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A new class of simple noetherian V-domains

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ABSTRACT

If R is any simple left noetherian, left hereditary, left V-domain, it is proven that the localization of R at any hereditary torsion theory τ that is cogenerated by a nonzero semisimple module, yields a ring of quotients R_τ with the same aforementioned properties. Examples of left V-domains R possessing (up to isomorphism) a single simple left R-module have been constructed by Cozzens (in 1970), and possessing infinitely many simple left R-modules, by Osofsky (in 1971). The methods developed in this paper can be used to construct V-domains possessing any prescribed number (finite or infinite) of simples. This answers in the affirmative a question posed by Cozzens and Faith in their book *Simple Noetherian rings* (Cambridge University Press, 1975).

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Contents

1. Introduction

In [4, pages 5–6] the authors note that all known examples of simple left noetherian domains correspond with one of two types:

Type 1. The ring of differential polynomials in a finite number of commuting derivations $\delta_1, \delta_2, \ldots, \delta_n$ over a simple left noetherian domain of characteristic 0.

Type 2. Suitable localizations (in the classical sense) of the skew polynomial ring $R = D[x, \sigma, \delta]$ where D is a division ring, σ an endomorphism on D, and δ a σ-derivation of D.

If F is a field of characteristic 0 and $\delta : F \to F$ is an outer derivation of F, then the ring $F[x, \delta]$ of differential polynomials in δ over F is known to be a simple (see [4, Theorem 3.2, pages 43–44]) left and right principal ideal domain (see [4, page 43]). Cozzens [3, Theorem 1.4, page 77] (see also [4, page 94]) has shown that for certain choices of field F and derivation δ, $F[x, \delta]$ is also a left V-domain (a domain for which all simple left modules are injective). The rings constructed by Cozzens, however, possess, up to isomorphism, only a single simple left module.

Working with rings of Type 2, Osofsky [10, Example (a), page 606] later showed that a field F with F-automorphism σ can be chosen such that the twisted (finite) Laurent polynomial ring $R = F[x, x^{-1}, \sigma]$ is a simple, left principal ideal, left V-domain that possesses infinitely many simple left R-modules. A self-contained account of twisted Laurent polynomial rings, culminating in a full description of Osofsky's example, is given in the final section of this paper.

The following two questions are posed in [4, Open problems (7) and (8), page 114]: do there exist left V-domains R with a prescribed (finite) number of simple left R-modules? and, do there exist simple noetherian domains that are neither of Type 1 nor Type 2? This paper answers both questions in the affirmative. We shall produce new examples of simple left noetherian, left hereditary domains by showing that the class of all such rings is closed under the formation of rings of quotients at any proper hereditary torsion theory (Corollary 9). If, moreover, the hereditary torsion theory is chosen to be one cogenerated by a direct sum M of injective simple modules, then the resulting ring of quotients will be a left V-domain possessing as many isomorphism classes of simples as there are components in the semisimple module M (Theorem 14).

Thus, starting with Osofsky's example of a simple, left principal ideal (and thus left noetherian, left hereditary) left V-domain admitting infinitely many simples, the method described above can be used to produce examples of left V-domains with any specified finite number of simples.

In a related investigation, Resco [11, Theorem, page 429] has characterized those division rings D with centre K such that $R = D \otimes_K K(x)$ is a left V-domain possessing, up to isomorphism, a unique simple left R-module. We point out that $R = D \otimes_K K(x)$ is the localization of the polynomial ring $D[x]$ with respect to the left (and right) denominator subset $K[x] \setminus \{0\}$ of $D[x]$, and is thus of Type 2.

Section 2 introduces the basic set, ring and module theoretic notational conventions used throughout the paper.

We shall assume familiarity on the part of the reader with standard ring theoretic notions including, but not limited to, simple[3] and semisimple rings and modules, essential extensions, module extensions, injective modules and the injective hull, noetherian and hereditary rings, the Ore condition and Ore domains. Definitions of the aforementioned notions can be found in a standard text on ring and module theory such as [1].

In contrast, we shall not assume the same familiarity with the methods of torsion theory which are fundamental to this paper. We have accordingly devoted Section 3 to a brief introduction to this topic. Many results are stated with a reference to one of the standard texts on the subject such as [5] or [12].

Section 4 contains the paper's main results.

Given the importance of Osofsky's example of a left V-domain with infinitely many simples (it constitutes the parent ring from which our class of examples is constructed) we have chosen to devote Section 5 of the paper to an exposition on twisted Laurent polynomial rings, this being the class of rings to which Osofsky's example belongs. The approach we shall adopt follows [4] rather than that of Osofsky in her original paper [10]. However, since the former is somewhat sparse in detail, we shall provide a thorough development of the necessary background material.

2. Ring and module theoretic preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If X is any nonempty set and $n \in \mathbb{N}$, then $X^{(n)}$ shall denote the cartesian product of n copies of X.

Throughout this paper R will denote an associative ring with identity. All modules are unital and R-Mod shall denote the category of all (unital) left R-modules. If $N, M \in R$-Mod we write $N \leqslant M$ if N is a submodule of M and $N \lesssim M$ if N is embeddable in M. If $A, B \in R$-Mod, then $\mathrm{Hom}_R(A, B)$ shall denote the abelian group of all R-homomorphisms from A to B.

[3] We shall assume a simple module is nonzero.

3. Torsion theoretic preliminaries

3.1. Hereditary torsion theories

A *hereditary torsion theory* on R-Mod is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of nonempty classes of left R-modules such that:

(T1) $\mathrm{Hom}_R(A, E(B)) = 0$ for all $A \in \mathcal{T}$ and $B \in \mathcal{F}$;
(T2) If $\mathrm{Hom}_R(X, E(B)) = 0$ for all $B \in \mathcal{F}$, then $X \in \mathcal{T}$;
(T3) If $\mathrm{Hom}_R(A, E(Y)) = 0$ for all $A \in \mathcal{T}$, then $Y \in \mathcal{F}$.

We shall denote by $\mathrm{Tors}\,_R R$ the collection[4] of all hereditary torsion theories on R-Mod.

Let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}\,_R R$. In this situation the class $\mathcal{T}$ is called the *torsion class* of τ and $\mathcal{F}$ the *torsion-free class* of τ. We call $M \in R$-Mod τ*-torsion* [resp. τ*-torsion-free*] if $M \in \mathcal{T}$ [resp. $M \in \mathcal{F}$]. A submodule L of M is said to be τ*-dense* [resp. τ*-pure*] in M if M/L is τ-torsion [resp. τ-torsion-free]. We shall frequently make use of the easily established fact that if M is τ-torsion-free, then every τ-dense submodule of M must be essential in M.

If $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}\,_R R$, then $\mathcal{T}$ is closed under submodules, homomorphic images, direct sums, and module extensions. Conversely, if $\mathcal{C}$ is any nonempty class of left R-modules that is closed under submodules, homomorphic images, direct sums, and module extensions, then there exists a unique $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}\,_R R$ such that $\mathcal{T} = \mathcal{C}$ [5, Proposition 1.7, page 5].

Dually, a nonempty class $\mathcal{D}$ of left R-modules is closed under submodules, injective hulls, direct products, and module extensions iff $\mathcal{D}$ is the τ-torsion-free class of some $\tau \in \mathrm{Tors}\,_R R$ [5, Propositions 1.10, page 7 and 1.12, page 8].

Let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}\,_R R$. Since $\mathcal{T}$ is closed under homomorphic images and direct sums, each $M \in R$-Mod has a largest submodule, denoted $t_\tau(M)$, that belongs to $\mathcal{T}$. We call $t_\tau(M)$ the τ*-torsion submodule* of M. Observe that M is τ-torsion-free iff $t_\tau(M) = 0$. A consequence of $\mathcal{T}$ being closed under module extensions, is that $t_\tau(M)$ is always τ-pure in M, that is to say, $t_\tau(M/t_\tau(M)) = 0$ (see (R3) below).

Dually, since $\mathcal{F}$ is closed under submodules and direct products, M has a smallest τ-pure submodule L, say, which must coincide with $t_\tau(M)$ since $t_\tau(M)/L$ is both τ-torsion and τ-torsion-free.

For each $\tau \in \mathrm{Tors}\,_R R$, the map $t_\tau(\_)$ which assigns to each $M \in R$-Mod its τ-torsion submodule $t_\tau(M)$, is an instance of a *left exact radical* functor, for it possesses the following three defining properties [5, Proposition 23.1, page 213]:

(R1) $f[t_\tau(M)] \subseteq t_\tau(N)$ for all $M, N \in R$-Mod and $f \in \mathrm{Hom}_R(M, N)$;

[4] $\mathrm{Tors}\,_R R$ can, in fact, be regarded as a set for there is a bijective correspondence between members of $\mathrm{Tors}\,_R R$ and certain families of left ideals of the ring R called Gabriel topologies.

(R2) $t_\tau(L) = L \cap t_\tau(M)$ for all $L \leqslant M \in R$-Mod;
(R3) $t_\tau(M/t_\tau(M)) = 0$ for all $M \in R$-Mod.

The set $\mathrm{Tors}\,_R R$ admits a natural partial ordering: if $\tau = (\mathcal{T}, \mathcal{F})$ and $\sigma = (\mathcal{T}', \mathcal{F}')$ are members of $\mathrm{Tors}\,_R R$, then

$$\tau \leqslant \sigma \text{ iff } \mathcal{T} \subseteq \mathcal{T}', \text{ or equivalently}, \mathcal{F} \supseteq \mathcal{F}' \\ \text{iff } t_\tau(M) \subseteq t_\sigma(M) \text{ for all } M \in R\text{-Mod}.$$

Let $\mathcal{C}$ be a nonempty class of left R-modules. Define:

$$\mathcal{F} \stackrel{\text{def}}{=} \{B \in R\text{-Mod} : \mathrm{Hom}_R(A, E(B)) = 0 \ \forall A \in \mathcal{C}\}, \text{ and}$$
$$\mathcal{T} \stackrel{\text{def}}{=} \{A \in R\text{-Mod} : \mathrm{Hom}_R(A, E(B)) = 0 \ \forall B \in \mathcal{F}\}.$$

It is easily shown that $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on R-Mod, and is the smallest member of $\mathrm{Tors}\,_R R$ whose torsion class contains $\mathcal{C}$ (see [12, page 139]). In this situation we call τ the *hereditary torsion theory generated by* $\mathcal{C}$, and denote it $\xi(\mathcal{C})$.

Dually, if:

$$\mathcal{T} \stackrel{\text{def}}{=} \{A \in R\text{-Mod} : \mathrm{Hom}_R(A, E(B)) = 0 \ \forall B \in \mathcal{C}\}, \text{ and}$$
$$\mathcal{F} \stackrel{\text{def}}{=} \{B \in R\text{-Mod} : \mathrm{Hom}_R(A, E(B)) = 0 \ \forall A \in \mathcal{T}\}, \text{ then}$$

$\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on R-Mod, and is the largest member of $\mathrm{Tors}\,_R R$ whose torsion-free class contains $\mathcal{C}$ (see [12, page 139]). We call such a τ the *hereditary torsion theory cogenerated by* $\mathcal{C}$, and denote it $\chi(\mathcal{C})$. If $\mathcal{C} = \{M\}$ is a singleton, we write $\xi(M)$ [resp. $\chi(M)$] in place of $\xi(\{M\})$ [resp. $\chi(\{M\})$].

It is shown in [12, Proposition 3.7, page 142] that every $\tau \in \mathrm{Tors}\,_R R$ is cogenerated by an injective left R-module, namely

$$E = \prod\{E(R/K) : K \text{ is } \tau\text{-pure in } {}_R R\}.$$

3.2. The localization functor

Let $\tau \in \mathrm{Tors}\,_R R$. We call $M \in R$-Mod *τ-injective* if, given any $L \in R$-Mod, every R-homomorphism from a τ-dense submodule of L to M extends to an R-homomorphism from L to M.

For each $\tau \in \mathrm{Tors}\,_R R$ and $M \in R$-Mod there is a left R-module $Q_\tau(M)$, called the *module of quotients of M at τ*, and an R-homomorphism $\lambda_\tau^M : M \to Q_\tau(M)$ that satisfy the following:

(Q1) $Q_\tau(M)$ is τ-torsion-free and τ-injective;

(Q2) $\operatorname{Ker}\lambda_\tau^M$ and $\operatorname{Coker}\lambda_\tau^M$ are τ-torsion. (In fact, in the presence of (Q1), the requirement that $\operatorname{Ker}\lambda_\tau^M$ be τ-torsion is equivalent to $\operatorname{Ker}\lambda_\tau^M = t_\tau(M)$.)

Note that since $\operatorname{Im}\lambda_\tau^M$ is a τ-dense submodule of the τ-torsion-free module $Q_\tau(M)$, $\operatorname{Im}\lambda_\tau^M$ must be essential in $Q_\tau(M)$.

If $M, N \in R$-Mod and $f \in \operatorname{Hom}_R(M, N)$, then there exists a unique R-homomorphism, denoted $Q_\tau(f)$, which makes the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\lambda_\tau^M"'] & N \arrow[d, "\lambda_\tau^N"] \\ Q_\tau(M) \arrow[r, dotted, "Q_\tau(f)"] & Q_\tau(N) \end{tikzcd} \tag{1}$$

commute.

For each $M \in R$-Mod, the pair of properties (Q1) and (Q2) characterizes $Q_\tau(M)$ to within isomorphism, as the following result shows.

Proposition 1 *([5, Proposition 26.9, page 245]). Let $\tau \in \operatorname{Tors}_R R$. The following statements are equivalent for $M, N \in R$-Mod and $f \in \operatorname{Hom}_R(M, N)$:*

(a) *N is τ-torsion-free and τ-injective, and $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are τ-torsion;*
(b) *There exists a unique R-homomorphism $\gamma : N \to Q_\tau(M)$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\lambda_\tau^M"'] & N \arrow[dl, dotted, "\gamma"] \\ Q_\tau(M) & \end{tikzcd}$$

commutes. Moreover, γ is an isomorphism.

For each $\tau \in \operatorname{Tors}_R R$ the *quotient category of* R-Mod *at* τ, denoted (R, τ)-Mod, is defined to be the full subcategory of R-Mod comprising all τ-torsion-free, τ-injective left R-modules. It is known that (R, τ)-Mod is a Grothendieck category. Indeed, by the Popescu-Gabriel Theorem (see [12, Theorem 4.1, page 220]) every Grothendieck category is equivalent to (R, τ)-Mod for some ring and hereditary torsion theory τ on R-Mod.

Let $\tau \in \operatorname{Tors}_R R$ and $M \in R$-Mod. Suppose N and $\overline{N}$ are submodules of M such that $N \subseteq \overline{N}$ and $t_\tau(M/N) = \overline{N}/N$. In this situation we call $\overline{N}$ the *τ-purification of* N

in M. Observe that N is τ-dense in $\overline{N}$ and $\overline{N}$ τ-pure in M (by (R3)). The map $N \mapsto \overline{N}$ constitutes a closure operator on the lattice of submodules of M, whose range comprises the set of all τ-pure submodules of M [12, page 207].

We shall make frequent use of the fact that the complete lattice of τ-pure submodules of any $M \in R$-Mod is isomorphic to the complete lattice of subobjects of $Q_\tau(M)$ in the quotient category (R, τ)-Mod [12, Corollary 4.4, page 208]. This has the consequence that any chain condition on the submodule lattice of M is inherited by the subobject lattice of $Q_\tau(M)$ in (R, τ)-Mod. In particular, if M is noetherian, then $Q_\tau(M)$ is a noetherian object in (R, τ)-Mod [12, Corollary 2.2, page 264].

If $\tau \in \mathrm{Tors}\,_R R$, then a nonzero τ-torsion-free left R-module containing no proper nonzero τ-pure submodule (this is equivalent to the requirement that every nonzero submodule be τ-dense) is called *τ-cocritical*. Observe that the simple objects in the quotient category (R, τ)-Mod are precisely the τ-injective, τ-cocritical left R-modules.

The following result shows that the injective objects in (R, τ)-Mod are precisely those objects in (R, τ)-Mod that are injective as left R-modules.

Proposition 2 *([12, Proposition 1.7, page 215]). Let $\tau \in \mathrm{Tors}\,_R R$. The following statements are equivalent for $M \in (R, \tau)$-Mod:*

(a) *M is injective as an object in (R, τ)-Mod;*
(b) *M is injective as a left R-module.*

The uniqueness of the map $Q_\tau(f)$ in Diagram (1) allows us to interpret Q_τ as an additive functor from R-Mod to (R, τ)-Mod which we call the *localization functor at τ*. Note that with this interpretation, Diagram (1) can be seen as defining a natural transformation λ_τ from the identity functor on R-Mod to Q_τ.

For each $\tau \in \mathrm{Tors}\,_R R$ the ring

$$R_\tau \overset{\mathrm{def}}{=} \mathrm{End}_R(Q_\tau(_R R))$$

is called the *ring of quotients of R at τ*.

Taking $M = N = {}_R R$ in Diagram (1) and noting that the functor Q_τ is additive, the map $f \mapsto Q_\tau(f)$ constitutes a ring homomorphism from $\mathrm{End}_R R$ to R_τ. Composing this ring homomorphism with the canonical ring isomorphism from R to $\mathrm{End}_R R$ yields a ring homomorphism

$$\varphi_\tau : R \to R_\tau.$$

This ring homomorphism induces a canonical embedding of R_τ-Mod into R-Mod.

Each τ-torsion-free, τ-injective left R-module admits a left R_τ-module structure that is compatible with its R-module structure [5, Proposition 26.33, page 256]. We thus have the containments

$$(R,\tau)\text{-Mod} \subseteq R_\tau\text{-Mod} \subseteq R\text{-Mod}.$$

Consider the following diagram in R-Mod

$$\begin{tikzcd}
{}_RR \arrow[r, "\text{canon.}", "\text{iso}"'] \arrow[d, "\lambda_\tau^R"'] & \mathrm{End}_R R \arrow[r] \arrow[d, "\mathrm{Hom}_R({}_RR,\lambda_\tau^R)"'] & R_\tau \arrow[dl, "\text{iso}"', "\mathrm{Hom}_R(\lambda_\tau^R,Q_\tau({}_RR))"] \\
Q_\tau({}_RR) \arrow[r, "\text{canon.}", "\text{iso}"'] & \mathrm{Hom}_R({}_RR, Q_\tau({}_RR)) &
\end{tikzcd}$$

Observe that the composition of maps in the top row of the above diagram corresponds with φ_τ, which in the context of the above diagram, is an R-homomorphism from ${}_RR$ to ${}_R(R_\tau)$. Notice too that the natural map $\mathrm{Hom}_R(\lambda_\tau^R, Q_\tau({}_RR))$ is indeed an isomorphism since $Q_\tau({}_RR)$ is τ-torsion-free and τ-injective. We thus obtain the commutative diagram

$$\begin{tikzcd}
{}_RR \arrow[r, "\varphi_\tau"] \arrow[d, "\lambda_\tau^R"'] & {}_R(R_\tau) \arrow[dl, "\text{iso}"] \\
Q_\tau({}_RR) &
\end{tikzcd} \tag{2}$$

in R-Mod. Thus ${}_R(R_\tau)$ and $Q_\tau({}_RR)$ are isomorphic as left R-modules (see [5, Proposition 26.23, page 252]).

We shall have need for the following result.

Proposition 3 *([5, Proposition 26.34, page 257]). Let $\tau, \sigma \in \mathrm{Tors}\,{}_RR$ with $\tau \leqslant \sigma$. There exists a unique ring homomorphism $\gamma : R_\tau \to R_\sigma$ which makes the following diagram of rings and ring homomorphisms commute*

$$\begin{tikzcd}
& R \arrow[dl, "\varphi_\tau"'] \arrow[dr, "\varphi_\sigma"] & \\
R_\tau \arrow[rr, dotted, "\gamma"] & & R_\sigma
\end{tikzcd}$$

3.3. Perfect torsion theories

Let $\tau \in \mathrm{Tors}\,{}_RR$. Denote by ζ_τ the natural transformation from the identity functor on R-Mod to the change of rings functor $R_\tau \otimes_R \_\!\_$. Recall that for each $M \in R$-Mod, $\zeta_\tau^M : M \to R_\tau \otimes_R M$ is defined by

$$\zeta_\tau^M(u) \overset{\text{def}}{=} 1_{R_\tau} \otimes u \ \forall u \in M.$$

The transformation λ_τ factors through ζ_τ in the sense that there is a natural transformation η_τ from $R_\tau \otimes_R \_\_$ to Q_τ such that for each $M \in R$-Mod, the diagram

$$\begin{tikzcd} & M \arrow[dl, "\zeta_\tau^M"'] \arrow[dr, "\lambda_\tau^M"] & \\ R_\tau \otimes_R M \arrow[rr, "\eta_\tau^M"] & & Q_\tau(M) \end{tikzcd}$$

commutes [5, page 265].

We call τ *perfect* if η_τ is a natural equivalence, which is to say the functors $R_\tau \otimes_R \_\_$ and Q_τ are naturally equivalent. In this situation the subcategories R_τ-Mod and (R, τ)-Mod of R-Mod coincide. We thus have

$$(R, \tau)\text{-Mod} = R_\tau\text{-Mod} \subseteq R\text{-Mod}.$$

It is known that $\tau \in \operatorname{Tors}{}_R R$ is perfect iff the functor Q_τ is exact (in general, Q_τ is only left exact [5, Proposition 26.5, page 243]) and there exists a set $\mathcal{A}$ of finitely generated left ideals of R such that the class of τ-torsion modules is generated by the family $\{R/A : A \in \mathcal{A}\}$ [5, Proposition 45.1, page 416].

We shall need the following.

Proposition 4 *([5, Corollary 45.6, page 418]). The following statements are equivalent for a perfect $\tau \in \operatorname{Tors}{}_R R$:*

(a) *$Q_\tau({}_R R)$ is a noetherian object in (R, τ)-Mod, that is to say, the lattice of subobjects of $Q_\tau({}_R R)$ in (R, τ)-Mod satisfies the ACC;*
(b) *The ring R_τ is left noetherian.*

4. The main results

Our first objective shall be to prove that if R is a simple ring, then so is R_τ for all proper $\tau \in \operatorname{Tors}{}_R R$. We provide a proof of this routine fact in the absence of a suitable reference. A preparatory lemma is needed.

Let R and T be rings and $\varphi : R \to T$ a ring monomorphism. We shall call φ *left essential* if the image of R in T is essential as a left R-submodule of ${}_R T$.

We omit the proof of the following easy lemma.

Lemma 5. *Let R and T be rings and $\varphi : R \to T$ a left essential ring monomorphism. If R is simple then so is T.*

Proposition 6. *If R is a simple ring, then so is R_τ for all proper $\tau \in \mathrm{Tors}\,_R R$.*

Proof. Consider the ring homomorphism $\varphi_\tau : R \to R_\tau$. Since R is simple, φ_τ must be monic. It suffices, in light of the previous lemma, to show that φ_τ is left essential. With reference to Diagram (2), it is easily seen that $\mathrm{Im}\,\varphi_\tau$ must be essential in $_R(R_\tau)$ because $\mathrm{Im}\,\lambda_\tau^R$ is essential in $Q_\tau(_R R)$. We conclude that φ_τ is a left essential ring monomorphism, as required. □

Our next objective is to show that the left Ore domain property is passed from a ring R to its ring of quotients R_τ at any proper $\tau \in \mathrm{Tors}\,_R R$.

Let S be the set of all regular elements of a ring R. Recall that R is said to be a *left Ore ring* if $Sr \cap Rs \neq \emptyset$ for all $r \in R$ and $s \in S$.

If R is any left Ore ring, then the *Classical torsion theory* μ_{cl} is defined by

$$t_{\mu_{\mathrm{cl}}}(M) \overset{\mathrm{def}}{=} \{x \in M : sx = 0 \text{ for some } s \in S\} \ \forall M \in R\text{-Mod}.$$

The ring of quotients of R at μ_{cl} is the familiar *Classical left ring of quotients* of R [5, Example 26.25, page 253].

Now suppose R is a left Ore domain. For such a ring R, $S = R\backslash\{0\}$ and $R_{\mu_{\mathrm{cl}}}$ is a division ring. We claim that if τ is any proper member of $\mathrm{Tors}\,_R R$, then $\tau \leqslant \mu_{\mathrm{cl}}$. Indeed, let $M \in R$-Mod and $x \in t_\tau(M)$. Note that x cannot have trivial left annihilator since this would imply $_R R \lesssim t_\tau(M)$, contradicting the fact that τ is proper. Hence $sx = 0$ for some $s \in R\backslash\{0\}$, so $x \in t_{\mu_{\mathrm{cl}}}(M)$. This shows that $t_\tau(M) \subseteq t_{\mu_{\mathrm{cl}}}(M)$ establishing our claim.

Proposition 7. *If R is a left Ore domain, then so is R_τ for all proper $\tau \in \mathrm{Tors}\,_R R$.*

Proof. Suppose R is a left Ore domain with μ_{cl} the Classical torsion theory on R-Mod. Take any proper $\tau \in \mathrm{Tors}\,_R R$. As noted above, $\tau \leqslant \mu_{\mathrm{cl}}$. It follows from Proposition 3 that there is a ring homomorphism $\gamma : R_\tau \to R_{\mu_{\mathrm{cl}}}$ which makes the diagram

$$\begin{tikzcd} & R \arrow[dl, "\varphi_\tau"'] \arrow[dr, "\varphi_{\mu_{\mathrm{cl}}}"] & \\ R_\tau \arrow[rr, dotted, "\gamma"] & & R_{\mu_{\mathrm{cl}}} \end{tikzcd}$$

commute. Since τ is proper, every $r \in t_\tau(_R R)$ must have a nonzero left annihilator, but since R is a domain, this is only possible if $r = 0$. Thus $t_\tau(_R R) = \mathrm{Ker}\,\varphi_\tau = 0$, whence φ_τ is monic. Similarly, $\varphi_{\mu_{\mathrm{cl}}}$ is also monic.

An argument similar to that used in the proof of Proposition 6, shows that φ_τ is a left essential monomorphism. Inasmuch as

$$0 = \operatorname{Ker} \varphi_{\mu_{\rm cl}} = \varphi_\tau^{-1}[\operatorname{Ker} \gamma],$$

we must have $\operatorname{Im} \varphi_\tau \cap \operatorname{Ker} \gamma = 0$, whence $\operatorname{Ker} \gamma = 0$, so γ is monic. Since $R_{\mu_{\rm cl}}$ is a Classical left ring of quotients for R and $R \subseteq R_\tau \subseteq R_{\mu_{\rm cl}}$, $R_{\mu_{\rm cl}}$ is also a Classical left ring of quotients for R_τ. Thus R_τ is a left Ore domain. □

Theorem 8. *Let R be a left noetherian left hereditary ring. Then:*

(a) *Every $\tau \in \operatorname{Tors}{}_R R$ is perfect.*
(b) *R_τ is left noetherian, left hereditary for every $\tau \in \operatorname{Tors}{}_R R$.*

Proof. (a) is [12, Corollary 3.6, page 232].

(b) Since ${}_R R$ is noetherian, $Q_\tau({}_R R)$ must be a noetherian object in (R, τ)-Mod. Since τ is perfect by (a), Proposition 4 implies that R_τ must be left noetherian.

[12, Proposition 3.10, page 232] asserts that if τ is perfect, then the left global dimension (l.gl.dim) of R_τ is less than or equal to that of R. This clearly has the consequence that if R is left hereditary, that is to say, $\operatorname{l.gl.dim} R \leqslant 1$, then $\operatorname{l.gl.dim} R_\tau \leqslant 1$, i.e., R_τ is left hereditary. □

Inasmuch as every left noetherian domain is left Ore (see [12, Proposition 1.7, page 53]), the following corollary is an immediate consequence of Propositions 6 and 7 and the previous theorem. Observe that Proposition 7, in particular, guarantees that the domain property is passed from R to R_τ.

Corollary 9. *If R is a simple left noetherian, left hereditary domain, then so is R_τ for every proper $\tau \in \operatorname{Tors}{}_R R$.*

The next main result shows that given any ring R, and any cardinal $\mathfrak{m}$ not exceeding the cardinality of a representative set of simples in R-Mod, a suitable quotient category (R, τ)-Mod can be chosen that admits, up to isomorphism, precisely $\mathfrak{m}$ simple objects. We require a preliminary lemma.

Lemma 10. *Let $\tau \in \operatorname{Tors}{}_R R$ with E an injective cogenerator for τ so that $\tau = \chi(E)$. If U is any simple object in the quotient category (R, τ)-Mod, then U embeds as a left R-module in E.*

Proof. Let U be a simple object in (R, τ)-Mod. Thus U is a τ-injective, τ-cocritical left R-module. Since U is not τ-torsion, $\operatorname{Hom}_R(U, E) \neq 0$. Pick $0 \neq f \in \operatorname{Hom}_R(U, E)$. Since $U/\operatorname{Ker} f \lesssim E$ is τ-torsion-free, $\operatorname{Ker} f$ is τ-pure in U. Since U is τ-cocritical it cannot contain a proper nonzero τ-pure submodule. Thus $\operatorname{Ker} f = 0$ or $\operatorname{Ker} f = U$. The latter is not possible since $f \neq 0$. It follows that $\operatorname{Ker} f = 0$, whence $U \lesssim E$. □

Proposition 11. *Let $\mathcal{S}$ be an arbitrary nonempty family of nonisomorphic simple left R-modules. Put $E = E(\bigoplus \mathcal{S})$ and $\tau = \chi(E)$. Then the map $S \mapsto Q_\tau(S)$ constitutes a bijection from $\mathcal{S}$ to a representative set of simple objects in the quotient category* (R,τ)-Mod*.*

Proof. Take $S \in \mathcal{S}$. Since E is τ-torsion-free and $S \lesssim E$, S is also τ-torsion-free. Since S is a simple left R-module, it contains no proper nonzero τ-pure submodule (indeed, it contains no proper nonzero submodules of any description). It follows that $Q_\tau(S)$ is a simple object in (R,τ)-Mod.

Now take $S' \in \mathcal{S}$ with $S' \neq S$ so that S and S' are nonisomorphic simples. Since the left R-module embeddings $\lambda_\tau^S : S \to Q_\tau(S)$ and $\lambda_\tau^{S'} : S' \to Q_\tau(S')$ are τ-dense, it is easily seen that $\mathrm{Hom}_R(Q_\tau(S), Q_\tau(S')) = 0$ from which we infer that $Q_\tau(S)$ and $Q_\tau(S')$ are nonisomorphic objects in (R,τ)-Mod.

It remains to show that every simple object in (R,τ)-Mod is isomorphic to $Q_\tau(S)$ for some $S \in \mathcal{S}$. To this end let U be an arbitrary simple object in (R,τ)-Mod. By the previous lemma, U embeds as a left R-module in E. Since $\bigoplus \mathcal{S}$ is essential in E, we must have $S \lesssim U$ for some $S \in \mathcal{S}$. Inasmuch as U is τ-cocritical, the embedding of S in U is τ-dense. Furthermore, U is τ-torsion-free and τ-injective because it is an object in (R,τ)-Mod. It follows from Proposition 1((a)⇒(b)) that $U \cong Q_\tau(S)$. □

Recall that a ring R is said to be a left *V-ring* if every simple left R-module is injective. More generally, we shall call a Grothendieck category $\mathcal{C}$ a *V-category* if every simple object in $\mathcal{C}$ is injective in $\mathcal{C}$. Note that R will be a left V-ring precisely if R-Mod is a V-category.

Remark 12. *Note that some Grothendieck categories are vacuously V-categories for the reason that they possess no simple objects.*

Theorem 13. *Let $\mathcal{S}$ be an arbitrary nonempty family of nonisomorphic injective simple left R-modules. Put $E = E(\bigoplus \mathcal{S})$ and $\tau = \chi(E)$. Then $\mathcal{S}$ is a representative set of simple objects in the category* (R,τ)-Mod *and* (R,τ)-Mod *is a V-category.*

Proof. Take $S \in \mathcal{S}$. Observe that S is τ-torsion-free (because E is τ-torsion-free and $S \lesssim E$) and τ-injective (because S is injective). Taking $M = N = S$ and f to be the identity map on S in Proposition 1, we see that $\gamma = \lambda_\tau^S : S \to Q_\tau(S)$ is an R-module isomorphism. Since $S \cong Q_\tau(S)$ for each $S \in \mathcal{S}$, $\mathcal{S}$ is a representative set of simple objects in (R,τ)-Mod by Proposition 11.

By Proposition 2, each member of $\mathcal{S}$ is an injective object in (R,τ)-Mod. Thus (R,τ)-Mod is a V-category. □

Question. *It would be interesting to know what Grothendieck V-categories arise in the above way, that is, as the quotient category of R-*Mod *at some hereditary torsion theory cogenerated by a direct sum of injective simple left R-modules. Is there perhaps a representation theorem along the lines of the Popescu-Gabriel Theorem [12, Theorem 4.1, page 220]?*

We are finally in a position to prove our main theorem.

Theorem 14. *Let R be a simple left noetherian, left hereditary domain. Let $\mathcal{S}$ be an arbitrary nonempty family of nonisomorphic injective simple left R-modules. Put $E = E(\bigoplus \mathcal{S})$ and $\tau = \chi(E)$. Then the ring of quotients R_τ of R at τ is a simple left noetherian, left hereditary, left V-domain with $\mathcal{S}$ a representative family of simple left R_τ-modules.*

Proof. That R_τ is a simple left noetherian, left hereditary domain is a consequence of Corollary 9.

It follows from Theorem 13 that (R, τ)-Mod is a V-category and $\mathcal{S}$ a representative set of simple objects in (R, τ)-Mod. By Theorem 8(a), τ is perfect. Hence the quotient category (R, τ)-Mod coincides with the module category R_τ-Mod. Thus R_τ is a left V-domain with $\mathcal{S}$ a representative family of simple left R_τ-modules. □

Remark 15. *If, in Theorem 14, R is chosen to be Osofsky's example of a simple, left principal ideal (and thus left noetherian, left hereditary), left V-domain possessing infinitely many simple left R-modules, then given any finite cardinal $\mathfrak{m}$, there will exist, by Theorem 14, a hereditary torsion theory τ on R-*Mod *such that the ring of quotients R_τ of R at τ, is a simple left noetherian, left hereditary, left V-domain with a representative set of simple left R_τ-modules, of cardinality $\mathfrak{m}$. This answers [4, Question 7, page 114] in the affirmative.*

Later, we shall see as a consequence of Theorem 36 that rings of the type constructed by Osofsky may be produced admitting arbitrarily large representative sets of simples. This fact, viewed in conjunction with Theorem 14, allows us to infer the existence of, for each cardinal $\mathfrak{m}$ (not necessarily finite), a simple left noetherian, left hereditary, left V-domain R with a representative set of simple left R-modules of cardinality precisely $\mathfrak{m}$.

5. Twisted Laurent polynomial rings

Osofsky's example of a simple left V-domain with infinitely many simples is a twisted Laurent polynomial ring over a carefully chosen field F and field automorphism σ on F. Given its importance as the starting point for the torsion theoretic construction detailed in the first four sections of this paper, we provide a self-contained account of Osofsky's construction, starting with a brief exposition on twisted Laurent polynomial rings.

5.1. Background theory

For the remainder of this paper:

- ▷ F shall denote a field; and
- ▷ σ shall denote a field automorphism of F of infinite order.

Recall that the *twisted Laurent polynomial ring* $F[x, x^{-1}, \sigma]$ comprises all formal sums of the form

$$\sum_{i=-m}^{k} a_i x^i \tag{3}$$

where m and k are nonnegative integers and $a_i \in F$ for $-m \leqslant i \leqslant k$.

Addition of formal sums is natural while multiplication is induced by the rule

$$xa = \sigma(a)x \ \forall a \in F.$$

The *twisted polynomial ring* $F[x, \sigma]$ is the subring of $F[x, x^{-1}, \sigma]$ comprising sums of the form

$$\sum_{i=0}^{k} a_i x^i.$$

The proofs of properties listed in the next three theorems that are not routine may be found in the early pages of standard texts such as [6], [7] and [9].

Theorem 16. *Let $T = F[x, \sigma]$. Then:*

(a) *T admits a degree function ∂ defined in the usual manner.*
(b) *T satisfies a left [resp. right] Division Algorithm: if $s, t \in T$ with $t \neq 0$, then there exist unique $q, r \in T$ such that*

$$s = qt + r \ \text{[resp. } s = tq + r\text{]}$$

where $r = 0$ or $\partial r < \partial t$.
(c) *T is a left and right principal ideal domain.*
(d) *Every proper nonzero ideal of T has the form Tx^m for some $m \in \mathbb{N}$.*

We shall say that $s = \sum_{i=0}^{n} b_i x^i \in F[x, \sigma]$ has *standard form* if $b_0 \neq 0$ and $b_n = 1$.

Theorem 17. *Let $R = F[x, x^{-1}, \sigma]$. Then:*

(a) *Every nonzero principal left [resp. right] ideal of R has the form Rr [resp. rR] for some $r \in F[x, \sigma]$ in standard form.*
(b) *R is a left and right principal ideal domain.*
(c) *If $s \in F[x, \sigma]$ has nonzero constant term and $\partial s = n \geqslant 1$, then $\dim_F(R/Rs) = n$ with $\{x^i + Rs : 0 \leqslant i \leqslant n-1\}$ a basis for ${}_F(R/Rs)$.*
(d) *R is a simple ring.*

We shall make frequent use of the fact that if $R = F[x, x^{-1}, \sigma]$, then every left R-module is canonically a left F-module for the reason that F is a subring of R. Furthermore, if $M, N \in R$-Mod, then every R-homomorphism $f : M \to N$ is also an F-homomorphism, so that $\mathrm{Hom}_R(M, N) \subseteq \mathrm{Hom}_F(M, N)$.

Remark 18. *If $R = F[x, x^{-1}, \sigma]$, then F is a left R-module with action defined by*

$$rc \stackrel{\text{def}}{=} \sum_{i=-m}^{k} a_i \sigma^i(c) \ \forall c \in F, \ \forall r = \sum_{i=-m}^{k} a_i x^i \in R.$$

Theorem 19. *Let $R = F[x, x^{-1}, \sigma]$. Suppose M, M', N, N' are left R-modules, $f \in \mathrm{Hom}_R(M, M')$ and $g \in \mathrm{Hom}_R(N, N')$. Then:*

(a) $\mathrm{Hom}_F(M, N)$ *is a left R-module with scalar multiplication defined by*

$$(r\varphi)(u) \stackrel{\text{def}}{=} \sum_{i=-m}^{k} a_i x^i \varphi(x^{-i} u) \ \forall r = \sum_{i=-m}^{k} a_i x^i \in R, \ \forall \varphi \in \mathrm{Hom}_F(M, N), \ \forall u \in M.$$

Moreover, the canonical F-homomorphisms $\mathrm{Hom}_F(f, N) : \mathrm{Hom}_F(M', N) \to \mathrm{Hom}_F(M, N)$ *and* $\mathrm{Hom}_F(M, g) : \mathrm{Hom}_F(M, N) \to \mathrm{Hom}_F(M, N')$ *are R-homomorphisms.*

(b) *$M \otimes_F N$ is a left R-module with scalar multiplication defined by*

$$r(u \otimes v) \stackrel{\text{def}}{=} \sum_{i=-m}^{k} a_i (x^i u \otimes x^i v) \ \forall r = \sum_{i=-m}^{k} a_i x^i \in R, \ \forall u \in M, \ \forall v \in N.$$

Moreover, the canonical F-homomorphisms $f \otimes_F N : M \otimes_F N \to M' \otimes_F N$ and $M \otimes_F g : M \otimes_F N \to M \otimes_F N'$ are R-homomorphisms.

Let $R = F[x, x^{-1}, \sigma]$. Take $M, N, L \in R$-Mod, $u \in M$ and $\varphi \in \mathrm{Hom}_F(M \otimes_F N, L)$. It is easily checked that the map $\varphi_u : N \to L$ defined by

$$\varphi_u(v) \stackrel{\text{def}}{=} \varphi(u \otimes v) \ \forall v \in N$$

is an F-homomorphism. The canonical map

$$\Theta_{(M,N,L)} : \mathrm{Hom}_F(M \otimes_F N, L) \to \mathrm{Hom}_F(M, \mathrm{Hom}_F(N, L))$$

defined by

$$(\Theta_{(M,N,L)}(\varphi))(u) \stackrel{\text{def}}{=} \varphi_u \ \forall \varphi \in \mathrm{Hom}_F(M \otimes_F N, L), \forall u \in M$$

is known to be an isomorphism of abelian groups [2, Proposition 5.2′, page 28].

Proof of the following result is left to the reader.

Theorem 20. *Let $R = F[x, x^{-1}, \sigma]$. For each $M, N, L \in R$-Mod, the canonical isomorphism $\Theta_{(M,N,L)}$ restricts to an isomorphism from $\mathrm{Hom}_R(M \otimes_F N, L)$ onto $\mathrm{Hom}_R(M, \mathrm{Hom}_F(N, L))$ that is functorial in each of its arguments M, N and L.*

Theorem 21. *Let $R = F[x, x^{-1}, \sigma]$. If L is an injective left R-module, then so is $\mathrm{Hom}_F(N, L)$ for all left R-modules N.*

Proof. Let $N, L \in R$-Mod with L injective. We claim that $\_ \otimes_F N$ is an exact functor from R-Mod to R-Mod. Indeed, suppose

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is any short exact sequence in R-Mod. Clearly, this sequence will also be exact in F-Mod. Since F is a field, N is flat in F-Mod, so the sequence

$$0 \longrightarrow A \otimes_F N \xrightarrow{\alpha \otimes_F N} B \otimes_F N \xrightarrow{\beta \otimes_F N} C \otimes_F N \longrightarrow 0$$

is exact in F-Mod. It follows from Theorem 19(b) that the above sequence is a sequence in R-Mod. It is clear too that exactness in F-Mod implies exactness in R-Mod. Thus $\_ \otimes_F N$ is exact which establishes our claim.

Since $\_ \otimes_F N$ is an exact covariant functor from R-Mod to R-Mod and $\mathrm{Hom}_R(\_, L)$ is an exact contravariant functor from R-Mod to the category of abelian groups $\mathbb{A}b$ (because L is injective in R-Mod), the composition of functors $\mathrm{Hom}_R(\_ \otimes_F N, L)$ is an exact contravariant functor from R-Mod to $\mathbb{A}b$.

Since the isomorphism $\Theta_{(M,N,L)}$ is functorial in M (by Theorem 20), the functors $\mathrm{Hom}_R(\_ \otimes_F N, L)$ and $\mathrm{Hom}_R(\_, \mathrm{Hom}_F(N, L))$ are naturally equivalent. Thus $\mathrm{Hom}_R(\_, \mathrm{Hom}_F(N, L))$ is an exact contravariant functor from R-Mod to $\mathbb{A}b$. This entails $\mathrm{Hom}_F(N, L)$ is injective in R-Mod. □

5.2. Proper cyclic modules and PCI-rings

The next result aids computation in the cyclic module R/Rs where $s \in R = F[x, x^{-1}, \sigma]$. We first introduce some notation.

If A is the $m \times n$ matrix over F whose (i, j)th entry is a_{ij}, we define $\sigma(A)$ to be the $m \times n$ matrix whose (i, j)th entry is $\sigma(a_{ij})$. (This notation shall apply to elements of $F^{(n)}$ interpreted as $1 \times n$ matrices over F.)

If $s = \sum_{i=0}^{n} b_i x^i \in F[x, \sigma]$ has standard form with $\partial s = n \geqslant 1$, we shall denote by $\mathbb{M}_s$ the $n \times n$ *companion matrix* of s, that is,

$$\mathbb{M}_s \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & -b_0 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & -b_1 \\ 0 & 1 & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & -b_{n-1} \end{bmatrix}.$$

Note that $\mathbb{M}_s$ is invertible since $\det \mathbb{M}_s = (-1)^n b_0 \neq 0$.

If A is an invertible $n \times n$ matrix over F and i is any integer, we define:

$$\omega(i, \sigma, A) \stackrel{\text{def}}{=} \begin{cases} A\sigma(A) \ldots \sigma^{i-1}(A), & \text{if } i \geqslant 1 \\ I_{n \times n}, & \text{if } i = 0 \\ \sigma^{-1}(A^{-1})\sigma^{-2}(A^{-1}) \ldots \sigma^{i}(A^{-1}), & \text{if } i < 0. \end{cases}$$

Given $\sum_{i=0}^{n-1} c_i x^i \in F[x, \sigma]$, we shall use the bold-face letter $\mathbf{c}$ to denote the n tuple $(c_0, c_1, \ldots, c_{n-1}) \in F^{(n)}$. The transpose of $\mathbf{c}$ is denoted $\mathbf{c}^{\mathrm{T}}$.

Proposition 22. *Let $R = F[x, x^{-1}, \sigma]$. Suppose $s = \sum_{i=0}^{n} b_i x^i \in F[x, \sigma]$ has standard form with $\partial s = n \geqslant 1$, $t = \sum_{i=0}^{n-1} c_i x^i \in F[x, \sigma]$ and $r = \sum_{i=-m}^{k} a_i x^i \in R$. Then*

$$rt \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs\text{) where } \mathbf{d}^{\mathrm{T}} = \sum_{i=-m}^{k} a_i \omega(i, \sigma, \mathbb{M}_s) \sigma^i(\mathbf{c})^{\mathrm{T}}.$$

Proof. Note first that $s = \sum_{i=0}^{n} b_i x^i \equiv 0$ (modulo Rs), whence

$$x^n \equiv -\sum_{i=0}^{n-1} b_i x^i \text{ (modulo } Rs\text{)}. \tag{4}$$

Then

$$\begin{aligned} xt &= \sum_{i=0}^{n-1} \sigma(c_i)x^{i+1} \\ &= \sum_{i=0}^{n-2} \sigma(c_i)x^{i+1} + \sigma(c_{n-1})x^n \\ &\equiv \sum_{i=0}^{n-2} \sigma(c_i)x^{i+1} + \sigma(c_{n-1})\left(-\sum_{i=0}^{n-1} b_i x^i\right) \text{ (modulo } Rs) \quad \text{[by (4)]} \\ &= \sum_{i=1}^{n-1} \sigma(c_{i-1})x^i - \sum_{i=0}^{n-1} \sigma(c_{n-1})b_i x^i \\ &= -\sigma(c_{n-1})b_0 + \sum_{i=1}^{n-1} [\sigma(c_{i-1}) - \sigma(c_{n-1})b_i]x^i. \end{aligned}$$

A routine calculation shows that for each $i \in \{0, 1, \ldots, n-1\}$, the coefficient of x^i in the above sum coincides with the ith entry of $\mathbb{M}_s\sigma(\mathbf{c})^{\mathrm{T}}$. Thus

$$xt \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs) \text{ where } \mathbf{d}^{\mathrm{T}} = \mathbb{M}_s\sigma(\mathbf{c})^{\mathrm{T}}. \tag{5}$$

A repetition of the above with xt in place of t yields

$$x^2 t \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs) \text{ where } \mathbf{d}^{\mathrm{T}} = \mathbb{M}_s\sigma(\mathbb{M}_s)\sigma^2(\mathbf{c})^{\mathrm{T}}.$$

More generally, for each $j \in \mathbb{N}$

$$x^j t \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs) \text{ where } \mathbf{d}^{\mathrm{T}} = \omega(j, \sigma, \mathbb{M}_s)\sigma^j(\mathbf{c})^{\mathrm{T}}. \tag{6}$$

For negative powers of x, we note first that if

$$x^{-1}t \equiv \sum_{i=0}^{n-1} d'_i x^i \text{ (modulo } Rs),$$

then

$$t = \sum_{i=0}^{n-1} c_i x^i \equiv x\left(\sum_{i=0}^{n-1} d'_i x^i\right) \text{ (modulo } Rs),$$

whence $\mathbf{c}^{\mathrm{T}} = \mathbb{M}_s\sigma(\mathbf{d}')^{\mathrm{T}}$ (by (5)) and so $\mathbf{d'}^{\mathrm{T}} = \sigma^{-1}(\mathbb{M}_s^{-1}\mathbf{c}^{\mathrm{T}}) = \sigma^{-1}(\mathbb{M}_s^{-1})\sigma^{-1}(\mathbf{c})^{\mathrm{T}}$.

More generally, for each $j \in \mathbb{N}$

$$x^{-j}t \equiv \sum_{i=0}^{n-1} d'_i x^i \text{ (modulo } Rs\text{) where } \mathbf{d'}^{\mathrm{T}} = \omega(-j, \sigma, \mathbb{M}_s)\sigma^{-j}(\mathbf{c})^{\mathrm{T}}. \tag{7}$$

Collating the formulas in (6) and (7) we obtain

$$rt = \sum_{i=-m}^{k} a_i x^i t \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs\text{) where } \mathbf{d}^{\mathrm{T}} = \sum_{i=-m}^{k} a_i \omega(i, \sigma, \mathbb{M}_s)\sigma^i(\mathbf{c})^{\mathrm{T}}. \quad \square$$

Remark 23. *Choosing $s = x - 1$, $t = c \in F$ and $r = \sum_{i=-m}^{k} a_i x^i$ in Proposition 22, we see that $\mathbb{M}_s = 1$, whence*

$$rc \equiv d \text{ (modulo } R(x-1)\text{) where } d = \sum_{i=-m}^{k} a_i \omega(i, \sigma, \mathbb{M}_s)\sigma^i(c) = \sum_{i=-m}^{k} a_i \sigma^i(c).$$

In light of Remark 18, we infer from the above that, as left R-modules,

$$R/R(x-1) \cong {}_R F.$$

Let $s = \sum_{i=0}^{n} b_i x^i \in F[x, \sigma]$ with $b_0, b_n \neq 0$ so that $\partial s = n$. Define $s^* \in F[x, \sigma]$ by

$$s^* \stackrel{\text{def}}{=} \sum_{i=0}^{n} b'_i x^i \text{ where } b'_i = \sigma^i(b_{n-i}) \text{ for all } i \in \{0, 1, \ldots, n\}.$$

In the definition of s^* above, since $b'_0, b'_n \neq 0$, s^{**} is defined and

$$s^{**} = \sum_{i=0}^{n} b''_i x^i \text{ where } b''_i = \sigma^i(b'_{n-i}) = \sigma^i(\sigma^{n-i}(b_i)) = \sigma^n(b_i) \text{ for all } i \in \{0, 1, \ldots, n\}.$$

Thus

$$\begin{aligned} x^{-n}s^{**}x^n &= x^{-n}\left(\sum_{i=0}^{n} \sigma^n(b_i)x^i\right)x^n \\ &= \sum_{i=0}^{n} x^{-n}\sigma^n(b_i)x^n x^i \\ &= \sum_{i=0}^{n} b_i x^i \\ &= s, \end{aligned}$$

whence

$$s^{**} = x^n s x^{-n}. \tag{8}$$

Proposition 24. *Let $R = F[x, x^{-1}, \sigma]$. For each $s \in F[x, \sigma]$ with nonzero constant term,*

$$\mathrm{Hom}_F(R/Rs, F) \cong R/Rs^* \text{ and } \mathrm{Hom}_F(R/Rs^*, F) \cong R/Rs,$$

as left R-modules.

Proof. Suppose $s \in F[x, \sigma]$ has nonzero constant term. If $\partial s = 0$, then $Rs = R = Rs^*$ and there is nothing to prove, so let us suppose that $\partial s = n \geq 1$.

Suppose first that s is monic and thus has standard form. By Theorem 17(c), $\{x^i + Rs : 0 \leqslant i \leqslant n-1\}$ is a basis for ${}_F(R/Rs)$. For each $j \in \{0, 1, \dots, n-1\}$ let $\pi_j \in \mathrm{Hom}_F(R/Rs, F)$ denote the canonical projection map defined by

$$\pi_j\left(\sum_{i=0}^{n-1} c_i x^i + Rs\right) \stackrel{\text{def}}{=} c_j.$$

Put $t = \sum_{i=0}^{n-1} c_i x^i \in F[x, \sigma]$. Taking $r = x^{-1}$ in Proposition 22, we see that

$$x^{-1}t \equiv \sum_{i=0}^{n-1} d_i x^i \text{ (modulo } Rs) \text{ where } \mathbf{d}^{\mathrm{T}} = \sigma^{-1}(\mathbb{M}_s^{-1})\sigma^{-1}(\mathbf{c})^{\mathrm{T}}.$$

A routine calculation shows that[5]

$$d_j = \sigma^{-1}(c_{j+1}) + \sigma^{-1}(-b_{j+1}b_0^{-1})\sigma^{-1}(c_0) \;\forall j \in \{0, 1, \dots, n-1\}. \tag{9}$$

Now[6]

$$\begin{aligned}
(x\pi_j)(t + Rs) &= x(\pi_j(x^{-1}t + Rs)) \\
&= xd_j \\
&= \sigma(d_j) \quad \text{[see Remark 18]} \\
&= c_{j+1} - (b_{j+1}b_0^{-1})c_0 \quad \text{[by (9)]} \\
&= (\pi_{j+1} - (b_{j+1}b_0^{-1})\pi_0)(t + Rs).
\end{aligned}$$

Thus

$$x\pi_j = \pi_{j+1} - (b_{j+1}b_0^{-1})\pi_0 \;\forall j \in \{0, 1, \dots, n-1\}. \tag{10}$$

Solving the recursive formula (10) yields:

[5] For convenience, we shall take $c_n = 0$.
[6] For convenience, we shall take $\pi_n = 0$.

$$\pi_1 = (x + b_1 b_0^{-1})\pi_0;$$

$$\pi_2 = x\pi_1 + b_2 b_0^{-1}\pi_0 = (x^2 + x(b_1 b_0^{-1}) + b_2 b_0^{-1})\pi_0;$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\pi_{n-1} = (x^{n-1} + x^{n-2}(b_1 b_0^{-1}) + \cdots + b_{n-1} b_0^{-1})\pi_0. \tag{11}$$

Since $\mathrm{Hom}_F(R/Rs, F)$ is spanned as an F-space by $\{\pi_j : 0 \leqslant j \leqslant n-1\}$, the system of equations (11) tells us that $\mathrm{Hom}_F(R/Rs, F)$ is a cyclic left R-module generated by π_0.

Taking $j = n-1$ in (10) and invoking (11), we obtain

$$x\pi_{n-1} = -(b_n b_0^{-1})\pi_0 = x(x^{n-1} + x^{n-2}(b_1 b_0^{-1}) + \cdots + b_{n-1} b_0^{-1})\pi_0,$$

whence

$$\begin{aligned} 0 &= (x^n + x^{n-1}(b_1 b_0^{-1}) + \cdots + x(b_{n-1} b_0^{-1}) + b_n b_0^{-1})\pi_0 \\ &= (x^n b_0 + x^{n-1} b_1 + \cdots + x b_{n-1} + b_n) b_0^{-1}\pi_0 \\ &= (\sigma^n(b_0)x^n + \sigma^{n-1}(b_1)x^{n-1} + \cdots + \sigma(b_{n-1})x + b_n) b_0^{-1}\pi_0 \\ &= s^* b_0^{-1}\pi_0. \end{aligned}$$

It follows that $\mathrm{Hom}_F(R/Rs, F) = R\pi_0 = Rb_0^{-1}\pi_0$ is an epimorphic image of R/Rs^*. Since $\dim_F(\mathrm{Hom}_F(R/Rs, F)) = \partial s = n = \partial s^* = \dim_F(R/Rs^*)$, this epimorphism must be an isomorphism. Thus $\mathrm{Hom}_F(R/Rs, F) \cong R/Rs^*$.

If s is not monic, write $s = bs'$ with $0 \neq b \in F$ and $s' \in F[x, \sigma]$ in standard form. The isomorphism $\mathrm{Hom}_F(R/Rs, F) \cong R/Rs^*$ follows noting that $R/Rs = R/Rs'$ and $R/Rs^* = R/R(s')^* b \cong R/R(s')^*$ (because b is a unit of R).

Now

$$\begin{aligned} \mathrm{Hom}_F(R/Rs^*, F) &\cong R/Rs^{**} \\ &= R/Rx^n s x^{-n} \quad \text{[by (8)]} \\ &= R/Rsx^{-n} \\ &\cong R/Rs \quad \text{[because } x^{-n} \text{ is a unit of } R]. \quad \square \end{aligned}$$

Recall that a ring R is said to be a left *PCI-ring* if every proper cyclic left R-module (this is a cyclic left R-module that is not isomorphic to ${}_R R$) is injective in R-Mod. Clearly every left PCI-ring is a left V-ring.

Remark 25. *Since $R = F[x, x^{-1}, \sigma]$ is a left principal ideal domain, a left R-module M will be injective precisely if it is divisible, which is to say, $rM = M$ for all nonzero $r \in R$. This characterization of injectivity can be sharpened further; since every nonzero left ideal of R has the form Rr for some nonzero $r \in F[x, \sigma]$ (Theorem 17(a)), to show M is injective, it suffices to show that $rM = M$ for all nonzero $r \in F[x, \sigma]$.*

The following theorem is the analogue for twisted Laurent polynomial rings of a theorem by Cozzens and Faith for differential polynomial rings [4, Theorem 5.21, page 93].

Theorem 26. *Let $R = F[x, x^{-1}, \sigma]$. The following statements are equivalent:*

(a) *For every $b \in F$ and nonzero $r = \sum_{i=0}^{k} a_i x^i \in F[x, \sigma]$, there exists $c \in F$ such that*

$$rc = \sum_{i=0}^{k} a_i \sigma^i(c) = b;$$

(b) *${}_R F$ is injective;*
(c) *R is a left PCI-ring.*

Proof. (a)⇔(b) In Remark 25 we noted that ${}_R F$ will be injective in R-Mod iff $rF = F$ for all nonzero $r = \sum_{i=0}^{k} a_i x^i \in F[x, \sigma]$, which is to say, for every $b \in F$, there exists $c \in F$ such that

$$rc = \left(\sum_{i=0}^{k} a_i x^i \right) c = \sum_{i=0}^{k} a_i \sigma^i(c) = b,$$

which is Statement (a).

(c)⇒(b) is an immediate consequence of the fact that ${}_R F$ is proper cyclic (see Remark 23).

(b)⇒(c) Let N be a proper cyclic left R-module. By Theorem 17(a), $N \cong R/Rs$ for some $s \in F[x, \sigma]$ in standard form. It follows from Proposition 24 that $N \cong \mathrm{Hom}_F(R/Rs^*, F)$ which is injective by Theorem 21. □

5.3. Simple modules over $F[x, x^{-1}, \sigma]$

Proposition 27. *Let $R = F[x, x^{-1}, \sigma]$. The following statements are equivalent for a nonzero $b \in F$ and $r = \sum_{i=0}^{k} a_i x^i \in F[x, \sigma]$ that is a non-unit of R:*

(a) *$\mathrm{Hom}_R(R/Rr, R/R(x - b)) \neq 0$;*
(b) *There exists a nonzero $c \in F$ such that $rc \equiv 0$ (modulo $R(x - b)$);*
(c) *There exists a nonzero $c \in F$ such that*

$$\sum_{i=0}^{k} a_i\omega(i,\sigma,b)\sigma^i(c) = 0.$$

Proof. (a)⇔(b) is a consequence of the fact that if I is a proper left ideal of an arbitrary ring R, and $M \in R$-Mod, then $\mathrm{Hom}_R(R/I, M) \neq 0$ iff there exists a nonzero $u \in M$ such that $Iu = 0$.

(b)⇔(c) Taking $s = x - b$ and $t = c \in F$ in Proposition 22, we see that

$$rc \equiv d \text{ (modulo } R(x-b)) \text{ where } d = \sum_{i=0}^{k} a_i\omega(i,\sigma,b)\sigma^i(c).$$

Thus

$$rc \equiv 0 \text{ (modulo } R(x-b)) \text{ iff } \sum_{i=0}^{k} a_i\omega(i,\sigma,b)\sigma^i(c) = 0.$$

The equivalence of (b) and (c) follows. □

It is clear that Statement (a) of Proposition 27 will hold for all $r \in F[x,\sigma]$ that are non-units of R, precisely if $R/R(x-b)$ is, up to isomorphism, the unique simple left R-module. Taking $b = 1$ in Proposition 27, so that $\omega(i,\sigma,b) = \omega(i,\sigma,1) = 1$ for all $i \in \{0,1,\dots,k\}$, and noting that $R/R(x-1) \cong {}_RF$ (see Remark 23), we obtain the corollary below. (For an analogous result see [4, Theorem 5.21, page 93].)

Corollary 28. *Let $R = F[x,x^{-1},\sigma]$. The following statements are equivalent:*

(a) *For every $r = \sum_{i=0}^{k} a_ix^i \in F[x,\sigma]$ that is a non-unit of R, there exists a nonzero $c \in F$ such that*

$$\sum_{i=0}^{k} a_i\sigma^i(c) = 0;$$

(b) *${}_RF$ is, up to isomorphism, the unique simple left R-module.*

Let $R = F[x,x^{-1},\sigma]$. For each nonzero $a \in F$, define

$$S_a \stackrel{\text{def}}{=} R/R(x-a).$$

It is clear from Theorem 17 that every simple left R-module S with $\dim_F S = 1$, will have the form S_a for some nonzero $a \in F$.

If F and σ are understood, we define

$$\Gamma \stackrel{\text{def}}{=} \{\sigma(c)c^{-1} : c \in F\backslash\{0\}\}.$$

It is easily checked that Γ is a subgroup of the multiplicative group $F\backslash\{0\}$ of nonzero elements of F.

Proposition 29. *Let $R = F[x, x^{-1}, \sigma]$. For all nonzero $a, b \in F$,*

$$S_a \cong S_b \text{ iff } a\Gamma = b\Gamma.$$

Proof.

$$\begin{aligned} &S_a = R/R(x-a) \cong R/R(x-b) = S_b \\ &\quad \text{iff } \operatorname{Hom}_R(R/R(x-a), R/R(x-b)) \neq 0 \quad [\text{because } S_a \text{ and } S_b \text{ are simple}] \\ &\quad \text{iff there exists a nonzero } c \in F \text{ such that } (-a)\omega(0,\sigma,b)c + 1\cdot\omega(1,\sigma,b)\sigma(c) = 0 \\ &\quad [\text{taking } r = x-a \text{ in Proposition 27}] \\ &\quad \text{iff there exists a nonzero } c \in F \text{ such that } -ac + b\sigma(c) = 0 \\ &\quad \text{iff there exists a nonzero } c \in F \text{ such that } b^{-1}a = \sigma(c)c^{-1} \\ &\quad \text{iff } b^{-1}a \in \Gamma \\ &\quad \text{iff } a\Gamma = b\Gamma. \quad \square \end{aligned}$$

Remark 30. *It follows from the previous result that the index $[F\backslash\{0\} : \Gamma]$ of Γ in $F\backslash\{0\}$ coincides with the number (possibly infinite) of isomorphism classes of 1-dimensional simple left R-modules. In particular, if $[F\backslash\{0\} : \Gamma]$ is infinite, then R will possess, up to isomorphism, infinitely many simple left R-modules.*

5.4. Fields of characteristic $p > 0$—Osofsky's example

Let F be a field of characteristic $p > 0$. Henceforth, we shall denote by σ_p the *Frobenius endomorphism* on F defined by

$$\sigma_p(a) \stackrel{\text{def}}{=} a^p \ \forall a \in F.$$

Recall that F is said to be *perfect* if σ_p is onto and thus an automorphism on F. It is clear that every algebraically closed field of characteristic $p > 0$ is perfect.

Let F be a perfect field of characteristic $p > 0$. Taking $R = F[x, x^{-1}, \sigma_p]$ in Theorem 26, the equation in Statement (a) reads

$$\sum_{i=0}^{k} a_i c^{p^i} = b. \tag{12}$$

Observe that (12) is a nontrivial (because $\sum_{i=0}^{k} a_i x^i$ is nonzero) polynomial equation in c over F. Thus if F is algebraically closed, (12) will have a solution for $c \in F$, thus fulfilling Statement (a).

Similarly, if, in Corollary 28, R is chosen to be $F[x, x^{-1}, \sigma_p]$, then the equation in Statement (a) reads

$$\sum_{i=0}^{k} a_i c^{p^i} = 0. \tag{13}$$

Since $r = \sum_{i=0}^{k} a_i x^i$ is a non-unit of R, if F is algebraically closed, (13) will have a nonzero solution for $c \in F$, thus fulfilling Statement (a).

We have thus proven the following theorem which combines [10, Proposition 4, page 601 and Proposition 8, page 603] (see also [3, Theorem 2.3, page 78]).

Theorem 31. *Let F be an algebraically closed field of characteristic $p > 0$. Then $R = F[x, x^{-1}, \sigma_p]$ is a left PCI-ring such that ${}_RF$ is, up to isomorphism, the unique simple left R-module.*

We shall show presently that the requirement that F be algebraically closed can be relaxed in a manner that retains the left PCI property for R, but which allows for a multiplicity of nonisomorphic simples.

The notion of 'q-field' defined below differs slightly from that introduced in [10, page 600].

Let $q \in \mathbb{N}$ be a prime and F an algebraic field extension of a field K. We call F a *q-field extension* of K if $\deg(\alpha, K)$ is relatively prime to q for all $\alpha \in F$.

The following lemma shows that q-field extensions are transitive with respect to fields of characteristic not equal to q.

Lemma 32. *Let q be a prime and K a field whose characteristic is not equal to q. If K' is a q-field extension of K and K'' is a q-field extension of K', then K'' is a q-field extension of K.*

Proof. If $K' = K''$, there is nothing to prove, so suppose $K' \subset K''$ and pick $\alpha \in K'' \backslash K'$. Let $f = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$ be the minimum polynomial for α over K'. Since K'' is a q-field extension of K', $n = \partial f = \deg(\alpha, K')$ is relatively prime to q.

Define $E = K(a_0, \ldots, a_{n-1})$ so that $f \in E[x]$ and $\deg(\alpha, E) = \deg(\alpha, K') = n$.

Let $\widehat{K}$ denote the separable closure of K in E. By the Primitive Element Theorem [8, Theorem 40, page 49], $\widehat{K} = K(\beta)$ for some $\beta \in E$. We have

$$\begin{aligned}[E(\alpha) : K] &= [E(\alpha) : E] \times [E : \widehat{K}] \times [\widehat{K} : K] \\ &= n \times [E : \widehat{K}] \times \deg(\beta, K).\end{aligned} \tag{14}$$

Observe that in (14), n and $\deg(\beta, K)$ are both relatively prime to q.

If K has characteristic 0, then E is separable, so $\widehat{K} = E$, whence $[E : \widehat{K}] = 1$.

If K has characteristic $p > 0$, then $[E : \widehat{K}]$ is a power of p [8, Exercise 5, page 59].

In both cases it follows from (14) that $[E(\alpha) : K]$ is relatively prime to q. Inasmuch as $[K(\alpha) : K]$ divides $[E(\alpha) : K]$, it follows that $[K(\alpha) : K]$ is relatively prime to q. We conclude that K'' is a q-field extension of K, as required. □

Let q be a prime. We say field F is *q-closed* if every polynomial over F whose degree over F is relatively prime to q, has a zero in F.

Proposition 33. *Let q be a prime and K a field whose characteristic is not equal to q. Then K has a q-closed, q-field extension.*

Proof. Let $\overline{K}$ be the algebraic closure of K. Denote by $\mathcal{F}$ the family of all subfields of $\overline{K}$ that are q-field extensions of K. Certainly $\mathcal{F}$ is nonempty since $K \in \mathcal{F}$. It is easily checked that the union of any chain in $\mathcal{F}$ is again a member of $\mathcal{F}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal member F, say. It remains to show that if f is a polynomial over F whose degree is relatively prime to q, then f has a zero in F. To this end, write f as a product of irreducible polynomials over F:

$$f = g_1 g_2 \dots g_k.$$

Since ∂f is relatively prime to q, for some $i \in \{1, 2, \dots, k\}$, g_i has degree that is relatively prime to q. Choosing a zero, β say, for g_i in $\overline{K}$, we see that $F(\beta)$ is a q-field extension of F. By the transitivity of q-field extensions (Lemma 32), $F(\beta)$ is a q-field extension of K, so $F(\beta) \in \mathcal{F}$. This contradicts the maximality of F unless $\beta \in F$. Thus f has a zero in F, as required. □

Theorem 34. *Let q be a prime and F a q-closed field of characteristic $p > 0$ where $p \neq q$. Then:*

(a) *The Frobenius endomorphism $\sigma_p : F \to F$ is onto and thus an automorphism, whence F is a perfect field.*
(b) *$R = F[x, x^{-1}, \sigma_p]$ is a left PCI-ring.*

Proof. Since F is q-closed and $p \neq q$, every polynomial equation over F of degree a nonnegative power of p, will have a zero in F. This has two consequences. The first is that every element of F has a pth root in F, i.e., σ_p is onto. Thus (a) holds.

The second is that every (nontrivial) equation of the type shown in (12), has a solution for $c \in F$. This entails Statement (a) of Theorem 26 is satisfied, whence R is a left PCI-ring. Thus (b) holds. □

If F is a perfect field of characteristic $p > 0$, observe that

$$\begin{aligned}\Gamma &= \{\sigma_p(c)c^{-1} : c \in F\backslash\{0\}\} \\ &= \{c^{p-1} : c \in F\backslash\{0\}\} \\ &= F^{p-1}\backslash\{0\}.\end{aligned} \tag{15}$$

The following result is inspired by [10, Example (a), page 606].

Proposition 35. *Let p be an odd prime and let F be a 2-field extension of the field of rational functions $K(X)$ in indeterminate X over a field K of characteristic p. Then*

$$[F\backslash\{0\} : F^{p-1}\backslash\{0\}] \geqslant |K|.$$

Proof. Suppose a and b are distinct elements of K. Consider the degree 1 polynomials $\pi_a = X + a$ and $\pi_b = X + b$ in $K[X]$. We claim that the cosets $\pi_a(F^{p-1}\backslash\{0\})$ and $\pi_b(F^{p-1}\backslash\{0\})$ of $F^{p-1}\backslash\{0\}$ in $F\backslash\{0\}$, are distinct. Suppose not, so that $\pi_a\pi_b^{-1} = \beta^{p-1}$ for some $\beta \in F\backslash\{0\}$. Since p is odd, $\frac{p-1}{2}$ is integral. Put $\gamma = \beta^{\frac{p-1}{2}}$, so that $\pi_a\pi_b^{-1} = \gamma^2$. Since F is a 2-field extension of $K(X)$ and $\gamma^2 \in K(X)$, we must have $\gamma \in K(X)$. Since π_a and π_b are prime elements of $K[X]$, it is easily shown that there can exist no rational function γ in X over K such that $\pi_a\pi_b^{-1} = \gamma^2$. This establishes our claim.

We conclude that $\{\pi_a(F^{p-1}\backslash\{0\}) : a \in K\}$ is a family of cosets of $F^{p-1}\backslash\{0\}$ in $F\backslash\{0\}$ that is equipotent with K. Hence $[F\backslash\{0\} : F^{p-1}\backslash\{0\}] \geqslant |K|$, as required. □

We are now in a position to state the main theorem.

Theorem 36. *Let:*

- ▷ *p be an odd prime;*
- ▷ *K be any field of characteristic p;*
- ▷ *F be a 2-closed, 2-field extension of the field of rational functions $K(X)$ in X over K;*
- ▷ *$\sigma_p : F \to F$ be the Frobenius automorphism;*
- ▷ *$R = F[x, x^{-1}, \sigma_p]$.*

Then:

(a) *R is a simple, left principal ideal, left PCI-domain; and*
(b) *R possesses, up to isomorphism, no fewer than $|K|$ simple left R-modules of dimension 1 over F.*

Proof. Note first that a 2-closed, 2-field extension of the field $K(X)$ is certain to exist by Proposition 33.

Note also that the given conditions on F imply, by Theorem 34(a), that the Frobenius endomorphism σ_p on F is indeed an automorphism as stated in the theorem.

(a) R is a simple, left principal ideal domain by Theorem 17. That R is also a left PCI-ring is a consequence of Theorem 34(b).

(b) is an immediate consequence of (15), Proposition 35 and Remark 30. □

Remark 37. *Since there is no upper bound on the cardinality of the field K in Theorem 36, we may infer from Statement (b) of this theorem that there is no upper bound on the size of a representative family of simple left R-modules for simple, left principal ideal, left PCI-domains R. As explained in Remark 15, this has the consequence that for each cardinal $\mathfrak{m}$ (not necessarily finite) there exists a simple, left noetherian, left hereditary, left V-domain R with a representative set of simple left R-modules of cardinality precisely $\mathfrak{m}$.*

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