DIVISIBILITY THEORY OF ARITHMETICAL RINGS WITH ONE MINIMAL PRIME IDEAL

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Abstract. Continuing the study of divisibility theory of arithmetical rings started in [1] and [2] we show that the divisibility theory of arithmetical rings with one minimal prime ideal is axiomatizable as Bezout monoids with one minimal m-prime filter. In particular, every Bezout monoid with one minimal m-prime filter is order-isomorphic to the partially ordered monoid with respect to inverse inclusion, of principal ideals in a Bezout ring with a smallest prime ideal. Although this result can be considered as a satisfactory answer to the divisibility theory of both semi-hereditary domains and valuation rings, the general representation theory of Bezout monoids is still open.

1. Introduction

One of the main results in the general valuation theory developed by Krull is a dictionary between valued fields and ordered abelian groups. This dictionary was later extended by Jaffard, Kaplansky and Ohm to the larger class of Bezout domains and lattice-ordered abelian groups. This one-to-one correspondence can be considered in some sense as a local theory because all rings considered, being domains, have just one minimal prime ideal! The aim of the present note is to make this local theory complete by showing that the divisibility theory of arithmetical rings with one minimal prime ideal can be axiomatized as Bezout monoids with one minimal m-prime filter. The subclass of Bezout rings with one minimal prime ideal appears naturally in the solution of Kaplansky's question on describing commutative rings whose finitely generated modules decompose into direct sums of cyclics (see e.g. [5] .) To achieve our goal, we develop first the structure theory of Bezout monoids with one minimal m-prime filter and then, with a fairly good structural description in hand, we can complete the job by constructing, for each Bezout monoid S with one minimal m-prime filter, a Bezout ring whose monoid of divisibility is order-isomorphic to S. Our approach is primarily influenced by the treatment of Kaplansky's problem on valuation rings presented in [7]. For the general, axiomatic theory of divisibility we refer to the fairly up-to-date presentation in Halter-Koch's book [8].

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A word about terminology. All structures are commutative. Rings have the identity element 1. The *monoid of divisibility*, or *divisibility theory* of a ring is simply the monoid of principal ideals under ideal multiplication partially ordered by reverse containment. A ring is called an *arithmetical ring* if its ideals form a distributive lattice. Every *Bezout ring*, i.e., a ring whose finitely generated ideals are principal, is arithmetical, but arithmetical rings needn't be Bezout rings. The set of *nontrivial* (i.e. different from 1 and 0) elements of a subset X in a monoid is denoted by $X^\star$. Moreover, $X^\bullet$ will be the extension of X by the new extra zero element 0. Thus $1^\bullet$ is the monoid consisting of exactly two elements 1 and 0. The *positive cone* of a lattice-ordered abelian group is the set of elements bigger than or equal to the identity.

2. Basic notions and preliminary results

For the benefit of the reader and for the sake of completeness we recall some easy, but basic results, notation and definitions from [1] and [2]. The development of and comments on this supporting material can be found in full detail in [1] and partly in [2].

Definition 2.1 (cf. Definition 1.1 [2])**.** A *Bezout monoid* S (in short a *B-monoid*) is a commutative monoid S with 0 such that the divisibility relation $a|b \iff aS \supseteq bS$ is a partial order, called a *natural partial order*, inducing a distributive lattice on S, with a multiplication distributive on both meets and joins; and S is *hyper-normal*, meaning that for any a, b, $d = a \wedge b \in S$, $a = da_1$ there is $b_1 \in S$ satisfying $a_1 \wedge b_1 = 1$, $b = db_1$. A monoid with 0 is called *0-cancellative* if $ax = ay \neq 0 \Rightarrow x = y$.

By Proposition 1.1 [2] the divisibility theory of a Bezout ring is a B-monoid. Moreover, by Corollary 3.4 [2] a monoid S is the divisibility theory of a semi-hereditary ring if and only if it is a semi-hereditary B-monoid. Using hyper-normality, one can easily see

Corollary 2.1 (cf. Proposition 1.3 [1])**.** *If 1 is a* meet-irreducible *element of a B-monoid S, that is, $s \wedge t = 1 \Rightarrow s = 1$ or $t = 1$, then S is a 0-cancellative naturally totally ordered monoid.*

This statement corresponds to a well-known result that local Bezout rings are valuation (or chained) rings. For this reason, a B-monoid is called *local* if 1 is meet-irreducible. Recall that a *filter* F of a B-monoid S is a subset closed under $\wedge$ such that $a \in F$ and $a \leq b \in S$ imply $b \in F$. Hence filters are also ideals in the usual sense, but the converse is in general not true. For example, the set of multiples of 3 or 5 is an ideal in the B-monoid of non-negative integers under multiplication, but it is not a filter. A filter F is called an *m-prime filter* if $ab \in F$ implies $a \in F$ or $b \in F$. It turns out (see [1] and [2]) that m-prime filters play a decisive role in working with B-monoids. As in the case of rings, one can define the *nil radical* of S as the intersection of all m-prime filters, and is also the intersection of all minimal m-prime filters (with respect to inclusion). As shown for commutative rings by Krull, one can see in the setting of B-monoids that the nil radical is just the set of all nilpotent elements. The general theory of m-prime filters in B-monoids can be found in [1], Sections 2 and 3.

Proposition 2.2 (cf. Theorem 2.10 [1])**.** *If F is an m-prime filter of a B-monoid S, then*

$$\forall\, x, y \in S : x \sim y \iff \exists\, s \in S \setminus F : x \leq ys \ \&\ y \leq xs$$

defines a congruence whose factor S_F *is a local B-monoid, and the congruence class of* 0 *is the set* $\{x \in S \,|\, \exists\, s \notin I : sx = 0\}$.

S_F is called the *localization of* S at an m-prime filter F or at the complement $S \setminus F$ (by identifying its elements with 1). Using localization one can make clear the relation between annihilators and cancellation in the next assertion.

Proposition 2.3 (cf. Proposition 2.14 [1]). *For arbitrary elements* v, x, y *of a B-monoid* S *the equality* $xv = yv$ *holds if and only if there is* $s \in v^{\perp} = \{z \in S \,|\, vz = 0\}$ *such that* $x \wedge s = y \wedge s$.

Factor lattices associated to filters are also B-monoids in view of the following result.

Proposition 2.4 (cf. Theorem 2.15 [1]). *For every filter* F *of a B-monoid* S *a relation*

$$x \;\sim\; y \Longleftrightarrow \exists\, s \in F : \, x \wedge s = y \wedge s$$

defines a congruence whose factor, denoted as S/F, *is a B-monoid, called the* factor B-monoid by a filter F *of* S.

The key notions in our investigation are the following.

Definition 2.2. A *factor set* of a local B-monoid Σ in a lattice-ordered abelian group G is is the function $\mathfrak{f} : \Sigma \times \Sigma \longrightarrow G^{\bullet}$ satisfying

(FS1) $\mathfrak{f}(\alpha, \beta) = \mathfrak{f}(\beta, \alpha) \;\; \forall\, \alpha,\, \beta \in \Sigma$

(FS2) $\mathfrak{f}(\alpha, \beta\gamma)\mathfrak{f}(\beta, \gamma) \;=\; \mathfrak{f}(\alpha, \beta)\mathfrak{f}(\alpha\beta, \gamma) \;\; \forall\, \alpha,\, \beta,\, \gamma \in \Sigma$

(FS3) $\mathfrak{f}(1, \alpha) = 1, \;\; \mathfrak{f}(0, \alpha) = 0 \;\; \forall\, \alpha \in \Sigma$

(FS4) $\mathfrak{f}(\alpha, \beta) = 0$ if and only if $\alpha\beta = 0$.

Two factor sets $\mathfrak{f}$ and $\mathfrak{h}$ are called *associated* if there is a function $\mathfrak{g} \,:\, \Sigma \longrightarrow G^{\bullet}$ with $\mathfrak{g}(1) = 1, \;\mathfrak{g}(\alpha) = 0 \Longleftrightarrow \alpha = 0$ such that

$$\mathfrak{h}(\alpha, \beta) = \frac{\mathfrak{g}(\alpha)\mathfrak{g}(\beta)}{\mathfrak{g}(\alpha\beta)}\mathfrak{f}(\alpha, \beta)$$

holds in the case $\alpha\beta \neq 0$.

Let T be the *positive cone* of G, i.e., the set of those elements in G which are equal or bigger than the identity element of G. The *(generalized) crossed product* $T \underset{\mathfrak{f}}{*} \Sigma$ is

$$S = T \underset{\mathfrak{f}}{*} \Sigma := \{(1, t) \,|\, t \in T\} \;\dot{\cup}\; \{(\alpha, g) \,|\, \alpha \in \Sigma^{\star},\, g \in G\} \;\dot{\cup}\; 0,$$

a disjoint union of 0, T and copies of G indexed by $\Sigma^{\star}$ endowed with the multiplication

$$(\alpha, x) \times (\beta, y) = \begin{cases} (\alpha\beta, \mathfrak{f}(\alpha, \beta)xy), & \text{if } \; \alpha\beta \neq 0; \;\; \alpha,\, \beta \;\in \Sigma \\ 0, & \text{if } \;\; \alpha\beta = 0; \;\; \alpha,\, \beta \;\in \Sigma. \end{cases}$$

One can easily check that $T \underset{\mathfrak{f}}{*} \Sigma$ is a B-monoid with one minimal m-prime filter where the natural partial order is exactly the lexicographic order. Two crossed products $T \underset{\mathfrak{f}}{*} \Sigma$ and $T \underset{\mathfrak{h}}{*} \Sigma$ are isomorphic if and only if $\mathfrak{f}$ and $\mathfrak{h}$ are associated. $F = S \setminus T$ after identifying $t \in T$

with $(1,t)$, is an m-prime (but not necessarily minimal) filter of S and elements of T are non-zero-divisors. $T \underset{\mathfrak{f}}{*} \Sigma$ is also a pullback in view of the next diagram

$$\begin{array}{ccc} S = T \underset{\mathfrak{f}}{*} \Sigma & \xrightarrow{\mathfrak{r}} & T \\ \downarrow \mathfrak{l} & & \downarrow \mathfrak{l} \\ \Sigma & \xrightarrow[\mathfrak{r}]{} & 1^{\bullet} \end{array}$$

where $\mathfrak{l}$ denotes the localization maps and $\mathfrak{r}$ is the canonical map to a Rees factor.

Remark 2.3. Generalized crossed products can be formally defined as universal constructions by using factor sets in terms of arbitrary monoids Σ and $T \subseteq U$.

3. Structure theory

Throughout this section S is always a B-monoid with one minimal m-prime filter M, with one exception in Corollary 3.2, and $T = S \setminus M$. By Proposition 2.1 the localization Σ of S at M is a local B-monoid such that all non-unital elements are nilpotent. The set $Z = \{x \in S \mid \exists\, s \notin M : sx = 0\}$ is precisely the set of elements mapped to 0 in Σ. Let $N = M \setminus Z$. Although N is a distributive sublattice of S; it is not, in general, a subsemigroup, i.e., not closed under multiplication. Its induced partial order also fails to be natural. We shall write a_σ for the image in Σ of $a \in S$ and denote elements of Σ by Greek letters α, β, ... A complete set of (pairwise different) elements of S representing Σ is called a *transversal* of Σ in S. A transversal is *normalized* if $1, 0 \in S$ are representatives of $1, 0 \in \Sigma$, respectively. In what follows we chose an arbitrary normalized transversal $\{a_\alpha \in S \mid \alpha \in \Sigma,\ a_1 = 1,\ a_0 = 0\}$ and fix this transversal throughout this section. Moreover, for each $\alpha \in \Sigma$ put $S_\alpha = \{b \in S \mid b_\sigma = \alpha\}$. Thus $S_1 = T$, $S_0 = Z$ hold. Furthermore, for an element $a \in S$ we also use the notation $S_a = \{x \in S \mid x_\sigma = a_\sigma = S_{a_\sigma}\}$.

Proposition 3.1. *$t < m$, or equivalently $M \subseteq St$ for all $m \in M$, $t \in T$.*

Proof. Put $d = t \wedge m = d(t_1 \wedge m_1)$, $t = dt_1 \in T$, $m = dm_1 \in M$, $t_1 \wedge m_1 = 1$. $d \leq t \notin M$ and $dm_1 = m \in M$ imply $d \notin M$ and $m_1 \in M$ because M is an m-prime filter. Thus $m_1^n = 0$ for some $n \in \mathbb{N}$. Hence $t_1 = t_1 \wedge m_1^n \leq (t_1 \wedge m_1)^n = 1$ whence $t = dt_1 = d < m$. □

A careful analysis of the above proof leads to the following more general result.

Corollary 3.2. *Let S be a B-monoid with the nil radical N. Then for each element $m \in N$ and $t \in S$ which is not an element of the union of minimal m-prime filters of S, $t < m$, or equivalently, $N \subseteq St$. In particular, if t is a* non-zero-divisor *(i.e., $t^\perp = 0$), then $N \subseteq St$.*

Proof. The only somewhat non-obvious claim to prove is that a non-zero-divisor t is not contained in any minimal m-prime filter. Assume indirectly that there is a non-zero-divisor t and a minimal m-prime filter F with $t \in F$. Then the image t_φ of t in the localisation S_φ is nilpotent contradicting the fact that $t_\varphi^n \neq 0$ for every $n \in \mathbb{N}$. □

Proposition 3.3. *T is* cancellative*, that is, for any a, x, $y \in T$ an equality $ax = ay$ implies $x = y$.*

Proof. Suppose $ax = ay$ for $a,\ x,\ y \in T$. By putting $b = x \wedge y,\ c = x \vee y$ we have $b,\ c = bd \in T$ for some $d \in T$ because M is m-prime. Thus $ab = abd$, i.e., $de = e$ for $e = ab$. Assume indirectly $x \neq y$. Then $b \neq c$ and hence $d \neq 1$. Since $e \notin M$, one obtains $e^{\perp} \subseteq M$. Therefore by Proposition 3.1, $e^{\perp} \subseteq dS$. If F is a maximal proper filter containing dS, then in the localisation S_{φ} of S at F we have $d_{\varphi} \neq 1$, $d_{\varphi}e_{\varphi} = e_{\gamma} \neq 0$, a contradiction to Corollary 2.1. ☐

Corollary 3.4. *The quotient group G of T is lattice-ordered and T is the positive cone of G.*

Proof. By Proposition 3.3 the quotient group G of T exists. Write elements of G in the form $s^{-1}t;\ s,\ t \in T$. It is a tedious but routine task to check that meets and joins in G obtained by putting

$$s^{-1}t \wedge u^{-1}v = (su)^{-1}(tu \wedge sv), \quad s^{-1}t \vee u^{-1}v = (su)^{-1}(tu \vee sv)$$

are well-defined, i.e., independent of the way we write $g = s^{-1}t, g \in G$; and to verify a flock of axioms ensuring that G is lattice-ordered. If $s^{-1}t \wedge 1 = s^{-1}(s \wedge t) = 1$, then $s = s \wedge t \leq t$ whence $t = su$ for some $u \in T$. Consequently $s^{-1}t = u \in T$, thus T is indeed, the positive cone of G. ☐

Proposition 3.5. *The equality $ZM = 0$ holds.*

Proof. Let $z \in Z, x \in M$. Then there is $s \notin M$ with $sz = 0$. Since $sM = M$, there is $y \in M$ with $sy = x$. Hence $zx = z(sy) = (sz)y = 0$. ☐

Proposition 3.6. *If $x \in M \setminus Z, z \in Z$, then $x < z$.*

Proof. Put $x \wedge z = y$ and $x = yx_1, z = yz_1, x_1 \wedge z_1 = 1$. $y \notin Z$ holds by $y \leq x \notin Z$. $z \in Z$ implies $tz = 0$ for some $t \in T$. If $z_1 \notin M$, then $0 = tz = (tz_1)y$ shows $y \in Z$ because $tz_1 \in T$, a contradiction. Thus $z_1 \in M$ and hence $x_1 \notin M$. By Proposition 3.1 $x_1 \leq z_1$ and hence $x_1 = x_1 \wedge z_z = 1$ whence $x = y < z$ holds. ☐

Proposition 3.5 implies that Z can be considered as a $B-$act over both S and $T^{\bullet}$ in the following sense.

Definition 3.1. Let U be a B-monoid. A *Bezout act over U*, shortly a *B-act*, is a distributive lattice A with the greatest element denoted also by 0 and a multiplication $U \times A \longrightarrow A : (u, a) \mapsto ua \in A$ such that

(BA0) $u(va) = (uv)a$ for all $u,\ v \in U;\ a \in A$,
(BA1) $1a = a, 0a = 0$ and $u0 = 0$ for every $a \in A$ and $u \in U$,
(BA2) $a \leq b;\ a,\ b \in A$ if and only if $b = sa$ for some $s \in U$,
(BA3) $\forall\, s, t \in U;\ a,\ b \in A :\ s(a \wedge b) = sa \wedge sb,\ \ s(a \vee b) = sa \vee sb,\ \ (s \wedge t)a = sa \wedge sb,\ \ (s \vee t)a = sa \vee ta$,
(BA4) for any two $a, b \in A$ and $d = a \wedge b$ and $u \in U$ with $a = ud$ there is $v \in U$ that satisfies $u \wedge v = 1, b = vd$.

A B-act A is *divisible* if $uA = A$ for each non-zero-divisor $u \in U$. A is called *cyclic* if $A = Ua$ holds for some $a \in A$, called a *generator* of A. A *U-act map* or *morphism* $f : A \to C$ between two acts A and C over U is a lattice morphism satisfying $f(ua) = uf(a)$ for all $u \in U, a \in A$. In particular, a B-act morphism between cyclic B-acts over U is obviously a multiplication by an appropriate element of U.

Remark 3.2. Although the same symbol 0 is used for the greatest element in different structures, there is no confusion of its meaning in particular cases.

In view of Definition 3.1 Z is a *divisible* B-act over $T^\bullet$. In particular, Z is a *waist filter* of S in the sense that every principal filter either contains Z or is a subset of Z. On the other hand, if Z is any divisible B-act over $T^\bullet$ where T is a positive cone of an abelian lattice-ordered group, then the disjoint union $S = T \dot\cup Z$ becomes a B-monoid with the smallest m-prime filter Z with respect to the extended multiplication by putting $Z^2 = 0$. This B-monoid S is called the *trivial extension* of T by Z (see Example 1.9 [7].) It is important to emphasize the divisibility of Z which makes all elements of T smaller than those of Z. This means that the trivial extension of a B-monoid S by an S-act A can be defined without difficulty by taking the disjoint union of S and M after identifying their greatest elements 0 and obviously extending the multiplication; but this new monoid is, in general, not a B-monoid. Conversely, every B-monoid with one minimal m-prime filter M satisfying $M = Z = \{s \in S \,|\, \exists t \in T = S \setminus M : ts = 0\}$, is clearly a trivial extension of T by a B-act $Z = M$ over $T^\bullet$. Proposition 3.6 can be sharpened as

Proposition 3.7. *If $x, y \in S$ such that $x_\sigma < y_\sigma$, then $x < y$.*

Proof. Put $u = x \wedge y = u(x_1 \wedge y_1), x = ux_1, y = uy_1, x_1 \wedge y_1 = 1$. Then one of x_1, y_1 is in M otherwise we have $y_\sigma = x_\sigma \vee y_\sigma = (x \vee y)_\sigma = (ux_1y_1)_\sigma = u_\sigma = x_\sigma$, a contradiction. On the other hand, one of x_1, y_1 is not in M by $x_1 \wedge y_1 = 1$. Thus, either $x_1 = 1$ or $y_1 = 1$. The latter is impossible because $y = uy_1 = u$ would imply $y_\sigma = u_\sigma = x_\sigma$, a contradiction. Thus $x_1 = 1$ and hence $x < y$. □

The following weak cancellation property is important in the structural study of B-monoids with one minimal m-prime filter.

Proposition 3.8. *Let $x \in M \setminus Z$, i.e., $x^\perp \subseteq M$. For any $s, t \in T$ there is precisely one $y \in M$ satisfying $tx = sy$. Moreover, an equality $tx = sx$ for $t, s \in T$ implies $t = s$. In particular, $xy = y \notin Z$ implies $x = 1$.*

Proof. Assume indirectly $tx = sy_1 = sy_2$ and $y_1 \neq y_2$. If $v = y_1 \vee y_2 > u = y_1 \wedge y_2 \in M$, then $v = uz$ with $1 \neq z$. Furthermore, we have $su = sv = (su)z = tx \neq 0$. Therefore $(su)^\perp = (tx)^\perp \subseteq M$. The filter generated by $z \wedge M$ is proper because either $z \in M$ or $M \subseteq Sz$. Hence this filter can be extended to an m-prime filter F. In the localization S_F of S at F we have $(su)_\varphi z_\varphi = (su)_\varphi = (tx)_\varphi \neq 0$ with $z_\varphi \neq 1$, a contradiction. For the second assertion, Proposition 2.3 implies $t \wedge y = s \wedge y$ for some $y \in x^\perp$. Since $y < s$ and $y < t$ by Proposition 3.7, one has $t = s$. □

For every element $s \in T$ we have $M \subseteq sS$. Hence for any element $x \in M$ there is an element $y \in M$ with $x = sy$. Therefore, one can define the action of G on the distributive lattice $N = M \setminus Z$ by putting $gx = y$ for every $g = s^{-1}t \in G$ and $x \in N$ if $tx = sy$. This definition is well-defined, i.e., y is independent of the representation of g in the form $g = s^{-1}t$. For, if $g = s^{-1}t = u^{-1}v$ with $s,\ t,\ u,\ v \in T$, then $ut = sv$ holds and the equalities $tx = sy$, $vx = uz$ imply $y,\ z \in N$ and $svy = tvx = tuz = svz$ showing $y = z$ by Proposition 3.7. It is obvious that for each $a \in N$ the orbit of a is just S_a. Moreover, for any $y = gx = s^{-1}tx$, $a \in N$ we have $ta = sb$ for $b = ga \in N$. By putting $d = s \wedge t = d(s_1 \wedge t_1)$, $s = ds_1$, $t = dt_1$, $s_1 \wedge t_1 = 1$ we obtain $g = s^{-1}t = (ds_1)^{-1}(dt_1) = s_1^{-1}t_1$. Therefore one can assume without loss of generality that $g = s^{-1}t$ with $s \wedge t = 1$. Consequently we have

$$(gx)a = ya = ya(s \wedge t) = yas \wedge yat = txa \wedge yat =$$
$$= sbx \wedge sby = sbx \wedge btx = bx(s \wedge t) = xb = x(ag).$$

Note that an equality $(gx)a = g(xa)$ with $a, x \in N$ does in general, not hold. The reason is that although the left hand side is always meaningful, the right hand side is not defined for the case when $0 \neq xa \in Z$. In fact, it is impossible to define an action of G on Z in the case $Z \neq 0$. For example, there are B-monoids S with $Z \neq 0$ and some elements $a, x \in N$, $g \in G$ such that $0 \neq (ga)x = a(gx) \in Z$, but $ax = 0$. However, the equality $(gx)a = x(ag)$ always holds, as we have already seen above. Furthermore, each $x \in S_a$ defines a bijective map $\jmath_x : S_a \longrightarrow G$, sending $y = gx \in S_\alpha$ (i.e., y satisfies $tx = sy$ if $g = s^{-1}t$) to $g \in G$. We summarize these results in the following theorem.

Theorem 3.9. *For a B-monoid S with one minimal m-prime filter M put $N = M \setminus Z$. The quotient group G of $T = S \setminus M$ acts on N by putting $ga = b$ for $a \in N$, $g = s^{-1}t$ if $ta = sb$. The orbit of $a \in N$ is just S_a. Each $x \in S_a$ induces the lattice isomorphism*

$$\jmath_x : S_a \longrightarrow G : y = gx \in S_a \mapsto \jmath_x(y) = g.$$

Moreover, for every $g \in G$; a, $b \in N$ the action of G on N satisfies the equality

$$(ga)b = a(gb).$$

If $ab \notin Z$, then $(ga)b = g(ab) = a(bg)$.

The already fixed transversal $\{a_\alpha,\ \alpha \in \Sigma\}$ immediately defines a factor set $\mathfrak{f} : \Sigma \times \Sigma \longrightarrow G \,\dot\cup\, 0$ by putting $\mathfrak{f}(\alpha, \beta) = 0$ if $\alpha\beta = 0 \in \Sigma$, or equivalently $a_\alpha a_\beta \in Z$ and requiring $a_\alpha a_\beta = \mathfrak{f}(\alpha, \beta)a_{\alpha\beta}$ for the case $0 \neq \alpha\beta \in \Sigma$, or equivalently $a_\alpha a_\beta \notin Z$. If $\mathfrak{h}$ is a factor set induced by another normalized transversal $\{b_\alpha \mid \alpha \in \Sigma\}$, then for each element $\alpha \in \Sigma^\star$ there is a uniquely determined element $\mathfrak{g}(\alpha) \in G$ such that $b_\alpha = \mathfrak{g}(\alpha)a_\alpha$. If in addition we put $\mathfrak{g}(1) = 1$, $\mathfrak{g}(0) = 0$ we obtain the function $\mathfrak{g} : \Sigma \longrightarrow G^\bullet$. For $\alpha\beta \neq 0 \in \Sigma$ the equalities

$$\mathfrak{h}(\alpha, \beta)\mathfrak{g}(\alpha\beta)a_{\alpha\beta} = \mathfrak{h}(\alpha, \beta)b_{\alpha\beta} = b_\alpha b_\beta = g(\alpha)g(\beta)a_\alpha a_\beta =$$
$$= g(\alpha)g(\beta)\mathfrak{f}(\alpha, \beta)a_{\alpha\beta},$$

imply

$$\mathfrak{h}(\alpha, \beta) = \frac{\mathfrak{g}(\alpha)\mathfrak{g}(\beta)}{\mathfrak{g}(\alpha\beta)}\mathfrak{f}(\alpha, \beta),$$

hence $\mathfrak{f}$ and $\mathfrak{h}$ are associated. Therefore we have proved, in every detail, the next theorem.

Theorem 3.10. *The Rees factor R of S by Z is isomorphic to the generalized crossed product $T \underset{\mathfrak{f}}{*} \Sigma$ of T by Σ with the factor set $\mathfrak{f}$ induced by a normalized transversal of Σ in S. Conversely, if P is the positive cone of a lattice-ordered group G, Δ is a local B-monoid whose non-unital elements are nilpotent, and $\mathfrak{f}$ is a factor set of Δ in $G^\bullet$, then the crossed product $C = P \underset{\mathfrak{f}}{*} \Delta$ is a B-monoid having one minimal m-prime filter $M_C = C \setminus P$, the localisation C_δ of C at M_C satisfies $C_\delta \cong \Delta$, and every element of $a \in P \subseteq C$ is a non-zero-divisor.*

Remark 3.3. The Rees factor of S by Z is exactly the factor of S by the filter Z in view of Propositions 2.4 and 3.6.

The following result provides some more information about the structure of Z.

Proposition 3.11. *Let S be a B-monoid with the smallest m-prime ideal M. Let $Z = \{x \in S \mid \exists s \notin M : sx = 0\}$, $T = S \setminus M$. If there is $u \in M$ such that $su \neq 0$ for all $s \in T$, or equivalently $\Sigma \neq 1^\bullet$, then Z is a factor of the quotient group G of T by an appropriate filter F of G: $g, h \in G$ map to the same element of Z iff there is an element $k \in F$ satisfying $g \wedge k = h \wedge k$.*

Proof. The case $Z = 0$ is obvious. Thus without loss of generality one can assume that $Z \neq 0$. By the assumption as well as by Propositions 3.5, 3.6 there are two elements $a, b \in M \setminus Z$ with $0 \neq ab \in Z$. In view of Propositions 3.5 and 3.7 we have $ac = 0$ if $c_\sigma > b_\sigma$, and $aS_b = Z$ from which the assertion follows immediately by Proposition 2.3. □

The above statement suggests the divison of B-monoids with one minimal m-prime filter into three types.

Definition 3.4. Let S be a B-monoid with one minimal m-prime filter M and consider $Z = \{x \in S \mid \exists\, s \notin M :\ sx = 0\} \subseteq M$. S is said to be of

(1) *type I* if $Z = 0$,
(2) *type II* if $0 \neq Z = M$, and
(3) *type III* if $0 \neq Z \neq M$.

By definition S is of type II if every element of the least minimal m-prime filter $M \neq 0$ is annihilated by some element in $T = S \setminus M$. This is equivalent to saying that Σ is just the trivial B-monoid $1^\bullet$. By Proposition 3.5, $M^2 = MZ = 0$. Therefore M can be considered as a divisible B-act over $T^\bullet$ and S is a trivial extension of T by $M = Z$. Therefore every B-monoid with one minimal m-prime ideal of type II can be obtained in this manner. The structure of B-monoids with one minimal m-prime filter of type II can be described more precisely with the help of direct limits as follows. First note that the set $I = M^\star = Z^\star$ of nonzero elements of $M = Z$ can be identified with the set of all nonzero cyclic acts $Sa = Ta, 0 \neq a \in M$ and the partial order given by set-theoretic inclusion on I is just the reverse order of the original ordering on I. I is indeed a down-directed set and for any $a \leq b$ in I there is $t^a_b \in T$ with $at^a_b = b$. Therefore by assigning to each $a \in I$ the factor B-act T_a of T via $t \sim s \iff \exists u \in a^\perp \cap T : t \wedge u = s \wedge u$ and the B-act map $\tau^a_b : T_b \to T_a$ given by multiplication with t^a_b for any pair $a \leq b, b = bt^a_b, t^a_b \in T$ one obtains a direct system of B-acts over $T^\bullet$ whose limit is canonically isomorphic to M. In fact, the element $a \in I$ is identified with the equivalence class of the image of $1 \in T$ in T_a. We have therefore the following description of type II.

Proposition 3.12. *If S is a B-monoid with one minimal m-prime filter M of type II, then there is a direct system of nonzero cyclic B-acts $T_a, 0 \neq a \in M$ over $T^\bullet$ together with the injective B-act maps $\tau^a_b : T_b \to T_a$ given by multiplication with elements $t^a_b \in T$ such that $b = at^a_b$ for any pair of elements $a \leq b = at^a_b$ and S is isomorphic to the trivial extension $T \dot\cup \varinjlim T_a$ sending M to $\varinjlim T_a$. In particular, every B-monoid with one minimal m-prime filter of type II can be obtained in this manner.*

For type II we do not know if Z is also a factor (as a B-act over T) of the quotient group G of T. For type III we need the following preparation.

Proposition 3.13. *Let S be of type III, that is, $0 \neq Z = \{x \in S \mid \exists s \notin M : sx = 0\} \neq M$. Then for each $\alpha \in \Sigma^\star$ there is exactly one $\alpha_i \in \Sigma^\star$ such that for every $a \in S_\alpha$ there is $b \in S_{\alpha_i}$ with $0 \neq ab \in Z$. In particular, the assignment $\imath : \alpha \in \Sigma^\star \mapsto \alpha_i \in \Sigma^\star$ is a bijection of $\Sigma^\star$ satisfying $\imath^2 = \imath$.*

Proof. Let $a \in S_\alpha$ be an arbitrary element. Then $a \in N = M \setminus Z$ holds. By Proposition 3.6 $a < z$ for any non-zero element $z \in Z$, hence there is $b \in S$ with $ab = z \neq 0$. By Proposition 3.5, $b \notin Z$. On the other hand, if $b \notin M$, then $b_\sigma = 1$ and thus $0 = (ab)_\sigma = a_\sigma b_\sigma = a_\sigma \neq 0$, a contradiction. Consequently we have $b \in M$, hence $\alpha_i = b_\sigma \in \Sigma^\star$. If $\alpha_i < \beta$ and $c \in S_\beta$ is an arbitrary element, then by Proposition 3.7 there is $x \in M$ with $c = bx$, from which $ac = abx = 0$ holds by Proposition 3.5. Thus $ac = 0$ for all $a \in S_\alpha$ and $c \in S_\beta$ if $\alpha_i < \beta$. If $\beta < \alpha_i$ and $c \in S_\beta$, then by the same argument as above one obtains $ac \notin Z$, otherwise $ab = 0$ results which is impossible by the choice of b. Thus we have shown the uniqueness of α_i which completes the proof if we interchange the roles of α and α_i. □

As an immediate consequence of the above proof we obtain

Corollary 3.14. *Let $\Sigma^\blacklozenge = \Sigma \,\dot\cup\, \infty$ be the commutative extension of Σ by the greatest nonzero element ∞ subject to $\alpha\alpha_i = \infty$ and $\alpha\infty = 0 = \alpha\beta$ for all $\alpha \in \Sigma$, $\beta \in \Sigma, \alpha_i < \beta$. Then $\Sigma^\blacklozenge$ is a local B-monoid and its Rees factor by the filter $\{\infty, 0\}$ is Σ.*

For the sake of simplicity we will write $\jmath_\alpha$ instead of $\jmath_{a_\alpha}$ for maps $\jmath_x$ defined in Theorem 3.9. Note that $\{a_\alpha,\, \alpha \in \Sigma\}$ is the transversal of Σ in S already under consideration throughout this section. For each $\alpha \in \Sigma^\star$ we define, by using the bijection $\imath : \alpha \mapsto \alpha_i$ established in Proposition 3.13, the G-pairing

$$\mathfrak{m}_\alpha : S_\alpha \times S_{\alpha_i} \longrightarrow G : (a, b) \mapsto \mathfrak{m}_\alpha(a, b) = \jmath_\alpha(a)\jmath_{\alpha_i}(b) \in G$$

and the T-act epimorphism

$$\mathfrak{c}_\alpha : G \longrightarrow Z : g \in G \mapsto a_\alpha(g a_{\alpha_i}) \in Z.$$

In view of Propositions 3.13 and 2.3 one can see immediately that factors of G by the congruences $g \sim_\alpha h \iff \exists k \in C_\alpha : g \wedge k = h \wedge k$ are isomorphic to Z as B-acts over $T^\bullet$ where $C_\alpha = \{g \in G | \mathfrak{c}_\alpha(g) = 0\}$. By Theorem 3.9, for any $\alpha \in \Sigma^\star$ and arbitrary elements $a \in S_\alpha$, $b \in S_{\alpha_i}$ it follows that

$$\mathfrak{c}_\alpha(\mathfrak{m}_\alpha(a, b)) = \mathfrak{c}_\alpha(\jmath_\alpha(a)\jmath_{\alpha_i}(b)) = a_\alpha[\jmath_\alpha(a)(\jmath_{\alpha_i}(b)a_{\alpha_i})] = a_\alpha(\jmath_\alpha(a)b) = (a_\alpha \jmath_\alpha(a))b = ab \in Z.$$

If $\alpha < \beta; \alpha, \beta \in \Sigma^\star$, then $\beta = \alpha\gamma$ for some $\gamma \in \Sigma^\star$ and so $\alpha_i = \gamma\beta_i$ holds. Hence

$$a_\alpha a_\gamma = \mathfrak{f}(\alpha, \gamma) a_\beta \quad \& \quad a_{\beta_i} a_\gamma = \mathfrak{f}(\beta_i, \gamma) a_{\alpha_i}.$$

To aid the exposition, we write

$$h_\beta = \mathfrak{f}(\alpha, \gamma)^{-1} \mathfrak{f}(\beta_i, \gamma).$$

One has for each $g \in G$, the following equalities

$$\mathfrak{c}_\beta(g) = a_\beta(g a_{\beta_i}) = (\mathfrak{f}(\alpha, \gamma)^{-1} a_\alpha a_\gamma)(g a_{\beta_i}) = a_\alpha(\mathfrak{f}(\alpha, \gamma)^{-1} g a_\gamma a_{\beta_i}) = a_\alpha(\mathfrak{f}(\alpha, \gamma)^{-1} \mathfrak{f}(\beta_i, \gamma) g a_{\alpha_i}) =$$

$$= a_\alpha(h_\beta g a_{\alpha_i}) = \mathfrak{c}_\alpha(h_\beta g)$$

whence $\mathfrak{c}_\beta = \mathfrak{c}_\alpha h_\beta$ and $\mathfrak{c}_\alpha = \mathfrak{c}_\beta \mathfrak{f}(\alpha,\gamma)\mathfrak{f}(\beta_i,\gamma)^{-1}$ holds for all $\alpha < \beta; \alpha, \beta \in \Sigma^\star$. Therefore, for arbitary elements $\alpha, \beta \in \Sigma^\star$ one has

$$(*) \qquad \mathfrak{c}_\beta = \mathfrak{c}_\alpha h_\beta \text{ where } h_\beta = \begin{cases} \mathfrak{f}(\alpha,\gamma)^{-1}\mathfrak{f}(\beta_i,\gamma) & \text{if } \alpha < \beta = \alpha\gamma \\ \mathfrak{f}(\alpha_i,\gamma)^{-1}\mathfrak{f}(\beta,\gamma) & \text{if } \beta < \alpha = \beta\gamma \end{cases}$$

The above considerations allow us to define a new B-monoid $\mathfrak{S}$ with one minimal m-prime filter of type I having the epimorphic image S as follows. Consider the disjoint union $\mathfrak{S} = S \dot\cup S_\infty$ where $S_\infty = G$, and define $S_\beta S_\infty = 0$ for all $\beta \in \Sigma^\star$. For $s \in T = S_1$ and $g \in S_\infty$ define their product as the usual group product $sg \in S_\infty$. We fix now one (arbitrarily chosen) element $\alpha \in \Sigma^\star$ and define for each $\beta \in \Sigma^\star$ and arbitrary elements $x \in S_\beta$, $y \in S_{\beta_i}$ the (commutative) product $xy = h_\beta \jmath_\beta(x)\jmath_{\beta_i}(y)$ according to the equality $(*)$. Note that $S_\beta S_\gamma = 0$ for all $\gamma > \beta_i$ and $xy \in S \setminus Z$ for all $x \in S_\beta$, $y \in S_\gamma$ with $\gamma < \beta_i$. Furthermore, T is the set of all non-zero-divisors of $\mathfrak{S}$ and the localization of $\mathfrak{S}$ at T is $\Sigma^\blacklozenge = \Sigma \dot\cup \infty$ with the greatest non-zero element ∞. Moreover, for this fixed element $\alpha \in \Sigma^\star$, the factor of $\mathfrak{S}$ by the filter $C_\alpha = \{g \in G = S_\infty \,|\, \mathfrak{c}_\alpha(g) = 0\}$ is S. Note that this factor is in general, not a Rees factor unless G is a totally ordered abelian group. Thus we have shown that B-monoids with one minimal m-prime filter of type III are factors of particular B-monoids with one minimal m-prime filter of type I. Therefore one can describe a structure of B-monoids with one minimal m-prime filter as follows.

Theorem 3.15. *Let S be a B-monoid with one minimal m-prime filter M, $T = S \setminus M$ and $Z = \{x \in S \,|\, \exists s \notin M : sx = 0\}$. Let Σ be a localization of S at M. Then S belongs exactly to one of the following three cases.*

(1) $Z = 0$. *Then S is a crossed product of T with Σ.*
(2) $Z = M$. *Then $S \cong T \ltimes \varinjlim\{T_a,\, t^a_b\}$ where the latter term is a divisible direct limit of factors T_a of T by $x \cong y \iff \exists z \in L_a = a^\perp \cap T : x \wedge z = y \wedge z$ given by a down-directed set $I = M^\star = \{0 \neq a \in M\}$ with elements $t^a_b \in T$ such that $b = at^a_b$ for all $a < b$ in I.*
(3) $0 \neq Z \neq M$. *Then Σ is the factor of the local B-monoid $\Sigma^\blacklozenge$ extended by the greatest nonzero element ∞. Moreover S is the factor of the B-monoid $\mathfrak{S}$ with one minimal m-prime filter of type I by a filter contained in S_∞.*

4. A representation theorem

The aim of this section is to prove the following theorem, main result of this note.

Theorem 4.1. *The divisibility theory of an arithmetical ring with one minimal prime ideal is a Bezout monoid with one minimal m-prime filter. Conversely, every Bezout monoid with one minimal m-prime filter is order-isomorphic to the divisibility theory of an appropriate Bezout ring.*

Proof. Let R be an arithmetical ring with one minimal prime ideal I and $S(R)$ its monoid of divisibility. It is shown in [1] Proposition 1.1 that $S(R)$ is a B-monoid, and hence $S(R)$ has only one minimal m-prime filter F consisting of principal ideals contained in I, as is easy to check. For the sufficiency, let S be an arbitrary B-monoid with one minimal m-prime filter M and K an arbitrary field. As a first step we need

Lemma 4.2 (cf. Gauss' Lemma on primitive polynomials)**.** *Let R be the 0-contracted monoid algebra of S over K, i.e., R consists of all linear combinations of nonzero elements of S with non-zero coefficients from K where the zero elements of both S and K are identified. Then the set of primitive elements of R is multiplicatively closed.*

Remark 4.1. $\sum_{i=1}^{i=n} k_i s_i \in R\,(0 \neq k_i \in K^\star,\ i = 1, ..., n)$ is called *primitive* if $\wedge_{i=1}^n s_i = 1$. Primitive elements are in general not regular, that is, they are not necessarily non-zero-divisors. For example, if $st = t \neq 0$; $t,\ s \in S$, then $1 - s$ is obviously primitive and a zero-divisor by $t(1-s) = 0$. Let $\mathfrak{P}$ be the monoid of principal ideals of the ring $\mathbb{Z} \ltimes \mathbb{Q}/\mathbb{Z}$, i.e., $\mathfrak{P}$ is generated by $s_p,\ t_{p^{-n}}$; $n \in \mathbb{N}$, $p \in \mathbb{P} = \{\text{prime numbers}\}$ subject to $s_p t_p = t_p t_q = 0\,\forall\, p,\ q \in \mathbb{P}$ and $s_p t_{p^{-n}} = t_{p^{n-1}},\ s_p t_q = t_q\ \forall\, p \neq q \in \mathbb{P}$; $n > 1$. $\mathfrak{P}$ is a B-monoid with one minimal m-prime filter M generated by all $t_{p^{-n}}$ and every element of $\mathfrak{P}$ different from 0 and 1 is a zero-divisor! Thus in the 0-contracted monoid algebra $K\mathfrak{P}$ there are many primitive elements which are zero-divisors! For more on the important role of Gauss' Lemma in valuation theory, we refer to the forthcoming paper [3]

Proof. Since $M \in sS$ for all $s \notin M$, an element $r = \sum_{i=1}^{i=n} k_i s_i$ is primitive if and only if the image of r in the monoid algebra KT, $T = S \setminus M$ is also primitive. By this observation, the assertion follows immediately from the well-known corresponding statement for lattice ordered groups, see for example [9], Theorem 8.1. For another short and conceptual proof of this important result, see [3]. □

We are now in position to verify the main result. We have to provide a representation of S as a monoid of divisibility of a Bezout ring in three cases according to the type of S in view of Theorem 3.15.

Case 1: S is of type I. Let R be a 0-contracted monoid algebra of S over K. We claim that primitive elements of R are non-zero-divisors. Consider an arbitrary primitive element $a = \sum_{i=1}^{i=n} k_i s_i$, $(k_i \in K^\star,\ 1 \leq n \in \mathbb{N})$. Since products of primitive elements are again primitive by Lemma 4.2 and each element of R can be written as a product of a primitive element with a monoid element, it is enough to see $as \neq 0$ for any $s \in S^\star$. If $s \in T$, then $sx \neq sy$ for any two different elements x, y of S because S is of type I. Thus $as \neq 0$ for all $s \notin M$. Assume $0 \neq s \in M$. Write $a = a_1 + a_2$ where a_1 and a_2 are linear combinations of the $k_i s_i$ with $s_i \notin M$ and $s_i \in M$, respectively. Since a is primitive, $a_1 \neq 0$ whence $sa_1 \neq 0$ in view of Proposition 3.7. Observing that $(ss_i)_\alpha = s_\alpha$ for $s_i \notin M$ and $ss_i > s$ for $s_i \in M$, one gets $as \neq 0$. Thus every primitive element of R is a non-zero-divisor. Let A be the localization of R at the set of all primitive elements. It is clear that the monoid of principal ideals of A is isomorphic to S and S can be considered as a submonoid with respect to the multiplication of A.

Case 2: S is of type III. By Theorem 3.15 S is a factor of the B-monoid $\mathfrak{S}$ with one minimal m-prime filter of type I by a filter C contained in S_∞. According to Case 1 there is a Bezout ring R containing a submonoid $\mathfrak{S}$ such that principal ideals of R are exactly those generated by elements of $\mathfrak{S}$. The factor ring A of R by the ideal generated by elements of C obviously has divisibility theory isomorphic to S.

Case 3: S is of type II. We will use the notation and the description of S given in Theorem 3.15 (2). Let $T = S \setminus M$ and B be the localisation of the monoid algebra KT at the set of all primitive elements. Then B is a Bezout domain whose divisibility theory is $T^{\bullet}$. For each $0 \neq a \in M$ let B_a be the cyclic factor of B by the ideal generated by $t \in a^{\perp}$. Multiplication by $t_b^a \in T \subseteq KT$ induces an injective module homomorphism $t_b^a : B_b \longrightarrow B_a$ for any two non-zero elements $a \leq b = t_b^a b$ of M. Let C be the direct limit of the B-modules ${}_B B_a$. Then it is routine to check that ${}_B C$ is divisible, and the lattice of B-submodules of C is isomorphic to $M = Z$. Let Q be the trivial extension of B by C, then the divisibility theory of Q is just S. □

The above proof suggests a broader and sharper version of Kaplansky's original question on valuation rings as follows.

Kaplansky's problem. Describe all factors of Bezout domains or more generally, all factors of semi-hereditary Bezout rings as well as of semi-hereditary Bezout monoids. In particular, one can search for the description of all factors having one minimal prime ideal (m-prime filter). For details on the original problem of Kaplansky, we refer to the book [7].

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