} = \sum_{n=1}^{\infty} \sum_{d|n} f(d) g(n/d) n^{-s},$$

and rearranging the terms is verified by absolute convergence, recalling the elementary fact that $\tau_0(n) = n^{o(1)}$, where τ_0 stands for the divisor counting function.

Proposition 2.1.3. *Let $y < x$ be real numbers, and $f : [y, x] \to \mathbf{C}$ a continuously differentiable function. Then*

$$\sum_{n\in\mathbf{Z}:y\leqslant n\leqslant x} f(n) = \int_y^x f(t)dt + O\left(\int_y^x |f'(t)|dt\right) + O(|f(y)|).$$

Proof. First observe that, by Newton-Leibniz,

$$f(x), f(y), \int_y^{\lceil y\rceil} f(t)dt, \int_{\lfloor x\rfloor}^x f(t)dt = O\left(\int_y^x |f'(t)|dt\right) + O(|f(y)|),$$

hence we can assume that $x, y \in \mathbf{Z}$, and on the left-hand side of the statement, we take the summation on $y \leqslant n < x$. Now let $y \leqslant n < x$ be arbitrary. Then, again by Newton-Leibniz,

$$\sum_{n=y}^{x-1}\left(\int_n^{n+1} f(t)dt - f(n)\right) = \sum_{n=y}^{x-1}\int_n^{n+1} f'(t)dt \ll \sum_{n=y}^{x-1}\int_n^{n+1} |f'(t)|dt = O\left(\int_y^x |f'(t)|dt\right),$$

and the proof is complete. □

2.2 Versions of Perron's and Jensen's formulae

Let

$$\delta(y) = \begin{cases} 0, & \text{if } 0 < y < 1. \\ \frac{1}{2}, & \text{if } y = 1. \\ 1, & \text{if } y > 1. \end{cases}$$

Proposition 2.2.1. *Let $\sigma, T, y > 0$. Then*

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} y^s \frac{ds}{s} = \delta(y) + \begin{cases} O(y^\sigma \min(1, T^{-1}|\log y|^{-1})), & \text{if } 0 < y \neq 0, \\ O(\sigma T^{-1}), & \text{if } y = 1. \end{cases}$$

Proof. Let first $0 < y < 1$. Then we close the contour two different ways. One is $\sigma - iT, S - iT, S + iT, \sigma + iT$ with some very large S. On the horizontal segments, the integral is absolutely bounded by

$$\int_\sigma^S \frac{y^u}{T} du = \left[\frac{y^u}{T|\log y|}\right]_{u=\sigma}^S \leqslant \frac{y^\sigma}{T|\log y|}.$$

On the vertical segment, the integral is absolutely bounded by

$$\int_{S-iT}^{S+iT} \frac{y^S}{S}|ds| \leqslant \frac{Ty^S}{S},$$

which is smaller than the previous, if S is large enough. The other one is the arc centered at the origin of passing through $\sigma - iT$ and $\sigma + iT$, on the right of $\Re s = \sigma$. On the arc of this, we see that $|y^s| \leqslant y^\sigma$, and the measure $|ds|/|s|$ cancels the length $O(T)$. This gives the statement when $y < 1$. When $y > 1$, the argument is similar, but we choose S to be a negative number of large absolute value and use the left arc. Note that in this case, we have the pole at $s = 0$ of the integrand, where we apply the residue theorem, noting that

$$\frac{y^s}{s} = \frac{e^{s\log y}}{s} = \frac{1}{s} + \text{holomorphic in } s \in \mathbf{C}.$$

When $y = 1$, then one can easily see that the integral in question is $\theta/2\pi$, where θ is the angle between $\sigma - iT$ and $\sigma + iT$. This angle differs from π by $O(\sigma T^{-1})$. □

Corollary 2.2.2. *Let $f : \mathbf{N} \to \mathbf{C}$ satisfy $f(n) = O(n^{o(1)})$. Then, for any $x \geqslant 2$ and $1 < \sigma \leqslant 3$,*

$$\sum_{n=1}^\infty f(n)\delta(x/n) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \mathcal{D}f(s)x^s\frac{ds}{s} + O\left(x^\sigma \sum_{n\neq x}\frac{|f(n)|}{n^\sigma}\min\left(1, \frac{1}{T\log\frac{x}{n}}\right) + \frac{|f(x)|}{T}\right).$$

Further, if $\sigma = 1 + 1/\log x$, then, denoting by $\|x\|$ the distance of x and the set $\mathbf{Z} \setminus \{x\}$, the error term is

$$O\left(\frac{x}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^\sigma} + \left(\max_{x/2\leqslant n\leqslant 3x/2}|f(n)|\right)\left(\min\left(1, \frac{x}{T\|x\|}\right) + \frac{x\log x}{T}\right)\right).$$

Proof. Note that the sum on the left-hand side is actually a finite sum. Write each $f(n)\delta(n/x)$, for $n \in \mathbf{N}$, by Proposition 2.2.1, as

$$f(n)\delta(n/x) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} f(n)\frac{x^s}{n^s}\frac{ds}{s} + \begin{cases} O(|f(n)|(x/n)^\sigma \min(1, T^{-1}|\log(x/n)|^{-1})), & \text{if } n \neq x, \\ O(|f(n)|\sigma T^{-1}), & \text{if } n = x. \end{cases}$$

Summing this over $n \in \mathbf{N}$, we clearly get the first statement by the absolute convergence of $\mathcal{D}f(s)$ on $\Re s = \sigma$.

As for the estimate of the error, note that setting $\sigma = 1 + 1/\log x$, we have $x^\sigma = ex = O(x)$. In the sum over n, first consider the n's satisfying $|x - n| \geqslant x/2$. Then $|\log(x/n)| \geqslant \log(3/2)$, hence

$$x^\sigma \sum_{|n-x|\geqslant x/2}\frac{|f(n)|}{n^\sigma}\min\left(1, \frac{1}{T\log\frac{x}{n}}\right) \ll \frac{x}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^\sigma}.$$

When $|n - x| < x/2$, then we see from Taylor expansion that

$$\frac{1}{\left|\log\frac{x}{n}\right|} = \frac{1}{\left|\log\left(1 + \frac{n-x}{x}\right)\right|} = O\left(\frac{x}{n-x}\right).$$

In this case, $x^\sigma/n^\sigma = O(1)$, therefore

$$\begin{aligned} x^\sigma \sum_{|n-x|<x/2, n\neq x}\frac{|f(n)|}{n^\sigma}\min\left(1, \frac{1}{T\log\frac{x}{n}}\right) &\ll \left(\max_{x/2\leqslant n\leqslant 3x/2}|f(n)|\right)\left(\min\left(1, \frac{x}{T\|x\|}\right) + \frac{x}{T}\sum_{k\leqslant x/2}\frac{1}{k}\right) \\ &\ll \left(\max_{x/2\leqslant n\leqslant 3x/2}|f(n)|\right)\left(\min\left(1, \frac{x}{T\|x\|}\right) + \frac{x\log x}{T}\right). \end{aligned}$$

The term $|f(x)|/T$ is clearly dominated by this, and the proof is complete. □

Proposition 2.2.3. *Let f be a not identically zero meromorphic function (with all removable singularities removed) on a neighborhood of the closed disc $D = \{z \in \mathbf{C} : |z - z_0| \leqslant r\}$ with no zeros or poles on the boundary and at the center of D. Then*

$$\log|f(z_0)| = \int_0^1 \log|f(z_0 + re^{2\pi it})|dt + \sum_{\rho:|\rho-z_0|<r}\log\frac{|\rho - z_0|}{r} - \sum_{\zeta:|\zeta-z_0|<r}\log\frac{|\zeta - z_0|}{r},$$

where ρ, ζ run through the zeros and poles of f in D, respectively, with multiplicity.

Proof. We may obviously take $z_0 = 0$ and $r = 1$. First we consider the special case when f has no zero or pole on D. Then, since D is simply connected, there is a holomorphic logarithm $g = \log f$ of f on D. Then Cauchy's integral formula shows (where ∂D stands for the unit circle, oriented counter-clockwise)

$$g(0) = \frac{1}{2\pi i}\int_{\partial D}\frac{g(z)}{z}dz = \int_0^1 g(e^{2\pi it})dt.$$

Then taking the real part, we see

$$\log|f(0)| = \Re g(0) = \int_0^1 \Re g(e^{2\pi it})dt = \int_0^1 \log|f(e^{2\pi it})|.$$

Now take the general case, that is, f may have zeros and poles inside D. For any $\mu \in D$, consider the Blaschke factor

$$B_\mu(z) = \frac{z - \mu}{1 - \overline{\mu}z}.$$

One can easily see that $B_\mu(z)$ is holomorphic on a neighborhood of D, its only (simple) zero is μ, and for $|z| = 1$,

$$|B_\mu(z)| = \left|\frac{B_{\mu(z)}}{\overline{z}}\right| = \left|\frac{z-\mu}{\overline{z}-\overline{\mu}}\right| = 1.$$

(In fact, one can prove that $\{B_\mu : |\mu| < 1\}$ is the conformal automorphism group of D).

Then dividing f by B_ρ for a zero ρ of f satisfying $|\rho| < 1$ subtracts $\log|\rho|$ from $\log|f(0)|$, and cancels a zero ρ of f. Similarly multiplying f by B_ζ for a pole ζ of f satisfying $|\zeta| < 1$ adds $\log|\rho|$ to $\log|f(0)|$, and cancels a pole ζ of f. Cancelling all the zeroes and poles, we arrive at the previously considered special case (which is already proven), then adding back the zeros and the poles, the statement is proven. □

Proposition 2.2.4. *Let f be a not identically zero meromorphic function (with all removable singularities removed) on a neighborhood of the closed disc $D = \{z \in \mathbf{C} : |z - z_0| \leqslant r\}$ with no zeros or poles on the boundary of D and at a certain point μ in the interior of D. Then*

$$\begin{aligned}\log|f(\mu)| = &\int_0^1 \log|f(z_0 + re^{2\pi it})|\Re\frac{re^{2\pi it} + (\mu - z_0)}{re^{2\pi it} - (\mu - z_0)}dt \\ &+ \sum_{\rho:|\rho-\mu|<r} \log\frac{|\rho - z_0|}{\left|r - \frac{\overline{(\rho-z_0)}(\mu-z_0)}{r}\right|} \\ &- \sum_{\zeta:|\zeta-z_0|<r} \log\frac{|\zeta - z_0|}{\left|r - \frac{\overline{(\zeta-z_0)}(\mu-z_0)}{r}\right|},\end{aligned}$$

where ρ, ζ run through the zeros and poles of f in D, respectively, with multiplicity.

Proof. The proof is similar to that of Proposition 2.2.3. After normalizing to $z_0 = 0$, $r = 1$, and taking the special case of no zeros and poles in D, instead of Cauchy's integral formula, we apply Poisson's formula, which states that for a harmonic function u on a neighborhood of D, we have

$$u(\mu) = \int_0^1 u(e^{2\pi it})\Re\frac{e^{2\pi it} + \mu}{e^{2\pi it} - \mu}dt.$$

When eliminating the zeros and the poles, we use the same Blaschke factors as in the proof of Proposition 2.2.3. □

Proposition 2.2.5. *Assume that $0 < c_2 < c_1 < 1$ are fixed numbers. Let f be a not identically zero holomorphic function on a neighborhood of the closed disc $D = \{z \in \mathbf{C} : |z - z_0| \leqslant r\}$, satisfying that for $z \in \partial D$, $0 \neq |f(z)| \leqslant M^{O_{c_1,c_2}(1)}|f(z_0)|$ for some $M \geqslant 1$. We have then*

$$\frac{f'(z)}{f(z)} = \sum_{\rho:|\rho-z_0|\leqslant c_1 r} \frac{1}{z-\rho} + O_{c_1,c_2}\left(\frac{\log M}{r}\right)$$

for every z which is not a zero of f, and satisfies $|z - z_0| \leqslant c_2 r$. The ρ in the sum runs through the zeros of f with multiplicity. Further, the number of such zeros satisfying $|\rho - z_0| \leqslant c_1 r$ is $O_{c_1,c_2}(\log M)$.

Proof. We normalize such that $z_0 = 0$, $r = 1$. In the proof, all implied constants are allowed to depend on c_1, c_2. First observe that $f(0) \neq 0$, since $f(0) = 0$ would imply that f vanishes on ∂D, and then that f is identically zero by the maximum modulus principle.

First we apply Proposition 2.2.3 as follows. First, since each zero ρ with $|\rho| < 1$ contributes negatively in Proposition 2.2.3, and we have no poles, we can record

$$\log|f(0)| \leqslant \int_0^1 \log|f(e^{2\pi it})|dt.$$

Also, by the assumption $0 \neq |f(z)| \leqslant M^{O(1)}|f(z_0)|$, we have

$$\log|f(e^{2\pi it})| \leqslant \log|f(0)| + O(\log M), \qquad \int_0^1 \log|f(e^{2\pi it})|dt \leqslant \log|f(0)| + O(\log M).$$

These together give

$$\log|f(0)| = \int_0^1 \log|f(e^{2\pi it})|dt + O(\log M),$$

which, compared with Proposition 2.2.3, gives

$$\sum_{\rho:|\rho|<1} \log\frac{1}{|\rho|} = O(\log M).$$

This clearly implies that the number of zeros ρ with $|\rho| \leqslant c_1$ is $O(\log M)$.

Also, if ρ is a zero, we may factor f as gB_ρ, where

$$B_\rho(z) = \frac{z-\rho}{1-\overline{\rho}z}$$

is again a Blaschke factor. Then

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \frac{B'_\rho(z)}{B_\rho(z)} = \frac{g'(z)}{g(z)} + \frac{1}{z-\rho} - \frac{1}{z-\overline{\rho}^{-1}}.$$

For zeros satisfying $c_1 < |\rho| < 1$, and for $|z| \leqslant c_2$, we use that

$$\frac{1}{z-\rho} - \frac{1}{z-\overline{\rho}^{-1}} = O\left(\log\frac{1}{|\rho|}\right),$$

and recalling

$$\sum_{\rho:|\rho|<1} \log\frac{1}{|\rho|} = O(\log M),$$

we see that these contribute only the admissible $O(\log M)$. For zeros ρ with $|\rho| \leqslant c_1$, the contribution of each $-1/(z-\overline{\rho}^{-1})$ is $O(1)$, and their number is $O(\log M)$, while the contribution of each $1/(z-\rho)$ is the same as in the claim. Therefore, to complete the proof, it suffices to show that if f has no zeros in D, then

$$\frac{f'(z)}{f(z)} = O(\log M)$$

for all z satisfying $|z| \leqslant c_2$. Normalize f such that $f(0) = 1$, and note that there exists a holomorphic $\log f$ in a neighborhood of D (we choose the branch satisfying $\log f(0) = 0$). Then by Proposition 2.2.3, we see that

$$\int_0^1 \log|f(e^{2\pi it})|dt = 0,$$

and by assumption, $\log|f(e^{2\pi it})| \leqslant O(\log M)$, implying

$$\int_{t\in[0,1]:\log|f(e^{2\pi it})|\geqslant 0} \log|f(e^{2\pi it})|dt = O(\log M), \qquad \int_{t\in[0,1]:\log|f(e^{2\pi it})|<0} \log|f(e^{2\pi it})|dt = O(\log M).$$

The difference of these two is

$$\int_0^1 |\log|f(e^{2\pi it})||dt = O(\log M).$$

Then from Proposition 2.2.4, we see that

$$\log|f(z)| = \int_0^1 f(e^{2\pi it})\Re\frac{e^{2\pi it}+z}{e^{2\pi it}-z}dt$$

for $|z| \leqslant c_2$. This implies, in particular, that $\log|f(z)| = O(\log M)$, and further that its real and imaginary partial derivatives (i.e. $(\partial/\partial x)\log|f(x+iy)|$ and $(\partial/\partial y)\log|f(x+iy)|$) are also $O(\log M)$. Now recalling

$$\Re\log f(z) = \log|f(z)|,$$

we see that the (real and imaginary) partial derivatives of the real part of $\log f(z)$ are $O(\log M)$. Then by the Cauchy-Riemann partial differential equations, the same holds also for the imaginary part of $\log f(z)$, therefore,

$$\frac{f'(z)}{f(z)} = (\log f(z))' = O(\log M),$$

and the proof is complete. □

2.3 Analytic properties of the ζ function

Definition 2.3.1 (Riemann ζ for $\Re s > 1$)**.** We define $\zeta(s)$ to be $\mathcal{D}1(s)$ for $\Re s > 1$, where 1 stands for the identically 1 number-theoretic function.

Proposition 2.3.2. *Let $s \in \mathbf{C}$ such that $\Re s > 1$, $s = O(1)$. Then*

$$\zeta(s) = \frac{1}{s-1} + O(1), \qquad \zeta'(s) = -\frac{1}{(s-1)^2} + O(1).$$

Proof. To see the first statement, we apply Proposition 2.1.3 with the function $n \mapsto n^{-s}$ to obtain, with $\sigma = \Re s > 1$,

$$\sum_{1 \leqslant n \leqslant x} n^{-s} = \int_1^x t^{-s} dt + O\left(\int_1^x |s| t^{-\sigma-1} dt\right) + O(1) = \frac{1}{s-1}(1 - x^{1-s}) + O(1),$$

and letting $x \to \infty$, we obtain the statement.

The proof of the second statement is similar by noting the following. The primitive function of $(\log t)t^{-s}$ is

$$-\frac{t^{1-s}((s-1)\log t + 1)}{(s-1)^2},$$

and substituting $t = 1$, we obtain the main term $-(s-1)^{-2}$. The derivative of $(\log t)t^{-s}$ is

$$t^{-s-1}(1 - s\log t),$$

and the absolute integral on $[1, \infty)$ of this is unformly bounded on $\Re s > 1$. □

Corollary 2.3.3. *We have, on $\Re s > 1$, $s = O(1)$, that*

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + O(1).$$

Proposition 2.3.4 (Euler product)**.** *On the domain $\Re s > 1$, we have*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product on the right-hand side is a locally uniformly convergent product.

Proof. We have, by fundamental theorem of arithmetic and the summation of geometric series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{N\to\infty} \sum_{n \text{ has no prime divisor bigger than } N} \frac{1}{n^s}$$
$$= \lim_{N\to\infty} \prod_{p \leqslant N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \lim_{N\to\infty} \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Noting that for any $\sigma > 1$, if $\Re s \geqslant \sigma$,

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n \text{ has no prime divisor bigger than } N} \frac{1}{n^s}\right| \leqslant \sum_{n=N+1}^{\infty} \frac{1}{n^\sigma} = o_\sigma(1), \qquad N \to \infty,$$

the proof is complete. □

Proposition 2.3.5. *The function ζ extends to $\{s : \Re s > 0\}$ as a meromorphic function with a unique, simple pole at 1. More specifically, $\zeta(s) - 1/(s-1)$ is holomorphic on $\{s : \Re s > 0\}$ after its removable singularity at 1 is removed.*

Also, for any $\varepsilon > 0$, if $\Re s \geqslant \varepsilon$ and $|s-1| \geqslant \varepsilon$, then

$$\log|\zeta(s)| \leqslant O_\varepsilon(\log(2 + |s|)).$$

Proof. Let first $s \neq 1$ from the half-plane $\{s : \Re s > 0\}$. By Proposition 2.1.3 again with the function $n \mapsto n^{-s}$, we obtain, for any $1 \leqslant y < x$ that

$$\sum_{y \leqslant n \leqslant x} n^{-s} = \frac{y^{1-s} - x^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma} y^{-\sigma}\right).$$

Then by Proposition 1.1.2, we obtain that there exists some $\zeta(s) \in \mathbf{C}$ such that

$$\sum_{n \leqslant x} n^{-s} = \zeta(s) - \frac{x^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma} x^{-\sigma}\right).$$

When $\Re s > 1$, this $\zeta(s)$ must coincide with the earlier defined $\mathcal{D}1(s)$, since both $x^{1-s}/(s-1)$ and the error term tend to 0 as $x \to \infty$. Therefore, we really have the extension, our goal is to prove its meromorphic nature and what we claimed about the pole.

Rearranging, we obtain

$$\sum_{n \leqslant x} n^{-s} + \frac{x^{1-s} - 1}{s-1} = \zeta(s) - \frac{1}{s-1} + O\left(\frac{|s|}{\sigma} x^{-\sigma}\right).$$

Here, the left-hand side is holomorphic on $\{s : \Re s > 0\} \setminus \{1\}$ for any fixed $x \geqslant 1$, and converges there locally uniformly to the function $\zeta(s) - 1/(s-1)$ as $x \to \infty$, which is therefore holomorphic there, implying in particular the holomorphy of ζ on the domain $\{s : \Re s > 0\} \setminus \{1\}$. Applying again Proposition 2.1.3, we see that

$$\sum_{1 \leqslant n \leqslant x} n^{-s} = \frac{1 - x^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma}\right),$$

therefore,

$$\sum_{n \leqslant x} n^{-s} + \frac{x^{1-s} - 1}{s-1}$$

is locally uniformly bounded on the indicated domain, so is its limit $\zeta(s) - 1/(s-1)$. Therefore, the isolated singularity of $\zeta(s) - 1/(s-1)$ at 1 is removable, which completes the proof of the first statement.

As for the second one, observe that our calculations also give

$$\zeta(s) - \frac{1}{s-1} = O\left(\frac{|s|}{\sigma}\right),$$

from which the claim follows immediately. □

In fact, ζ can be meromorphically continued to the whole complex plane with the only pole at 1 (where we have already analyzed it). Since we do not need it for our number-theoretic investigations, we assume all along that $\Re s > 0$ whenever ζ is evaluated. Note also that

$$\zeta(\overline{s}) = \overline{\zeta(s)},$$

which follows immediately from the uniqueness principle of complex analysis.

Proposition 2.3.6. *If $\Re s > 1$, then $\zeta(s) \neq 0$.*

Proof. Since the convolution $1 * \mu$ is the characteristic function of 1, we see that

$$\frac{1}{\zeta(s)} = \mathcal{D}\mu(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

from which the statement is obvious. □

Proposition 2.3.7. *For any $\varepsilon > 0$ and $t_0 \in \mathbf{R}$, the number of zeros of ζ (counted with multiplicity) in the domain $\{\sigma + it : \sigma \geqslant \varepsilon, |t - t_0| \leqslant 1\}$ is $O_\varepsilon(\log(2 + |t_0|))$.*

Proof. By Proposition 2.3.6, each zero we have to count belongs in fact to the rectangle $\{\sigma+it : \varepsilon \leqslant \sigma \leqslant 1, |t-t_0| \leqslant 1\}$. We can clearly assume that $t \geqslant 0$ by conjugation, and also that $t \geqslant 10$ (say), since there is only a concrete number of zeros of ζ up to height 10.

For any $t_1 \in [t_0-1, t_0+1]$, consider the disc D of radius $2-\varepsilon/2$ centered at $2+it_1$, and assume ∂D contains no zero of ζ. Set further D' for the smaller disc of the same center and radius $2-3\varepsilon/4$. Clearly,

$$\log|\zeta(2+it_1)| \leqslant \log|\zeta(2)| = O(1),$$

and similarly,

$$\log|\zeta(2+it_1)| \geqslant \log\left(1-\frac{1}{2^2}-\frac{1}{3^2}-\dots\right) = O(1),$$

therefore, $\log|\zeta(2+it_1)| = O(1)$. Also, $s \in \partial D$, $\log|\zeta(s)| \leqslant O_\varepsilon(\log(2+|t_0|))$ by Proposition 2.3.5. Now we apply Proposition 2.2.3, noting that no poles contribute, and we obtain

$$O(1) = O_\varepsilon(\log(2+|t_0|)) + \sum_{\rho:|\rho-(2+it_1)|<2-\varepsilon/2} \log\frac{|\rho-(2+it_1)|}{2-\varepsilon/2}.$$

The contribution of each zero is negative, and for each zero ρ satisfying $|\rho-(2+it_1)| < 2-3\varepsilon/4$, the contribution is negative and $\Omega_\varepsilon(1)$. This means that the number of zeros ρ with $|\rho-(2+it_1)| < 2-3\varepsilon/4$ is $O_\varepsilon(\log(2+|t_0|))$. Clearly, we can cover our rectangle in question with $O_\varepsilon(1)$ such discs D', and this completes the proof. □

Proposition 2.3.8. *For any* $0<\varepsilon<C$, *we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = -\sum_{|\rho-s|<\varepsilon/2}\frac{1}{s-\rho} + \frac{1}{s-1} + O_{C,\varepsilon}(\log(2+|t|))$$

for $s=\sigma+it$, $\varepsilon \leqslant \sigma \leqslant C$, *where* ρ *runs through the zeros of* ζ *with multiplicity.*

Proof. First assume that $C \leqslant 4-\varepsilon$. Let $f(s)=(s-1)\zeta(s)$. Then f is holomorphic on $\Re s>0$, and we may apply Proposition 2.2.5 with discs of center $2+it$ and radii $2-\varepsilon/4$, $2-\varepsilon/2$, $2-\varepsilon$, and $M=2+|t|$ (for this M, we need $\log|\zeta(2+it)|=O(1)$, which follows immediately from $|\zeta(2+it)| \geqslant 1-1/2^2-1/3^2-\dots>0$, and on the boundary of the outermost disc, we use Proposition 2.3.5). Bounding the number of zeros away from s by at least $\varepsilon/2$ by Proposition 2.3.7, we obtain

$$\frac{f'(s)}{f(s)} = \sum_{|\rho-s|<\varepsilon/2}\frac{1}{s-\rho} + O_\varepsilon(\log(2+|t|)).$$

Writing this back to ζ, we obtain the statement for any s being in the $(2-\varepsilon)$-neighborhood of $2+it$. For $C>4-\varepsilon$, we can do the same, using $(C+\varepsilon)/2+it$ as a center instead of $2+it$. □

Corollary 2.3.9. *For any* $C,\varepsilon>0$, $t_0 \in \mathbf{R}$, *we have*

$$\int_\varepsilon^C \int_{t_0-1}^{t_0+1} \left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| dt d\sigma \ll_{C,\varepsilon} \log(2+|t_0|).$$

Proof. This follows immediately from Proposition 2.3.8, by noting that the error term there is obviously admissible here, also that $1/s$ is locally integrable in a small neighborhood of 0, implying that $1/(s-\rho)$, $1/(s-1)$ are locally integrable, and finally bounding the number of zeros by Proposition 2.3.7. □

2.4 Zero-free regions

We have already seen that there are no zeros of ζ in the domain $\Re s>1$. In this section, we are going to prove stronger results on zero-free regions.

Proposition 2.4.1. *For any* $\theta \in \mathbf{R}$,

$$3+4\cos\theta+\cos(2\theta) \geqslant 0.$$

Proof. One can readily check the identity

$$1-\cos(2\theta)=4(1+\cos\theta)-2(1+\cos\theta)^2,$$

which implies

$$1-\cos(2\theta)\leqslant 4(1+\cos\theta),$$

which is clearly equivalent to the statement. □

Proposition 2.4.2. *If* $\Re s\geqslant 1$*, then* $\zeta(s)\neq 0$.

Proof. In view of Proposition 2.3.6 and the fact that ζ has a pole at 1, it suffices to show that $\zeta(1+it)\neq 0$ for any $0\neq t\in\mathbf{R}$. Fix such a t.

Assume, by contradiction, that $1+it$ is a zero of order $k\geqslant 1$. At $1+2it$, we may have a zero or not, let the order be l there, the only thing we are going to use about it is $l\geqslant 0$. Then, for $1<\sigma<2$, we have, from the Laurent expansion of ζ'/ζ about 1, $1+it$, $1+2it$,

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)}=\frac{1}{\sigma-1}+O(1),\qquad -\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}=-\frac{k}{\sigma-1}+O_t(1),\qquad -\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}=-\frac{l}{\sigma-1}+O_t(1).$$

This means that for σ close enough to 1,

$$-3\Re\frac{\zeta'(\sigma)}{\zeta(\sigma)}-4\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}-\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}<0.$$

On the other hand, if we apply Proposition 2.4.1 with θ being the angle of n^{-it},

$$3+4\Re n^{-it}+\Re n^{-2it}\geqslant 0.$$

Multiplying this inequality by $\Lambda(n)/n^{\sigma}$, then summing it over $n\in\mathbf{N}$, we obtain

$$-3\Re\frac{\zeta'(\sigma)}{\zeta(\sigma)}-4\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}-\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\geqslant 0,$$

a contradiction. □

Proposition 2.4.3. *There exists a constant* $c>0$ *such that if* $\sigma>1-c/\log(2+|t|)$*, then* $\zeta(\sigma+it)\neq 0$.

Proof. Let $c>0$ be chosen later, and assume that

$$\zeta(\beta+it)=0,$$

for some $\beta>1-c/\log(2+|t|)$. Since at 1, ζ has a pole, we clearly have $t\gg 1$ (if c is chosen sufficiently small).

We apply Proposition 2.3.8 with $\varepsilon=1/2$ to see that

$$-\frac{\zeta'(s)}{\zeta(s)}=-\sum_{|\rho-s|<1/4}\frac{1}{s-\rho}+\frac{1}{s-1}+O(\log(2+|t|))$$

on $1/2<\Re s<2$. Then, for $1<\sigma<2$,

$$-\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\leqslant-\frac{1}{\sigma-\beta}+O(\log(2+|t|)).$$

Since at $1+2it$, ζ has no pole,

$$-\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\leqslant O(\log(2+|t|)),$$

and recall also

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)}=\frac{1}{\sigma-1}+O(1).$$

In the proof of Proposition 2.4.2, we proved

$$-3\Re\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \geqslant 0,$$

which implies

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} \geqslant O(\log(2+|t|)).$$

Take $\sigma = 1+4(1-\beta)$, say. Then

$$\left(\frac{3}{4}-\frac{4}{5}\right)\frac{1}{1-\beta} \geqslant O(\log(2+|t|)).$$

This is a contradiction for some small enough c. □

2.5 The prime number theorem

Proposition 2.5.1. *For any $0<\varepsilon<1$ and any $2 \leqslant T \leqslant x$, we have*

$$\sum_{n\leqslant x}\Lambda(n) = x - \sum_{\rho:\Re\rho\geqslant\varepsilon, |\Im\rho|\leqslant T}\frac{x^\rho}{\rho} + O_\varepsilon\left(x^\varepsilon\log^2 T + \frac{x\log^2 x}{T}\right),$$

where ρ runs through the zeros of ζ with multiplicity.

Proof. First we pick some $T' \in [T, T+1]$ such that

$$\int_{\varepsilon/4}^{2}\left|\frac{\zeta'(\sigma\pm iT)}{\zeta(\sigma\pm iT)}\right| d\sigma \ll_\varepsilon \log T.$$

The existence of such a T' is guaranteed by Corollary 2.3.9.

We apply Corollary 2.2.2 to see that, with $\sigma = 1+1/\log x$,

$$\sum_{n=1}^{\infty}\Lambda(n) = -\frac{1}{2\pi i}\int_{\sigma-iT'}^{\sigma+iT'}\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{x^s}{s}ds + O\left(\frac{x\log^2 x}{T}\right),$$

where in the estimate of the error term, we use $T\leqslant x$, and also

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+1/\log x}} = -\frac{\zeta'(1+1/\log x)}{\zeta(1+1/\log x)} = O(\log x).$$

Now we shift the contour to $[\sigma-iT', \varepsilon'-iT'], [\varepsilon'-iT', \varepsilon'+iT'], [\varepsilon+iT', \sigma+iT']$ in such a way that $\Re s = \varepsilon'$ avoids the poles of ζ'/ζ and also to satisfy

$$\int_{\varepsilon'-iT'}^{\varepsilon'+iT'}\left|\frac{\zeta'(s)}{\zeta(s)}\cdot\frac{x^s}{s}\right| ds \ll_\varepsilon x^\varepsilon\log^2 T.$$

The existence of such an $\varepsilon' \in [\varepsilon/2, \varepsilon]$ is again guaranteed by Corollary 2.3.9. The contribution of zeros of ζ with $T' \leqslant |\Im\rho| \leqslant T+1$ is $O(x\log T/T)$, and for zeros $\varepsilon' \leqslant \Re\rho \leqslant \varepsilon$ is $O(x^\varepsilon\log^2 T)$, borh follow from Proposition 2.3.7. Collecting all the error terms, and noting that passing through the pole of ζ gives the term x, while zeros ρ give the terms $-x^\rho/\rho$, the proof is complete. □

Theorem 2.5.2. *For some $c>0$, we have, for any $x\geqslant 2$,*

$$\sum_{n\leqslant x}\Lambda(n) = x + O(xe^{-c\sqrt{\log x}}).$$

Proof. If we apply Proposition 2.5.1 in such a way that $\varepsilon = 1 - c/\log T$ is chosen with an appropriate $c > 0$, Proposition 2.4.3 excludes zeros, that is,

$$\sum_{n \leqslant x} \Lambda(n) = x + O\left(x^{1-c/\log T} \log^2 T + \frac{x \log^2 x}{T}\right).$$

Taking now $T = e^{-c\sqrt{\log x}}$, we obtain the statement with an adjusted c. □

Theorem 2.5.3. *For some $c > 0$, we have, for any $x \geqslant 2$,*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(xe^{-c\sqrt{\log x}}),$$

where $\pi(x)$ is the number of primes up to x.

Proof. Introduce $\theta(n) = \Lambda(n) \cdot 1_{\text{primes}}(n)$. Then one can easily see that

$$\sum_{n \leqslant x} \theta(n) = \sum_{n \leqslant x} \Lambda(n) + O(x^{1/2} \log x) = x + O(xe^{-c\sqrt{\log x}}),$$

by Theorem 2.5.2. Then, with the notation $\sum_{n \leqslant x} \theta(n) = x + E(x)$,

$$\begin{aligned}\pi(x) = \int_{3/2}^x \frac{1}{\log t} d\left(\sum_{n \leqslant t} \theta(n)\right) &= \int_{3/2}^x \frac{dt}{\log t} + \int_{3/2}^x \frac{dE(t)}{\log t} \\ &= \int_{3/2}^x \frac{dt}{\log t} + \frac{E(x)}{\log x} - \frac{E(3/2)}{\log(3/2)} + \int_{3/2}^x \frac{E(t)}{t \log^2 t} dt.\end{aligned}$$

In the last integral, $E(t) \ll xe^{-c\sqrt{\log x}}$ uniformly for $3/2 \leqslant t \leqslant x$ by Theorem 2.5.2, and $\int_{3/2}^x (t \log^2 t)^{-1} dt = O(1)$. Switching $3/2$ to 2 is admissible. □

Corollary 2.5.4. *We have, for $x \geqslant 2$,*

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Proof. Assume that $x \geqslant 4$, say. The error term of Theorem 2.5.2 is clearly admissible, so we are left with the integral there. Integrating by parts,

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}.$$

Here,

$$\int_2^x \frac{dt}{\log^2 t} = \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t} = O(\sqrt{x}) + O\left(\frac{x}{\log^2 x}\right),$$

and the proof is complete. □

Remark 2.5.5. With further integrations by parts, we get similarly that for any $K \in \mathbf{N}$,

$$\pi(x) = x \sum_{k=1}^K \frac{(k-1)!}{\log^k x} + O_K\left(\frac{x}{\log^{K+1} x}\right).$$

Chapter 3
Primes in arithmetic progressions

3.1 Primitive characters

Definition 3.1.1 (quasiperiod of Dirichlet characters)**.** Given a Dirichlet character χ modulo q, we say it has quasiperiod d, if $\chi(m) = \chi(n)$ for any integers m, n satisfying $m \equiv n \bmod d$ and $\gcd(mn, q) = 1$.

Proposition 3.1.2. *Assume χ is a Dirichlet character modulo q. If d is a quasiperiod of χ, then* $\gcd(d, q)$ *is also a quasiperiod of χ.*

Proof. Let m, n be integers coprime to q which satisfy that $\gcd(d, q) \mid (m - n)$. Fix some integers u, v such that $m - n = du + qv$. Then

$$\chi(m) = \chi(m - qv) = \chi(n + du) = \chi(n),$$

and only the last equality requires explanation: we have to show that $n + du$ is coprime to q. And this indeed holds, since $n + du = m - qv$, and $m - qv$ is coprime to q. □

Proposition 3.1.3. *Assume χ is a Dirichlet character modulo q. If d_1, d_2 are quasiperiods of χ, then so is $d = \gcd(d_1, d_2)$.*

Proof. Given integers m, n which are both coprime to q and congruent to each other modulo d, our strategy is to construct a certain k such that $m \equiv k \bmod d_1$, $n \equiv k \bmod d_2$ and $\gcd(k, q) = 1$. This suffices, since

$$\chi(m) = \chi(k) = \chi(n).$$

To this aim, seek for k in the form $k = m + d_1 w_1 = n + d_2 w_2$, that is, we have to solve the diophantine equation

$$m - n = d_2 w_2 - d_1 w_1$$

for w_1, w_2. Clearly there are solutions, since $\gcd(d_1, d_2) \mid (m-n)$, and focusing on, say, w_1, it is determined modulo $d_2 / \gcd(d_1, d_2)$. Now choose an integer w_1 such that it is from the prescribed residue class modulo $d_2 / \gcd(d_1, d_2)$, and that for any prime divisor p of q not dividing $d_1 d_2$, $w_1 \not\equiv -m/d_1 \bmod p$. The existence of such integers is guaranteed by the Chinese Remainder Theorem. Then let $k = m + d_1 w_1 = n + d_2 w_2$ (with w_2 accompanying w_1), and we claim this does the job. The only thing we have to check is that no prime divisor of q divides k. And indeed, if a prime divisor p of q is coprime to $d_1 d_2$, then $k \not\equiv 0 \bmod p$, since $w_1 \not\equiv -m/d_1 \bmod p$; while if $p \mid d_1 d_2$, then $p \mid d_1$ implies $k \equiv m \bmod p$, and $p \mid d_2$ implies $k \equiv n \bmod p$, any of these excludes $p \mid k$ by the assumption $\gcd(mn, q) = 1$. □

Definition 3.1.4 (induction of characters)**.** Let χ^* be a Dirichlet character modulo d. Then for a multiple q of d, we can consider the Dirichlet character χ defined as

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } \gcd(n, q) = 1, \\ 0 & \text{if } \gcd(n, q) > 1. \end{cases}$$

Then χ is the character induced by χ^*.

Proposition 3.1.5. *Let χ be a Dirichlet character modulo q, and assume d is the smallest quasiperiod of q. Then there is a unique Dirichlet character χ^* modulo d which induces χ.*

Proof. For any n coprime to d, choose $k \in \mathbf{Z}$ such that $n + kd$ is coprime to q, and let $\chi^*(n) = \chi(n + kd)$ (the d-quasiperiodicity cancels the ambiguity in the choice of k, hence this is well-defined). For n not coprime to d, let $\chi^*(n) = 0$. Clearly χ^* induces χ. Uniqueness follows from that χ determines χ^* on residue classes coprime to d. □

Definition 3.1.6 (conductor, primitive character)**.** Given a Dirichlet character χ modulo q, its smallest quasiperiod d is called its conductor. We say that χ is primitive, if $d = q$.

Proposition 3.1.7. *Assume χ is a Dirichlet character modulo q. Then the following are equivalent:*

(i) χ is primitive;

(ii) for any $d \mid q$, $d < q$, there exists $c \equiv 1 \bmod d$, $\gcd(c, q) = 1$ such that $\chi(c) \neq 1$;

(iii) for any $d \mid q$, $d < q$, and any residue class $a \bmod d$,

$$\sum_{\substack{1 \leqslant x \leqslant q \\ x \equiv a \bmod d}} \chi(x) = 0.$$

Proof. Assume first that (i) holds. Then for any $d \mid q$, $d < q$, since d is not a quasiperiod, there exist m, n coprime to q, congruent modulo d such that $\chi(m) \neq \chi(n)$. Clearly $c \equiv mn^{-1} \equiv 1 \bmod d$, and $\chi(c) = \chi(m)/\chi(n) \neq 1$, so (ii) holds.

Assume now that (ii) holds. Given the input as in (iii), take c guaranteed by (ii). Then

$$\sum_{\substack{1 \leqslant x \leqslant q \\ x \equiv a \bmod d}} \chi(x) = \sum_{\substack{1 \leqslant x \leqslant q \\ x \equiv a \bmod d}} \chi(cx) = \chi(c) \sum_{\substack{1 \leqslant x \leqslant q \\ x \equiv a \bmod d}} \chi(x),$$

so the sum must vanish, as $\chi(c) \neq 1$, that is, (iii) holds.

Assume finally that (iii) holds. By contradiction, assume that $d \mid q$, $d < q$ is a quasiperiod. Then setting $a \equiv 1 \bmod d$,

$$\sum_{\substack{1 \leqslant x \leqslant q \\ x \equiv 1 \bmod d}} \chi(x) = \#\{1 \leqslant x \leqslant q : \gcd(x, q) = 1, x \equiv 1 \bmod d\},$$

which is strictly positive, since $x = 1$ is an element of the set on the right-hand side. This is a contradiction, so (i) holds. □

Proposition 3.1.8. *Let q_1, q_2 be coprime integers, and χ_1, χ_2 Dirichlet characters modulo q_1, q_2, respectively. Then the character χ modulo $q = q_1 q_2$ defined via $\chi(n) = \chi_1(n)\chi_2(n)$ is primitive if and only if χ_1, χ_2 are both primitive.*

Proof. It is easy to see that if χ_1, χ_2 have quasiperiods d_1, d_2, respectively, then χ has quasiperiod $d_1 d_2$, which shows that primitivity of χ implies that of χ_1, χ_2.

As for the converse, assume χ_1, χ_2 are primitive, and that d is a quasiperiod of χ. Set $d_1 = \gcd(q_1, d)$ and $d_2 = \gcd(q_2, d)$. Now take any integers m, n coprime to q_1 and congruent modulo d_1. Choose $m' \equiv m \bmod q_1$, $m' \equiv 1 \bmod q_2$, $n' \equiv n \bmod q_1$, $n' \equiv 1 \bmod q_2$. Then $m' \equiv n' \bmod d$, $\gcd(m'n', q) = 1$, hence $\chi(m') = \chi(n')$. Then the calculation

$$\chi_1(m) = \chi_1(m') = \chi_1(m')\chi_2(m') = \chi(m') = \chi(n') = \chi_1(n')\chi_2(n') = \chi_1(n') = \chi_1(n)$$

shows that χ_1 has quasiperiod d_1. By primitivity of χ_1, this implies $d_1 = q_1$. Similarly, $d_2 = q_2$. □

Therefore, to describe the primitive characters, it suffices to describe them for prime power moduli. This is not hard, using the explicit description of the multiplicative group of residue classes modulo prime power moduli (using their cyclic or "almost" cyclic structure).

3.2 Analytic properties of Dirichlet L-functions

For any Dirichlet character χ modulo $q \in \mathbf{N}$, we may define, for $\Re s > 1$,

$$L(s,\chi) = \mathcal{D}\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

since the Dirichlet series on the right-hand side is absolutely convergent in the indicated domain. Then, analogously to the ζ function, we have

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \qquad \Re s > 1.$$

Proposition 3.2.1. *Let $y \leqslant x$ be real numbers, and assume $f : \mathbf{R} \to \mathbf{C}$ is continuously differentiable. Then, for $\chi \neq \chi_0$, we have*

$$\sum_{y \leqslant n \leqslant x} \chi(n) f(n) \ll q \int_y^x |f'(t)| dt + q|f(x)|.$$

If $\chi = \chi_0$, then

$$\sum_{y \leqslant n \leqslant x} \chi(n) f(n) - \frac{\varphi(q)}{q} \int_y^x f(t) dt \ll q \int_y^x |f'(t)| dt + q|f(x)|.$$

Proof. First we consider the case when $\chi \neq \chi_0$. Write, for $n \in \mathbf{N}$, $X(n) = \sum_{j \leqslant n} \chi(j)$, then $\chi(n) = X(n) - X(n-1)$. Then

$$\sum_{y \leqslant n \leqslant x} \chi(n) f(n) = \sum_{y \leqslant n \leqslant x} (X(n) - X(n-1)) f(n) = \sum_{y+1 \leqslant n \leqslant x} X(n)(f(n) - f(n-1)) + qO(|f(y)|) + qO(|f(x)|),$$

using that $|X(n)| \leqslant q$. Here, by Newton-Leibniz, $f(n) - f(n-1) = \int_{n-1}^n f'(t) dt$, that is, applying again $|X(n)| \leqslant q$, the above is at most

$$q \int_y^x |f'(t)| dt + qO(|f(y)|) + qO(|f(x)|).$$

Clearly $|f(y) - f(x)| \leqslant \int_y^x |f'(t)| dt$, hence we can drop the term $qO(|f(y)|)$.

When $\chi = \chi_0$, we can repeat the above argument with the function $\tilde{\chi}(n) = \chi(n) - \varphi(q)/q$, which sums to 0 over a period of length q, and over all intervals, its sum in absolute value is at most q. We evaluate

$$\sum_{y \leqslant n \leqslant x} \frac{\varphi(q)}{q} f(n)$$

via Proposition 2.1.3. □

Proposition 3.2.2. *For any Dirichlet character $\chi \neq \chi_0$ modulo q, $L(s,\chi)$ extends to $\Re s > 0$ as a holomorphic function.*

Also, for any $\varepsilon > 0$, if $\Re s \geqslant \varepsilon$, then

$$\log |L(s,\chi)| \leqslant O_\varepsilon(\log(q(2 + |s|))).$$

If $\chi = \chi_0$, then $L(s,\chi_0) = \zeta(s) \prod_{p|q}(1 - p^{-s})$. In particular, $L(s,\chi_0)$ extends meromorphically to $\Re s > 0$ with a unique, simple pole at $s = 1$, where the residue is $\varphi(q)/q$. Also, for any $\varepsilon > 0$, if $\Re s \geqslant \varepsilon$ and $|s-1| \geqslant \varepsilon$,

$$\log |L(s,\chi_0)| \leqslant O_\varepsilon(\log(q(2 + |s|))).$$

Proof. Let first $\chi \neq \chi_0$. By Proposition 3.2.1 with the function $n \mapsto n^{-s}$, we obtain, for any $1 \leqslant y < x$ that

$$\sum_{y \leqslant n \leqslant x} n^{-s} = qO\left(\frac{|s|}{\sigma} y^{-\sigma}\right).$$

Then by Proposition 1.1.2, we obtain that there exists some $L(s,\chi) \in \mathbf{C}$ such that

$$\sum_{n \leqslant x} n^{-s} = L(s,\chi) + qO\left(\frac{|s|}{\sigma} x^{-\sigma}\right).$$

When $\Re s > 1$, this $L(s,\chi)$ must coincide with the earlier defined $\mathcal{D}\chi(s)$. Therefore, we really have the extension, and its holomorphic nature follows from the local uniform convergence of $\sum_{n \leqslant x} n^{-s}$ to $L(s,\chi)$.

To see the estimate, write $x = 1$, and then

$$L(s,\chi) = qO\left(\frac{|s|}{\sigma}\right),$$

from which estimate claim follows immediately.

If $\chi = \chi_0$, for $\Re s > 1$, we obtain the relation to $\zeta(s)$ from the Euler products, and then we apply Proposition 2.3.5 to extend $\zeta(s)$. As for the estimate, observe that each factor $1 - p^{-s}$ is at most 2 in absolute value, and the number of such factors (i.e. the number of prime divisors of q) is at most $\log_2 q$, which together give the claim. □

One can easily check that

$$\frac{1}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}, \qquad \Re s > 1,$$

which in particular implies that $L(s,\chi)$ does not vanish on $\Re s > 1$ (recall Proposition 2.3.6).

For convenience, introduce

$$E_0(\chi) = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

Proposition 3.2.3. *Let χ be a Dirichlet character modulo q. Then for any $\varepsilon > 0, t_0 \in \mathbf{R}$, the number of zeros of $L(s,\chi)$ (counted with multiplicity) in $\{s : \Re s \geqslant \varepsilon, |\Im s - t_0| \leqslant 1\}$ is $O_\varepsilon(\log(q(2+|t_0|)))$.*

Proof. This follows the same way as Proposition 2.3.7. □

Proposition 3.2.4. *Let χ be a Dirichlet character modulo q. For any $0 < \varepsilon < C$, we have*

$$-\frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{|\rho - s| < \varepsilon/2} \frac{1}{s-\rho} + \frac{E_0(\chi)}{s-1} + O_{C,\varepsilon}(\log(q(2+|t|)))$$

for $s = \sigma + it$, $\varepsilon \leqslant \sigma \leqslant C$, where ρ runs through the zeros of $L(s,\chi)$ with multiplicity.

Proof. This follows the same way as Proposition 2.3.8. □

Corollary 3.2.5. *Let χ be a Dirichlet character modulo q. For any $C, \varepsilon > 0$, $t_0 \in \mathbf{R}$, we have*

$$\int_\varepsilon^C \int_{t_0-1}^{t_0+1} \left|\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)}\right| dt d\sigma \ll_{C,\varepsilon} \log(q(2+|t_0|)).$$

Proof. This follows the same way as Corollary 2.3.9. □

3.3 Convolutions with quadratic characters

Proposition 3.3.1. *Let χ be a real nonprincipal character of modulus q. Then, for any $1/2 \leqslant s < 1$ and $x \geqslant 2$,*

$$\sum_{n\leqslant x}\frac{(1*\chi)(n)}{n^s}=\zeta(s)L(s,\chi)+\frac{x^{1-s}}{1-s}L(1,\chi)+O\left(\frac{q}{1-s}x^{1/2-s}\right).$$

Proof. We start by Dirichlet's hyperbola method to see that the quantity in question is

$$\sum_{d\leqslant\sqrt{x}}\frac{\chi(d)}{d^s}\sum_{m\leqslant x/d}\frac{1}{m^s}+\sum_{m\leqslant\sqrt{x}}\frac{1}{m^s}\sum_{\sqrt{x}<d\leqslant x/m}\frac{\chi(d)}{d^s}.$$

In the proof of Proposition 2.3.5 and that of Proposition 3.2.2, we proved that

$$\sum_{n\leqslant x}\frac{1}{n^s}=\zeta(s)-\frac{x^{1-s}}{s-1}+O(x^{-s}),\qquad \sum_{n\leqslant x}\frac{\chi(n)}{n^s}=L(s,\chi)+qO(x^{-s}).$$

Then on the one hand,

$$\begin{aligned}\sum_{d\leqslant\sqrt{x}}\frac{\chi(d)}{d^s}\sum_{m\leqslant x/d}\frac{1}{m^s}&=\sum_{d\leqslant\sqrt{x}}\frac{\chi(d)}{d^s}\left(\zeta(s)-\frac{(x/d)^{1-s}}{s-1}+O\left(\left(\frac{x}{d}\right)^{-s}\right)\right)\\&=L(s,\chi)\zeta(s)+O(qx^{-s/2})+\frac{x^{1-s}}{1-s}L(1,\chi)+O\left(\frac{qx^{1-3s/2}}{1-s}\right)+O(x^{1/2-s})\\&=L(s,\chi)\zeta(s)+\frac{x^{1-s}}{1-s}L(1,\chi)+O\left(\frac{qx^{1/2-s}}{1-s}\right).\end{aligned}$$

On the other hand,

$$\sum_{m\leqslant\sqrt{x}}\frac{1}{m^s}\sum_{\sqrt{x}<d\leqslant x/m}\frac{\chi(d)}{d^s}=\sum_{m\leqslant\sqrt{x}}\frac{1}{m^s}\cdot O(qx^{-s/2})=O\left(\frac{qx^{1/2-s}}{1-s}\right),$$

and the proof is complete. □

Proposition 3.3.2. *There exists some $c>0$ with the following property. If χ,χ' are distinct real primitive characters of moduli $q,q'>1$, respectively, then for any $1-c\leqslant s<1$ and $x\geqslant 2$,*

$$\begin{aligned}&\sum_{n\leqslant x}\frac{(1*\chi*\chi'*\chi\chi')(n)}{n^s}\\&=\zeta(s)L(s,\chi)L(s,\chi')L(s,\chi\chi')+\frac{x^{1-s}}{1-s}L(1,\chi)L(1,\chi')L(1,\chi\chi')+O\left(\frac{(qq')^2x^{1-c-s}}{1-s}\right).\end{aligned}$$

Proof. Record first that the distinct and primitive being of χ,χ' guarantees that none of $\chi,\chi',\chi\chi'$ is principal. Then we have to evaluate

$$\sum_{n\leqslant x}\sum_{d_1d_2d_3m=n}\frac{\chi(d_1)}{d_1^s}\cdot\frac{\chi'(d_2)}{d_2^s}\cdot\frac{(\chi\chi')(d_3)}{d_3^s}\cdot\frac{1}{m^s}.$$

The evaluation of this sum is based on a higher-dimensional version of Dirichlet's hyperbola method. Namely, we separate the part when $d_1,d_2,d_3\leqslant x^{1/4}$:

$$\sum_{d_1,d_2,d_3\leqslant x^{1/4}}\frac{\chi(d_1)}{d_1^s}\cdot\frac{\chi'(d_2)}{d_2^s}\cdot\frac{(\chi\chi')(d_3)}{d_3^s}\sum_{m\leqslant x/(d_1d_2d_3)}\frac{1}{m^s}.$$

From

$$\sum_{n\leqslant x}\frac{1}{n^s}=\zeta(s)-\frac{x^{1-s}}{s-1}+O(x^{-s}),\qquad \sum_{n\leqslant x}\frac{\chi(n)}{n^s}=L(s,\chi)+qO(x^{-s}),$$

one can easily check that

$$\sum_{d_1,d_2,d_3\leqslant x^{1/4}} \frac{\chi(d_1)}{d_1^s}\cdot\frac{\chi'(d_2)}{d_2^s}\cdot\frac{(\chi\chi')(d_3)}{d_3^s} \sum_{m\leqslant x/(d_1d_2d_3)} \frac{1}{m^s}$$
$$= \zeta(s)L(s,\chi)L(s,\chi')L(s,\chi\chi') + \frac{x^{1-s}}{1-s}L(1,\chi)L(1,\chi')L(1,\chi\chi') + O\left(\frac{(qq')^2x^{1-c-s}}{1-s}\right),$$

if $c>0$ is a sufficiently small fixed constant.

In the remaining part, we crudely apply dyadic decomposition as follows. For any $0\leqslant A,B,C<1$,

$$\sum_{x^A\leqslant d_1<2x^A}\frac{\chi(d_1)}{d_1^s} = O(qx^{-As}),\quad \sum_{x^B\leqslant d_2<2x^B}\frac{\chi'(d_2)}{d_2^s} = O(q'x^{-Bs}),\quad \sum_{x^C\leqslant d_3<2x^C}\frac{(\chi\chi')(d_3)}{d_3^s} = O((qq')x^{-Cs}),$$

and since in the remaining part, $\max(A,B,C)>1/4$, the proof is complete by the trivial bound $\sum_{m\leqslant x^{1-(A+B+C)}} m^{-s} = O(x^{1-s}/(1-s))$. □

3.4 Zero-free regions and zero repulsions

Proposition 3.4.1. *There exists a constant $c>0$ such that for any Dirichlet character χ modulo q, $L(s,\chi)$ has no zeros in the region*

$$\left\{\beta+it : \beta > 1-\frac{c}{\log(q(2+|t|))}\right\},$$

with the possible exception of a single real zero $1-c/\log(2q)<\beta<1$. If this exceptional real zero exists, it necessarily belongs to a real character.

Proof. We may freely assume that χ is nonprincipal, in particular, $q\geqslant 2$. All along, when we evaluate $L(s,\chi)$ or its logarithmic derivative, etc., we will assume that $\Re s<2$.

First consider the case when χ is complex, i.e. χ^2 is nontrivial. Assume, by contradiction, that $L(\beta+it,\chi)=0$ for some $\beta>1-c/\log(q(2+|t|))$. Then by Proposition 3.2.4, we have, for $\sigma>1$

$$-\Re\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} \leqslant -\frac{1}{\sigma-\beta} + O(\log(q(2+|t|))).$$

Also, since χ^2 is nonprincipal,

$$-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} \leqslant O(\log(q(2+|t|))),$$

and clearly

$$-\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} \leqslant \frac{1}{\sigma-1}+O(1).$$

Applying Proposition 2.4.1 to θ equaling the angle of $\chi(n)n^{-it}$, then summing it up over $n\in\mathbf{N}$ coprime to q with weights $\Lambda(n)$, we obtain

$$-3\Re\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - 4\Re\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} - \Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} \geqslant 0.$$

This means that

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta} \geqslant O(\log(q(2+|t|))).$$

Setting $\sigma=1+4(1-\beta)$, this is a contradiction, if $c>0$ is small enough (just like in the proof of Proposition 2.4.3). To sum up this part: if χ is nonreal, then there is no zero in the domain in question.

Assume now $\chi^2=\chi_0$. Then, for $\sigma>1$

$$-\Re\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} = -\Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} - \Re\sum_{p|q}\frac{\log p}{p^{\sigma+2it}-1},$$

and here,

$$\sum_{p|q} \frac{\log p}{p^{\sigma+2it}-1} = O\left(\sum_{p|q} \log p\right) = O(\log q),$$

hence, by Proposition 2.3.8,

$$-\Re \frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} \leqslant \Re \frac{1}{\sigma+2it-1} + O(\log(q(2+|t|))).$$

Now take some $c_1 > 0$, and assume that there are two zeros of $L(s,\chi)$ in the region

$$\left\{\beta + it : |t| \leqslant \frac{c_1}{\log q} : \beta > 1 - \frac{c}{\log(q(2+|t|))}\right\},$$

where the only assumption on c is $0 < c < c_1$. Letting these zeros be $\beta_1 + it_1, \beta_2 + it_2$, we have, by Proposition 3.2.4,

$$-\Re \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} \leqslant -\Re \frac{1}{\sigma-\beta_1-it_1} - \Re \frac{1}{\sigma-\beta_2-it_2} + O(\log q).$$

We also have, in the indicated domain,

$$\left|\frac{L'(\sigma,\chi)}{L(\sigma,\chi)}\right| \leqslant \frac{1}{\sigma-1} + O(1),$$

therefore,

$$\Re \frac{1}{\sigma-\beta_1-it_1} + \Re \frac{1}{\sigma-\beta_2-it_2} \leqslant \frac{1}{\sigma-1} + O(\log q).$$

Here, $|1-(\beta_{1,2}+it_{1,2})| \leqslant 2c_1/\log q$, hence if we take, say, $\sigma = 1 + 10c_1/\log q$, the left-hand side is at least $\log q/c_1 \cdot 20/12^2$, while the right-hand side is $\log q/c_1 \cdot 1/10 + O(\log q)$, which is a contradiction, if c_1 is small enough. To sum up this part: if χ is real, there might be at most one zero up to height $c_1/\log q$, this potential zero will be referred as the exceptional zero.

Fix such a c_1, and now consider the domain

$$\left\{\beta + it : |t| \geqslant \frac{c_1}{\log q} : \beta > 1 - \frac{c}{\log(q(2+|t|))}\right\},$$

and assume there is a zero $\beta + it$ of $L(s,\chi)$ here. Recalling the above-proven

$$-3\Re \frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - 4\Re \frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} - \Re \frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} \geqslant 0$$

and

$$-\Re \frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi^2)} \leqslant \Re \frac{1}{\sigma+2it-1} + O(\log(q(2+|t|))),$$

this means, by Proposition 3.2.4,

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + \Re \frac{1}{\sigma+2it-1} \geqslant O(\log(q(2+|t|))).$$

Setting here $\sigma = 1 + 4(1-\beta)$, this gives

$$-\frac{1}{20(1-\beta)} + \Re \frac{1}{4(1-\beta)+2it} \geqslant O(\log(q(2+|t|))).$$

This can be easily seen to give a contradiction, if we make $c > 0$ small enough. To sum up this part: even if χ is real, the indicated domain is zero-free above height $c_1/\log q$.

To complete the proof we still have to see that the exceptional zero (if exists) is necessarily real. But this is clear from the fact that for real characters χ, $L(\overline{s},\chi) = \overline{L(s,\chi)}$, hence zeros come in conjugate pairs. □

Proposition 3.4.2. *There is an absolute constant $c > 0$ such that the following holds. For any two distinct, primitive characters χ, χ' of moduli q, q', respectively, there is at most one real zero of $L(s, \chi)$ and $L(s, \chi')$ on the segment $[1 - c/\log(qq'), 1]$.*

Proof. First observe that for any $n \in \mathbf{N}$,

$$1 + \chi(n) + \chi'(n) + \chi\chi'(n) = (1 + \chi(n))(1 + \chi'(n)) \geqslant 0.$$

Multiplying this by $\Lambda(n)/n^\sigma$ for any $1 < \sigma < 2$, and summing up over $n \in \mathbf{N}$, we see that

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} - \frac{L'(\sigma, \chi)}{L(\sigma, \chi)} - \frac{L'(\sigma, \chi')}{L(\sigma, \chi')} - \frac{L'(\sigma, \chi\chi')}{L(\sigma, \chi\chi')} \geqslant 0.$$

On the other hand, assuming that $L(s, \chi)$ has a zero β, and that $L(s, \chi')$ has a zero β' on the segment $[1 - c/\log(qq'), 1]$, we obtain, via Proposition 2.3.8 and Proposition 3.2.4,

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + O(1), \qquad -\frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \leqslant -\frac{1}{\sigma - \beta} + O(\log q), \qquad -\frac{L'(\sigma, \chi')}{L(\sigma, \chi')} \leqslant -\frac{1}{\sigma - \beta'} + O(\log q').$$

Also, since χ and χ' are distinct and primitive, $\chi\chi'$ is nonprincipal, therefore

$$-\frac{L'(\sigma, \chi\chi')}{L(\sigma, \chi\chi')} \leqslant O(\log(qq')).$$

Summing up, we see that

$$\frac{1}{\sigma - 1} - \frac{1}{\sigma - \beta} - \frac{1}{\sigma - \beta'} \geqslant O(\log(qq')).$$

If, say, $\sigma = 1 + 10c/\log(qq')$, this is a contradiction, if c is small enough. □

Proposition 3.4.3. *For any $c > 0$, the following holds. If for a real character χ modulo q, $L(s, \chi)$ has a zero β on the segment $[1 - c/\log q, 1]$, then $L(1, \chi) \ll_c (1 - \beta)\log^2 q$.*

There exists an absolute $c > 0$ with the following property. If $L(1, \chi) \leqslant c/\log q$ for a real character χ modulo q, then there is a real zero β of $L(s, \chi)$ with $\beta \geqslant 1 - O(L(1, \chi))$.

Proof. We start with the proof of the first statement. Fix $c > 0$, and all the implied constants in $O(\dots)$ below depend only on c. Recall from the proof of Proposition 3.2.2 that

$$\sum_{n \leqslant x} n^{-s} = L(s, \chi) + qO\left(\frac{|s|}{\sigma} x^{-\sigma}\right).$$

Writing $x = q$, and $n^{-s} = O(n^{-1})$ for $\Re s > 1 - c/\log q$, we see that

$$L(s, \chi) = O\left(\sum_{n \leqslant q} \frac{1}{n}\right) + O(1) = O(\log q)$$

for $|s - 1| \leqslant 2c/\log q$. Then by Cauchy's integral formula, we have

$$L'(s, \chi) = O(\log^2 q)$$

for $|s - 1| \leqslant c/\log q$. Then if $L(\beta, \chi) = 0$ for some $\beta \in [1 - c/\log q, 1]$, the proof is completed by

$$L(1, \chi) = \int_\beta^1 L'(s, \chi) ds.$$

Now we prove the second statement. Take some large $C > 0$ (to be specified later) and some $c > 0$ small (to be specified later). Let $s = 1 - CL(1, \chi)$, then $3/4 \leqslant s < 1$, if c is small enough (in terms of C). It is easy to check that $(1 * \chi)(n) \geqslant 0$, and $(1 * \chi)(1) = 1$, therefore,

$$\sum_{n \leqslant x} \frac{(1 * \chi)(n)}{n^s} \geqslant 1.$$

Also, from Proposition 3.3.1, we see that the left-hand side here is

$$\zeta(s)L(s,\chi) + \frac{x^{CL(1,\chi)}}{C} + O(x^{-1/4}q/L(1,\chi)).$$

Now if $L(1,\chi) \leqslant c/\log q$, then by choosing $C > 0$ large, $c > 0$ small, this gives, for $x = q^{10}/L(1,\chi)^{10}$, that

$$\zeta(s)L(s,\chi) \geqslant 1/2.$$

It is easy to see that $\zeta(s) < 0$ on a segment $[1-a, 1]$ for some $a > 0$ (in fact, it it not hard to prove that this holds even for $a = 1$), hence $L(s,\chi) < 0$ either, if $s > a$ (forced by choosing c small enough after choosing C). Then the proof is complete by recalling $L(1,\chi) > 0$ and the fact that $L(s,\chi)$ is real-valued and continuous for $s > 0$. □

Proposition 3.4.4. *There exists some $c > 0$ with the following property. Let χ be a primitive real character of modulus q. Suppose that $L(\beta,\chi) = 0$ for some $1 - c \leqslant \beta < 1$. Then for any primitive real character χ' of modulus q', we have*

$$L(1,\chi)L(1,\chi') \gg \frac{1-\beta}{(qq')^{O(1-\beta)}\log(qq')}.$$

Proof. First, observe that

$$(1 * \chi * \chi' * \chi\chi')(n) \geqslant 0, \qquad (1 * \chi * \chi' * \chi\chi')(1) = 1.$$

Let $c > 0$ be small enough such that from Proposition 3.3.2, we obtain, if χ in the statement exists,

$$\frac{x^{1-\beta}}{1-\beta}L(1,\chi)L(1,\chi')L(1,\chi\chi') + O\left(\frac{(qq')^2}{1-\beta}x^{1-2c-\beta}\right) = \sum_{n\leqslant x}\frac{(1 * \chi * \chi' * \chi\chi')}{n^\beta} \geqslant 1.$$

Writing $x = (qq'/(1-\beta))^C$ for some $C > 0$, the error term is

$$O((qq')^{2+(1-c-\beta)C}(1-\beta)^{(c+\beta-1)C-1}),$$

which is less than $1/2$, if C is large enough. Also, $x^{1-\beta} \ll (qq')^{O(1-\beta)}$ with the implied constants depending only on C. Then the statement follows from $L(1,\chi\chi') = O(\log(qq'))$, which follows from

$$\sum_{n\leqslant x}\frac{(\chi\chi')(n)}{n} + qq'O(x^{-1}),$$

by setting $x = qq'$. □

Theorem 3.4.5 (Siegel). *For any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that*

$$L(1,\chi) \geqslant c_\varepsilon q^{-\varepsilon}.$$

For any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that $L(s,\chi)$ has no zero on the segment $[1 - c_\varepsilon q^{-\varepsilon}, 1]$ for any primitive real character of modulus q.

Proof. By Proposition 3.4.3, it suffices to prove the first statement. Let first $c'_\varepsilon > 0$ effectively computable (to be chosen later). If there is no primitive real character χ with a zero of $L(s,\chi)$ on the segment $[1 - c'_\varepsilon q^{-\varepsilon}, 1]$, then Proposition 3.4.3 gives the statement immediately.

Assume hence that there is a primitive real character χ with a zero β of $L(s,\chi)$ on the segment $[1 - c'_\varepsilon q^{-\varepsilon}, 1]$. Choose χ among such characters in such a way that q is minimal. Then for $q' < q$, χ' primitive real character of modulus q', we see that $L(s,\chi')$ has no zero on $[1 - c'_\varepsilon q'^{-\varepsilon}, 1]$. Let then $q' \geqslant q$ be arbitrary, χ' primitive real character of modulus q'.

By Proposition 3.4.3, we see that

$$L(1,\chi) \ll (1-\beta)\log^2 q.$$

Then by Proposition 3.4.4, we obtain that

$$L(1,\chi') \gg \frac{1}{q'^{O(1-\beta)}\log^3(q')} = q'^{O(\beta-1)}\log^{-3}(q'),$$

and here, the exponent of q' can be made arbitrarily close to 0, by choosing c'_ε sufficiently small, since $0 > \beta - 1 \geqslant -c'_\varepsilon$.

We finally adjust c_ε to make the statement true for $L(1,\chi)$. Indeed, we know it is nonzero from Theorem 1.4.5, and then its positive being follows from the following facts: $L(s,\chi) \in \mathbf{R}$ for $s > 1$, $\lim_{\mathbf{R}\ni s\to+\infty} L(s,\chi) = 1$, $L(s,\chi) \neq 0$ for $s > 1$, and $L(s,\chi)$ is continuous. □

3.5 Prime number theorem for arithmetic progressions

Theorem 3.5.1. *There exists an absolute constant $c > 0$ with the following property. For any $x \geqslant 2$, if χ_0 is the principal Dirichlet character modulo $q = O(e^{c\sqrt{\log x}})$, then*

$$\sum_{n\leqslant x} \Lambda(n)\chi_0(n) = x + O(xe^{-c\sqrt{\log x}}).$$

Proof. The proof is the same as that of Theorem 2.5.2 (see also Proposition 2.5.1, the only difference is that we refer to Propisition 3.2.3 and Corollary 3.2.5). □

Theorem 3.5.2. *There exists an absolute constant $c > 0$ with the following property. For any $x \geqslant 2$, if χ is a nonprincipal Dirichlet character modulo $q = O(e^{c\sqrt{\log x}})$ with no zero on the segment $[1 - c/\log q, 1]$, then*

$$\sum_{n\leqslant x} \Lambda(n)\chi(n) = O(xe^{-c\sqrt{\log x}}).$$

Proof. The same as that of Theorem 3.5.1, this time with no pole at 1. □

Theorem 3.5.3. *There exists an absolute constant $c > 0$ with the following property. For any $x \geqslant 2$, if χ is a nonprincipal Dirichlet character modulo $q = O(e^{c\sqrt{\log x}})$ with a zero β on the segment $[1 - c/\log q, 1]$, then*

$$\sum_{n\leqslant x} \Lambda(n)\chi(n) = -\frac{x^\beta}{\beta} + O(xe^{-c\sqrt{\log x}}).$$

Proof. The same as that of Theorem 3.5.2, this time with a pole at β. □

Corollary 3.5.4. *There exists an absolute constant $c > 0$ with the following property. For any $x \geqslant 2$, $q \leqslant e^{c\sqrt{\log x}}$ and any a coprime to q,*

$$\sum_{n\equiv a \bmod q} \Lambda(n) = \frac{x}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)}\frac{x^\beta}{\beta} + O(xe^{-c\sqrt{\log x}}),$$

where β is the exceptional zero of $L(s,\chi)$ on the segment $[1 - c/\log q, 1]$ (if c is small enough, χ is well-defined by Proposition 3.4.2). If there is no such character, then

$$\sum_{n\equiv a \bmod q} \Lambda(n) = \frac{x}{\varphi(q)} + O(xe^{-c\sqrt{\log x}}).$$

Theorem 3.5.5. *Let $c > 0$ be the minimum of the constants from Theorems 3.5.1-3.5.3. Then for any $A > 0$, there exists a constant $x_0(A)$ such that if $x \geqslant x_0(A)$, and χ is a nonprincipal character modulo q, then*

$$\sum_{n\leqslant x} \Lambda(n)\chi(n) = O(xe^{-c\sqrt{\log x}}).$$

Proof. We use the notation of Theorem 3.5.3, assuming that χ has a Siegel zero β. Then for any $\varepsilon > 0$, by Theorem 3.4.5,

$$x^\beta = xe^{-(1-\beta)\log x} \leqslant xe^{-c_\varepsilon q^{-\varepsilon}\log x}$$

for some $c_\varepsilon > 0$. Since $q \leqslant (\log x)^A$, if we choose $\varepsilon < 1/(2A)$, then the above is

$$xe^{-c_\varepsilon q^{-\varepsilon}\log x} \leqslant xe^{-c_\varepsilon(\log x)^{1-A\varepsilon}} \leqslant xe^{-c\sqrt{\log x}},$$

if x is large enough. Then the claim follows from Theorem 3.5.3. □

Corollary 3.5.6 (Siegel-Walfisz)**.** *For any $A > 0$, there exists a constant $c_A > 0$, such that for any $x \geqslant 2$, $q \leqslant (\log x)^A$ and any a coprime to q,*

$$\sum_{n \equiv a \bmod q} \Lambda(n) = \frac{x}{\varphi(q)} + O(xe^{-c_A\sqrt{\log x}}).$$