Some questions

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This is a list of questions that I am interested in. Many of them belong to other people and some are purely speculative. It may make sense to contact me before putting a lot of work in one of them as it may have been solved. Also feel free to contact me (abert (at) renyi.hu) if you do not understand a question or have comments.

I added a short text after some of the questions, but did not include who proved the particular results – that would also mean adding references, definitions etc. May change it later, if it is requested.

1 Groups acting on rooted trees

Question 1 (Sidki) *Can the odometer acting on the rooted binary tree be embedded in a free pro* 2*-group?*

It can be embedded in a free group.

Question 2 (many people) *Does the Baumslag-Pride theorem hold for pro p-groups? That is, is it true that if G is a pro p-group with $r+2$ generators and r relators, then G virtually surjects onto a non-Abelian free pro p-group?*

It is true in the discrete case.

Question 3 *Let G be a closed transitive subgroup of the automorphism group of a rooted tree. Is it true that every level stabilizer of G contains a fixedpoint-free element?*

Transitive and fixedpoint-free are understood with respect to the boundary of the tree. This is equivalent to the following old question of Jehne in field arithmetic: Do there exist fields $K \leq L \leq M$ with K is global, L/K finite separable and M/K infinite separable, such that all intermediate fields $L \leq M_0 \leq M$ of finite degree over K are Kronecker-conjugate to L? For pro-p groups it is true, though that is not so interesting from the number theoretic point of view.

Question 4 *Let Γ be a countable subgroup of $SL_2(\mathbb{Q}_p)$ that does not contain parabolic elements and let γ be a random element of $SL_2(\mathbb{Q}_p)$. Show that the group generated by Γ and γ does not contain any parabolic elements a.s.*

The analogous result is true for rooted tree actions.

Question 5 (A, Virag) *What is the length of the shortest law in the n times iterated wreath product of C_2 (the automorphism group of the rooted binary tree of length n)?*

Probably the shortest law is x^{2^n}.

2 Expanders

Question 6 (Benjamini) *Can the set of balls in an infinite Cayley graph form an expander family?*

Most likely not. I do not know why, but I really like this problem.

Question 7 *Suppose G and H are Cayley (or vertex transitive) expanders on the same number of vertices and you can almost match them in edge distance. Is it true that they are isomorphic?*

An analogous rigidity result holds for graphings of property (T) groups.

Question 8 *Suppose G is a finite vertex transitive expander. Is it true that every almost automorphism of G is almost an automorphism?*

An almost automorphism is a bijection of the vertex set that almost sends edges to edges. The statement is true for Cayley diagrams. Expansion is necessary.

Question 9 *Let Γ be a finitely generated group and let $\{H_n \mid n \geq 1\}$ be a property (τ) family of normal subgroups of finite index in Γ. Does the chain $\Gamma_n = \cap_{k=1}^{n} H_k$ have property (τ)?*

On graph theory language, this asks if the diagonal product of expander Cayley graphs keeps being expander on every connected component. It is not true if one omits normality.

Question 10 *Let Γ be a finitely presented group with a chain of normal subgroups of finite index and with trivial intersection. Assume that the chain has property (τ). Does Γ have a free subgroup?*

I hope to have a positive answer for this one – if that helps, assume that every index is a prime power. One would need an 'asymptotic ping-pong lemma' to start with.

Question 11 *Let G be an infinite d-regular Ramanujan graph (meaning, the Markov operator has the same spectral radius as of the d-regular tree). Is it true that for every C, the probability that a random walk on G ends touching a C-cycle tends to zero? That is, is it true that the random walk neighbourhood sampling on G converges to the d-regular tree?*

It is known that a unimodular random network that is d-regular, infinite and Ramanujan is the d-regular tree a.s.

3 Graph limits

Question 12 *Let (G_n) be a locally convergent sequence of bounded degree, integer labeled graphs. Is the normalized mod p rank of the adjacency matrix convergent?*

The same over $\mathbb{Q}$ is true and is equivalent to the Lück Approximation Theorem.

Question 13 (Elek) *For a graph sequence (G_n) let the edge measure be*

$$e((G_n)) = \liminf \frac{|E(G_n)|}{|V(G_n)|}$$

A graph sequence (H_n) defined on the same vertex set is equivalent to (G_n), if the bi-Lipschitz constants of the identity maps are bounded. Let the combinatorial cost of (G_n) be the infimum of edge measures of graph sequences equivalent to (G_n). Let (G_n) and (H_n) be graph sequences that locally converge to the same limit. Do they have the same combinatorial cost?

This is 'morally' equivalent to the Fixed Price problem, in the sense that once one is solved, the other is expected to be solved soon as well.

Question 14 (Lovasz) *By compactness, for any $\varepsilon > 0$ there exists $K > 0$ such that any finite graph can be approximated within the error ε by a finite graph of size K. Can we give an estimate for K in terms of ε?*

Question 15 *Let (G_n) be a locally convergent graph sequence and let μ_n be the probability distribution of the roots of the chromatic polynomial of G_n. Is it true that for all k, the limit*

$$\lim_{n\to\infty} \int_{\mathbb{C}} z^k \, d\mu_n$$

exists?

Question 16 *What can be the shape of the eigenvalue distribution of a finite d-regular graph? Find natural restrictions.*

One such non-trivial restriction is: if most of the measure is inside the Alon-Boppana bound, then we roughly know its shape.

Question 17 *Let G be a d-regular Cayley graph with spectral measure μ. Is μ a weak limit of spectral measures of finite d-regular graphs?*

A negative answer would solve the sofic problem negatively.

Question 18 (Bollobas) *Let G_n be a random d-regular graph on n points. Is there a limit of the independence ratio of G_n?*

It is known that for every n it is very concentrated. So, maybe the value its concentrated around changes with n?

Question 19 (Szegedy) *Let G_n be a random d-regular graph on n points. Does (G_n) converges to the (weak closure of) i.i.d.-s in local-global convergence?*

Question 20 *Is the i.i.d. action of the free group F_2 a local-global limit of finite actions of F_2?*

Question 21 (Bowen) *Let Γ be a Property (T) group and let G_n be a sofic approximation to G, that is, a sequence of finite graphs that converges to a Cayley graph of G. Surely, G_n does not have to be an expander family. But can we modify the sequence by an asymptotically vanishing amount (in the edit distance) to a sequence G'_n such that any subsequence of connected components of G'_n is an expander family? To put it another way, can one 'pullback' the ergodic decomposition of the invariant measure on the ultraproduct space?*

Question 22 *Is it true that any ergodic random unimodular network that is an infinite tree a.s. can be obtained as the limit of an expander family?*

4 Vertex transitive graphs

Question 23 *Let G be an infinite vertex transitive graph, let A be a finite set of vertices of G and b a vertex. Show that*

$$\sum_{x \in \partial A} d(b, x) \geq |A| .$$

where ∂A is the set of points of distance 1 from A.

Known for unimodular vertex transitive graphs.

Question 24 *One can define the first L^2 Betti number of an arbitrary vertex transitive graph G, from the expected degree of a free spanning forest of G. Does G and G^2 have the same first L^2 Betti number?*

When G is a Cayley graph for a group Γ, this number only depends on Γ, and not on the generating set, hence it is true there.

Question 25 *Can one show using free spanning forests any basic properties of the first L^2 Betti number? For instance, that it is multiplicative for taking a finite index subgroup.*

Question 26 (Lyons) *Let G be an infinite Cayley graph of a group that is not virtually cyclic. Show that there exists $p < 1$ such that the p-edge percolation of G has some infinite clusters.*

Not virtually cyclic is the same as not having linear word growth.

Question 27 *Let G be an infinite, connected Cayley graph. Then G has a perfect matching. Does G have an invariant random subgraph that is a perfect matching a.s.?*

Known if G is bipartite.

Question 28 (Babai) *Let G be an infinite Cayley (vertex transitive) graph. Does G have a spanning tree without leaves?*

Question 29 *Let G be an infinite Cayley (vertex transitive) graph. Does the set of dead ends in G have density 0?*

Question 30 *Are factors of i.i.d. on the 3-regular tree closed in the weak topology?*

Most likely not. In particular, look at the weak limit of majority functions on n-balls. Is that a factor of i.i.d.?

Question 31 *Let G be an infinite Cayley graph. Does there exist a G-invariant proper coloring with $\chi(G)$ colors?*

Question 32 *Let X be the space of k-regular Cayley graphs with the local convergence topology. It is easy to see that the set T of transient graphs is closed. Is the Green function (the expected number of returns to the identity) continuous on T?*

Question 33 *Let $k > 1$. Does there exist $C(k) < 1$ such that if G is a k-regular transient Cayley graph, then the probability of return is at most $C(k)$?*

Question 34 *Let Γ be a nonamenable group. Does $\{0,1\}^\Gamma$ factor onto $\{0,1,2\}^\Gamma$?*

5 Graphings

Question 35 *Can every d-regular graphing without multiple edges be properly edge colored by $d+1$ colors (measurable Vizing theorem)?*

Question 36 *Let G_n and H_n be graph sequences that converge to the same limit. Let T_n be a spanning tree in G_n that is convergent. Does there exist a spanning tree P_n in H_n that converges to the same limit as T_n?*

Question 37 *Let G be an expander (strongly ergodic) graphing of bounded degree that can be properly colored by C colors with arbitrarily small error. Can it be properly C-colored?*

Question 38 *Let G be an expander (strongly ergodic) graphing of bounded degree that weakly contains a finite graph H. Does G factor on H?*

True when H has two vertices.

6 Cost and rank gradient

Question 39 *Let Γ be a higher rank semisimple real lattice. Is the rank gradient of Γ zero?*

Known for non-uniform lattices.

Question 40 (Gaboriau) *Let A and B be countably infinite groups. Does $A \times B$ have fixed price 1?*

Known if A or B contains an infinite amenable (or arbitrarily large finite) subgroup.

Question 41 (Gaboriau) *Does every countable group have fixed price?*

Question 42 *Let Γ be a residually finite group with property (T). Does Γ have zero rank gradient?*

This is a baby version of the question, whether these groups have fixed price 1.

Question 43 *Let Γ act on X by p.m.p. maps and let H be subgroup of finite index in Γ. Is it true that*

$$\mathrm{cost}(H, X) - 1 = (\mathrm{cost}(\Gamma, X) - 1)\,|\Gamma : H| \text{ ?}$$

Already for i.i.d., this would show that every profinite free action of Γ has the same cost, so the rank gradient is independent of the chain.

Question 44 *Is it true that $\mathrm{cost}(\Gamma, X) = \mathrm{cost}(\Gamma, X^2)$ for free actions?*

By X^2 we mean the diagonal action of Γ on $X \times X$.

Question 45 *Let Γ be amenable, acting ergodically and essentially faithfully on X (every nontrivial element moves a set of positive measure). Is the groupoid cost of the action 1? If Γ is finitely presented, is it true for any infinite ergodic action?*

The second theorem is true for profinite actions.

Question 46 (Gaboriau) *Given a graphing of an equivalence relation, is there a subgraphing whose cost is almost the cost of the relation?*

What does this suggest for profinite actions?

7 Miscellaneous

Question 47 *Can a non-Abelian free group F have a nontrivial pseudocharacter that is invariant under* $\mathrm{Aut}(F)$*?*

Most likely not.

Question 48 *Are two independent random subsets of $\mathbb{Z}$ as metric spaces quasi-isometric a.s.? How about other Cayley graphs, like for $SL_3(\mathbb{Z})$?*

The answer is expected to be positive for $\mathbb{Z}$ and negative for $SL_3(\mathbb{Z})$.

Question 49 (Amit) *Is it true that if P is a finite p-group of order n and w is any word then the probability that w is satisfied is at least $1/n$?*

I tend to believe that its true. There are partial results, but nothing deep yet.

Question 50 *Let M be the set of measurable real functions. For $f \in M$ let $H_f, V_f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$H_f(x, y) = (x, y + f(x)) \text{ and } V_f(x, y) = (x + f(y), y).$$

Let $H = \{H_f \mid f \in M\}$ and $V = \{V_f \mid f \in M\}$. What is the group generated by H and V? Is there N such that every element of this group can be obtained as the product of N elements of H and V?

For the class of all functions over any Abelian group, it generates the full symmetric group in boundedly many steps. For continuous functions over $\mathbb{R}$, it is not bounded. For polynomials it is bounded.

Question 51 (A, Virag) *What is the length of the shortest nontrivial word which is satisfied in all groups of size 2^n?*

Probably it is 2^n. However, opposed to the similar question above, there are many candidates for such a law, using commutators.