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FUNCTIONAL ANALYSIS

Handwritten lecture notes based on chapters I, II, III, VI, VII and Suppl. of

Michael Read, Barry Simon: *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis* (Revised and enlarged edition), Academic Press, Inc. London, New York etc, 1980

2. Hilbert spaces: * inner products and the geometry of Euclidean spaces * Schwarz's and Bessel's inequalities * the Riesz lemma * orthonormal bases and Fourier series * applications to basic ergodic theory: measure preserving transformations, Koopman's representation, von Neumann's ergodic theorem *

Inner product spaces

vector spaces (finite dim): linear prop (alg.)
metric prop (top.)
$\boxed{\text{geometric prop — given the notion of angle between vectors}}$

Def inner product V complex vector space, $x, y, z \in V$, $\alpha, \beta \in \mathbb{C}$

$$(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$$

(i) $(x,x) \geq 0$ & $(x,x) = 0$ iff $x = 0$

(ii) $(x, \alpha y + \beta z) = \alpha (x,y) + \beta (x,z)$

(iii) $(x,y) = \overline{(y,x)}$

Examples

(1) $V = \mathbb{C}^n$; $(x,y) = \sum_{i=1}^{n} \overline{x_i}\, y_i$

(2) $V = \mathcal{C}([a,b], \mathbb{C})$, $(f,g) = \int_a^b \overline{f}(x)\, g(x)\, dx$

Def orthogonality: $x \perp y \iff (x,y) = 0$

orthonormal set $(x_\alpha)_{\alpha \in A}$ $\quad (x_\alpha, x_\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$

shorthand $\|x\| = \sqrt{(x,x)}$ "norm"

Th 1. (Pythagorean) $\{x_n\}_{n=1}^{N}$ finite o-n set in V

$(\forall x \in V)$:

$$\|x\|^2 = \sum_{n=1}^{N} |(x_n, x)|^2 + \left\| x - \sum_{n=1}^{N} (x_n, x) x_n \right\|^2$$

Proof

$$x = \underbrace{\sum_{n=1}^{N} (x_n, x)\, x_n}_{u} + \underbrace{\left(x - \sum_{n=1}^{N} (x_n, x)\, x_n \right)}_{v}$$

$$\|x\|^2 = \|u\|^2 + 2\,\mathrm{Re}\,(u, v) + \|v\|^2$$

$$\|u\|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} \overline{(x_n, x)}\, (x_m, x) \cdot (x_m, x_n) = \sum_{n=1}^{N} |(x_n, x)|^2$$

$$(u, v) = \sum_{n=1}^{N} \overline{(x_n, x)} \left(x_n, x - \sum_{m=1}^{N} (x_m, x)\, x_m \right) = 0$$

Problem 5

Corollary (Bessel's inequality): $\|x\|^2 \geq \sum_{n=1}^{N} |(x_n, x)|^2$

Corollary (Schwarz's inequality) $|(x, y)| \leq \|x\| \cdot \|y\|$

Proof. $N = 1$, $x_1 = \frac{y}{\|y\|}$

$$\|x\|^2 \geq \left| \left(\frac{y}{\|y\|}, x \right) \right|^2 = \frac{|(y, x)|^2}{\|y\|^2}$$

The parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Problem 4, think also about real inner prod space

theorem 2 $\|x\| = \sqrt{(x,x)}$ is indeed a norm (Schwarz $\Rightarrow$ triangle) HW: 2,4

thus: an inner prod space is a metric space

Def Complete inner prod space – Hilbert space $\mathcal{H}$.

Ex 1 is complete, Ex 2 is not

Examples problem 1

(2') $L_2[a,b] = \{[f] \mid \int_a^b |f(x)|^2 dx < \infty\}$

$$(f,g) = \int_a^b \bar{f}(x)\, g(x)\, dx$$

– is complete (adapt the proof of Riesz Fischer)

– $C[a,b]$ is dense in $L_2[a,b]$ Problem 2

(4) (M,μ) σ finite

$$L_2(M,\mu) = \{[f] \mid \int_M |f|^2 d\mu < \infty\}$$

$$(f,g) = \int \bar{f} g \, d\mu \qquad \text{Completeness: R-F.}$$

(3)

$$\ell_2 = \{\{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

$$(x, y) = \sum_{n=1}^{\infty} \bar{x}_n y_n$$

(actually the previous example with $M = \mathbb{N}$, $\mu(\{A\}) = \# A$)

⑤ direct sum $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces

$$\mathcal{H}_1 \oplus \mathcal{H}_2 = \{\langle x_1, x_2\rangle \in \mathcal{H}_1 \times \mathcal{H}_2 \mid (\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = (x_1, y_1)_1 + (x_2, y_2)_2\}$$

Countable direct sum: $\boxed{\text{problem 3}}$

$\mathcal{H}_n \quad n = 1, 2, \ldots$

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n = \{(x_1, x_2, \ldots) \mid x_i \in \mathcal{H}_i \cdot \sum_{n=1}^{\infty} \|x_n\|_n^2 < \infty\}$$

scalar product ..

⑥ vector valued functions $(M, d\mu)$, $\mathcal{H}$

$L_2(M, d\mu, \mathcal{H}) =$

$$\{f : M \to \mathcal{H} \text{ meas.} \mid \int_M \|f\|_{\mathcal{H}} \, d\mu < \infty\}$$

$$(f, g) = \int_M (f(x), g(x))_{\mathcal{H}} \, d\mu(x).$$

Isomorphism of Hilbert spaces

$U : \mathcal{H}_1 \to \mathcal{H}_2$ invertible:

$$(Ux, Uy)_{\mathcal{H}_2} = (x, y)_{\mathcal{H}_1}$$

$\{$ by polarization identity $\Leftrightarrow$ $\|Ux\| = \|x\|$. $\}$

Orthogonal projection, Riesz lemma

Def closed subspace : $\mathcal{M} \subset \mathcal{H}$
algebraically : subspace
topologically : closed

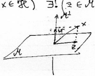
finite dim subspaces are always closed

orthogonal complement : $\mathcal{M}$ subspace

$$\mathcal{M}^\perp = \{ y \in \mathcal{H} \mid (\forall x \in \mathcal{M}) \ (x,y) = 0 \}$$

$\boxed{\text{Problem 6}}$ $\boxed{(\mathcal{M}^\perp)^\perp = \overline{\mathcal{M}}}$

Theorem 3 $\mathcal{M} = \overline{\mathcal{M}} \subset \mathcal{H}$
$(\forall x \in \mathcal{H}) \ \exists! \ (z \in \mathcal{M}, \ w \in \mathcal{M}^\perp) : \ x = z + w$



Lemma : $\mathcal{M} = \overline{\mathcal{M}}$ $\quad x \in \mathcal{H}$: $\exists! \ z \in \mathcal{M}$ closest to x.

Proof $d = \operatorname{dist}(x, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x - y\|$.

y_n : $\|x - y_n\| \longrightarrow d$

$$\|y_n - y_m\|^2 = \|(y_n - x) - (y_m - x)\|^2 \qquad \text{(Parallelogram law)}$$

$$= 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - \|y_n + y_m - 2x\|^2 =$$

$$= \underbrace{2\|y_n - x\|^2 + 2\|y_m - x\|^2}_{\downarrow \, 4d^2} - \underbrace{4\left\| \frac{y_n + y_m}{2} - x \right\|^2}_{\vee\!/ \, d^2}$$

~~Proof~~ of the Theorem: z the closest from $\mathcal{M}$ (according to Lemma)

$$w = x - z$$

we prove $w \in \mathcal{M}^\perp$: assume $(w, y) > 0$ for some $y \in \mathcal{M}$

$$\|x - \underbrace{(z + ty)}_{\in \mathcal{M}}\|^2 = \|(x-z) + ty\|^2 = d^2 - 2t\,(w, y) + t^2\|y\|^2$$

$< d^2$ for t small contradiction!

thus: $\mathcal{H} = \bar{\mathcal{M}} \oplus \mathcal{M}^\perp$ $\boxed{\text{uniqueness Problem 7}}$

Def $\mathcal{L}(\mathcal{H}, \mathcal{H}') = \{$bdd lin. ~~fr~~ transfos. from $\mathcal{H}$ to $\mathcal{H}'\}$

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|_{\mathcal{H}'} \quad \Big| \; \mathrm{Ker}\, T, \; \mathrm{Ran}\, T$$

$\mathcal{L}(\mathcal{H}, \mathbb{C}) = \mathcal{H}^* =$ dual space of $\mathcal{H}$

= bdd linear functionals

let $y \in \mathcal{H}$ (fixed)

$$T^{(y)} : \mathcal{H} \to \mathbb{C} \quad : \quad T^{(y)}x = (y, x)$$

linear, bdd (by Schwarz) $\Big\} \to T^{(y)} \in \mathcal{H}^*$

$$\|T^{(y)}\|_{\mathcal{H}^*} = \|y\|_{\mathcal{H}}$$

$x \mapsto (y, Ax)$ b.l.f.

y^* (y^*, Ax) $y^* = A^* y$

Converse : <u>The Riesz lemma</u> : $\forall T \in \mathcal{H}^* \ \exists!\, y_T \in \mathcal{H}$:

$$\forall x \in \mathcal{H} \qquad Tx = (y_T, x) \quad ; \quad \|y_T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}^*}$$

<u>Proof</u> assume $T \neq 0 \Rightarrow$

$$\operatorname{Ker} T = \{x \in \mathcal{H} \mid Tx = 0\} = \overline{\operatorname{Ker} T} \neq \mathcal{H}$$

$x_0 \in (\operatorname{Ker} T)^\perp \qquad x_0 \neq 0 \qquad$ (exists!)

claim $(\operatorname{Ker} T)^\perp = \{\alpha x_0 \mid \alpha \in \mathbb{C}\}$ one dimensional

why : for $x \in \mathcal{H}$ $\qquad x = \underbrace{\left(x - \frac{Tx}{Tx_0} \cdot x_0\right)}_{\in \operatorname{Ker} T} + \frac{Tx}{Tx_0} x_0$

let $x_1 \in (\operatorname{Ker} T)^\perp$

$x_1 = \left(x_1 - \frac{Tx_1}{Tx_0} x_0\right) + \frac{Tx_1}{Tx_0} x_0$

let $y_T = \frac{\overline{Tx_0}}{\|x_0\|^2} \cdot x_0$

$\forall x \in (\operatorname{Ker} T) \oplus (\operatorname{Ker} T)^\perp \qquad (y_T, x) = Tx$

Sesquilinear forms → $\boxed{\text{Corollary : problem 8}}$

$\boxed{\text{Problem 9}}$: Hahn Banach for Hilbert sp.

HW: 5, 9,

$\mathrm{I} : \mathcal{H} \to \mathcal{H}^*$ $\qquad \boxed{\text{antilinear isometry}}$ sesquilinear forms + adjoint

conjugate linear

Borel Probab Measures on $[0,1]$ $\quad \mu([0,1]) = 1$

$\mu([0,x]) = F(x)$

- monotone nondecr.
- continuous from right
- $F(1) = 1$.

$$\mu((a,b]) = F(b) - F(a)$$

$$\mu([a,b]) = \lim_{\varepsilon\to 0} \big(F(b) - F(a-\varepsilon)\big) \quad a>0$$

$$\mu((a,b)) = \lim_{\varepsilon\to 0} \big(F(b-\varepsilon) - F(a)\big)$$

$$\mu([a,b)) = \lim_{\varepsilon\to 0} \big(F(b-\varepsilon) - F(a-\varepsilon)\big) \quad a>$$

$$\mu(\{a\}) = \lim_{\varepsilon\to 0} \big(F(a) - F(a-\varepsilon)\big)$$

[illegible]

and vice versa: F determines μ

[illegible]

First: concrete examples on $[0,1]$ ← discrete / cont. sing

von Neumann's proof of { Radon-Nikodym & Lebesgue decomposition

$(M, \mathcal{F})$ measurable space ν, μ two measures

Def # ν, μ are mutually singular:

$$\exists A \in \mathcal{F} : \nu(A) = \mu(A^c) = 0 \qquad \nu \perp \mu$$

"ν and μ are completely concentrated on disjoint supports"

μ is absolutely continuous w.r.t. ν $(\mu \ll \nu)$

$$(\forall A \in \mathcal{F}) : \left(\nu(A) = 0\right) \to \left(\mu(A) = 0\right)$$

Radon-Nikodym Theorem: $\mu \ll \nu$ iff

$\exists$ measurable $\rho (\geq 0)$ such that $\forall A \in \mathcal{F}$

$$\mu(A) = \int \rho(x)\, \chi_A(x)\, d\nu(x)$$

f is uniquely determined μ - a.e.

Lebesgue decomposition: ν, μ two measures (ν = reference)

$$\mu = \mu_{ac} + \mu_{sing} : \quad \mu_{ac} \ll \nu, \quad (\nu, \mu_{sing} \text{ mut. sing})$$

the decomposition is unique

von Neumann's proof (by use of Riesz Lemma)

(J. von Neumann = Neumann János
Bp, 1903 — Wash., 1957)

Theorem (S1) $(M, \mathcal{F})$ μ, ν $\boxed{\text{finite}}$ measure.

$\exists A \in \mathcal{F}) : \nu(A^c) = 0$

$\rho \in L_1(M, d\nu)$ $\quad \rho \geq 0$

$(\forall B \subset A) \quad \mu(B) = \int_B \rho \, d\nu$

Remark $\mu_{ac}(\cdot) = \mu(\cdot \cap A)$

$\mu_{sing}(\cdot) = \mu(\cdot \cap A^c)$

Proof: $\alpha = \nu + \mu$ (finite!) $\quad \mathcal{H} = L_2(M, d\alpha)$ Real

$L \in \mathcal{H}^*$: $L(g) = \int_M g \, d\mu$

linear ✓

bdd: $|L(g)| = \left|\int 1 \cdot g \, d\mu\right| \leq \left(\int |g|^2 d\mu\right)^{1/2} \left(\int 1^2 d\mu\right)^{1/2}$

$\leq \left(\int |g|^2 d\alpha\right)^{1/2} \left(\int 1^2 d\alpha\right)^{1/2} \leq \|g\| \cdot \sqrt{\alpha(M)}$

$$\int g\,d\mu = \int g\,F\,dx \qquad \left|\ \text{in particular}\ \right| \qquad \mu(B) = \int_B F\,dx$$

$$\int g\,d\nu = \int g\,(1-F)\,dx \qquad\qquad \nu(B) = \int_B (1-F)\,dx$$

by Riesz: $\exists\, F \in \mathcal{H} = L_2(M, d\alpha)$:

$$L(g) = \int_M F \cdot g \, d\alpha \qquad \mu(B) = \int_B F\, d\alpha \;;\; \nu(B) = \int_B (1-F)\, d\alpha$$

claim: $0 \leq F \leq 1$ $\quad \alpha$ - a.e.

$$C_n = \{x \mid F(x) < -\tfrac{1}{n}\} \qquad C_\infty = \{x \mid F(x) < 0\} = \bigcup_{n=1}^{\infty} C_n$$

$$0 \leq \mu(C_n) = \int_{C_n} F\, d\alpha < -\tfrac{1}{n}\, \alpha(C_n)$$

$$\Rightarrow \alpha(C_n) = 0 \;;\; \alpha(C) = \lim \alpha(C_n) = 0$$

$$C_n = \{x \mid F(x) > 1 + \tfrac{1}{n}\}$$

$$0 \leq \nu(C_n) = \int_{C_n} (1-F)\, d\alpha < -\tfrac{1}{n}\, \alpha(C_n)$$

$$\Rightarrow \alpha(C_n) = 0.$$

Let: $A = \{x \mid 0 \leq F(x) < 1\}$; $A^c = \{x \mid F(x) = 1\}$

$$\rho(x) = \begin{cases} \dfrac{F(x)}{1-F(x)} & x \in A \\ 0 & x \in A^c \end{cases}$$

$\nu(A^c) = 0$ ✓ $\qquad \nu(A^c) = \int_{A^c} (1-F)\, d\alpha = 0.$

$B \subset A$:

$$\int_B \rho(x)\, \nu(dx) = \int_B \rho(x)\, (1 - F(x))\, d\alpha(x) = \int_B F(x)\, d\alpha(x)$$

$$= \int_B d\mu(x) = \mu(B)$$

$\rho \in L_1(M, d\mu)$ automatically

done

more about decomposition

$$\mu = \mu_{cont} + \mu_{pp}$$

on $[0,1]$

$$\mu = \mu_{ac} + \underset{\substack{\uparrow \\ \text{Devil's staircase (= Cantor's fct)}}}{\mu_{sing\,cont}} + \mu_{pp}$$

ON bases in Hilbert spaces: $\mathcal{O} = \{ S \mid S = $ o-n collection in $\mathcal{H} \}$ ordered by inclusion

Def O-N basis — maximal O-N collection (w.r.t. inclusion)

Thm $\mathcal{H}$ Hilbert sp, $\mathcal{H}$ has an O-N basis

Proof (juggling with Zorn's lemma)

$\mathcal{O} = \{ S : S = $ o-n set in $\mathcal{H} \}$ ordered by inclusion

$\{ S_\alpha : \alpha \in A \}$ linearly ordered $\Rightarrow \bigcup_{\alpha \in A} S_\alpha \in \mathcal{O}$

(i.e. $\forall$ linearly ordered subset has an upper bound in $\mathcal{O}$)

$\Rightarrow$ by Zorn $\exists$ max element in $\mathcal{O}$

Thm (why basis?) $\{ x_\alpha \}_{\alpha \in A}$ o-n basis in $\mathcal{H}$, $y \in \mathcal{H}$

then: $(x_\alpha, y) \neq 0$ for at most countably many α-s (call them $\alpha_1, \alpha_2, \dots$)

And

$$y = \lim_{n \to \infty} \sum_{i=1}^{n} \underbrace{(x_{\alpha_i}, y)}_{\text{Fourier coefficients}} x_{\alpha_i} = \sum_{i=1}^{\infty} (x_{\alpha_i}, y) x_{\alpha_i}$$

$$\|y\|^2 = \sum_{i=1}^{\infty} |(x_{\alpha_i}, y)|^2 \quad \text{(Parseval's formula)}$$

Conversely: given c_i $i = 1, 2, \dots$ $\sum |c_i|^2 < \infty$, α_i

then $\sum c_i x_{\alpha_i} = y \in \mathcal{H}$

Proof $A_n = \{ \alpha \in A \mid |(x_\alpha, y)| > \frac{1}{n} \}$

by Bessel $\cdot \|y\|^2 \geq \frac{1}{n^2} \# A_n \Rightarrow A_n$ is finite

$\{ \alpha \in A \mid |(x_\alpha, y)| \neq 0 \} = \bigcup_n A_n$

By Bessel : $\sum_{i=1}^{\infty} |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$

y $\quad y_n = \sum_{i=1}^{n} (x_{\alpha_i}, y) \, x_{\alpha_i}$ is Cauchy

why? : $\|y_n - y_m\|^2 = \sum_{i=n+1}^{m} |(x_{\alpha_i}, y)|^2$

thus $y_n \longrightarrow y'$

2) $y' = y$ why?

$(\forall \alpha \in A)$ $\quad (y' - y, x_\alpha) = 0$

(i) $\alpha = \alpha_\ell$: $\quad (\alpha_\ell, y') = (\alpha_\ell, y)$

(ii) $\alpha \neq \alpha_\ell$ $\quad (x_\alpha, y) = (x_\alpha, y') = 0$

but $\{x_\alpha\}$ is basis $\Rightarrow y - y' = 0$.

3) $\|y - y_n\|^2 = \|y\|^2 - \sum_{1}^{n} |(x_{\alpha_i}, y)|^2 \longrightarrow 0$.

Gram - Schmidt orthogonalization

Prop Given a countable lin. indep system $\{u_1, u_2, \ldots\}$ there is a unique (up to $\tilde{v}_k = e^{i\alpha_k} v_k$) o-n syste $\{v_1, v_2, \ldots\}$ such that

$(\forall n)$ span $\{u_1, \ldots, u_n\}$ = span $\{v_1, \ldots, v_n\}$

Procedure: orthogonalization — normalization

$w_1 = u_1$ $\qquad v_1 = \frac{w_1}{\|w_1\|}$

$w_2 = u_2 - (v_1, u_2) v_1$ $\qquad v_2 = \frac{w_2}{\|w_2\|}$

misprint in the book

$- - -$

$w_{k+1} = u_{k+1} - \sum_{1}^{k} (v_i, u_{k+1}) v_i$ $\qquad v_{k+1} = \frac{w_{k+1}}{\|w_{k+1}\|}$

(by linear independence you never get $\|w_k\| = 0$)

$\boxed{\text{probl. } 10}$: Legendre polynomials

Definition: separable metric space = it contains a countable dense subset

Theorem: $\mathcal{H}$ is separable $\iff$ $\mathcal{H}$ has an (at most) countable o-n b

Classification: $\dim(\mathcal{H}) = \# \{\text{o-n basis}\}$
of separable $\mathcal{H}$-s

$\dim \mathcal{H} = n < \infty \iff \mathcal{H} \cong \mathbb{C}^n$
$\dim \mathcal{H} = \infty \iff \mathcal{H} \cong l_2$ (isomorphism)

Proof. $(\Leftarrow)$ $\left\{ \sum_{i=1}^{N} c_i x_i \;\middle|\; N < \infty,\ c_i \in \mathbb{Q} + i\mathbb{Q} \right\}$
is dense and countable

$(\Rightarrow)$ $\{u_i\}_{i=1}^{\infty}$ lin indep
span$[\{u_i\}]$ is dense
apply Gram-Schmidt

Classification: $\{x_n\}_{n=1}^{\infty}$ o-n basis

$T: \mathcal{H} \to l_2$

$Ty = \left((x_1, y), (x_2, y), \dots \right)$ $l_{\mathbb{Z}}^2, L^2(\mathbb{R}), L^2((-1,1)) \dots$

All the natural problems are in separable $\mathcal{H}$-s
Then why do we need other Hilbert spaces?

more structure! $\boxed{\text{pr. 11}}$ ($L_2(\mathbb{R})$ is sep)

e.g. Δ on $l_2(\mathbb{Z}^2)$ $\boxed{\text{pr. 19}}$ construction of not sep $\mathcal{H}$

HW: 10, 11, 13, 25 see also $\boxed{\text{pr 25}}$

Fourier Series :

The Wave equation :

find $u : [a,b] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ $(\mathbb{C})$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$a = -\pi$
$b = \pi$

B.C : fixed endpoints $\longrightarrow$ (Dirichlet) $\quad u(a,t) = 0 = u(b,t) \quad \forall t$

free endpoints $\longrightarrow$ (Neumann) $\quad \frac{\partial u}{\partial x}(a,t) = 0 = \frac{\partial u}{\partial x}(b,t) \quad \forall t$

ring $\longrightarrow$ (periodic) $\quad u(a,t) = u(b,t) \quad \forall t$

I.C. $\begin{cases} u(x,0) = f(x) & x \in [a,b] \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$

interpretation

The Heat equation :

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \kappa > 0.$$

B.C same

I.C. $\quad u(x,0) = f(x).$

$[a,b] = [-\pi, \pi]$

~~solve~~ the heat equ. with periodic b.c.:

separation of variables: (forget about IC for the moment)

try. $u(x,t) = X(x) \cdot T(t)$ separation of variable

$$\frac{1}{\kappa} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad + \text{B.C.}$$

$$\frac{1}{\kappa} \cdot \frac{\dot{T}(t)}{T(t)} = -\lambda = \frac{X''(x)}{X(x)} \qquad \lambda \in \mathbb{C}$$

$$\dot{T}(t) = -\kappa \lambda T(t)$$

$$\begin{cases} X''(x) = -\lambda X(x) \\ X(-\pi) = X(\pi) \end{cases} \qquad X'' + \lambda X = 0$$

from general theory of ODE's:

two linearly indep solutions $X(x) = \exp \pm i \sqrt{\lambda} \cdot x$

Boundary cond: $\exp \pm 2\pi i \sqrt{\lambda} = 1$

$$\lambda_n = n^2$$

$$\boxed{e_n(x) = \frac{1}{\sqrt{2\pi}} \exp inx \qquad n \in \mathbb{Z}.}$$

$$T_n(t) = T_n(0) e^{-\kappa n^2 t}$$

linear superposition of solutions

$$u(x,t) = \sum_{n=-\infty}^{+\infty} b_n e^{-kn^2 t} \cdot \frac{1}{\sqrt{2\pi}} e^{inx} \text{ "}$$

watch the IC.

$$(u(x,0) =) \text{"} f(x) = \sum_{n=-\infty}^{\infty} b_n \cdot \frac{1}{\sqrt{2\pi}} e^{inx} \text{ "}$$

? what sense can we give to this?

$\{e_n\}_{n \in \mathbb{Z}}$ $\qquad e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$

is an ortho-normal $\boxed{\text{system}}$ of vectors in $L_2([-\pi,\pi], dx)$

$$b_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

is $\{e_n\}_{n \in \mathbb{Z}}$ a basis? YES $\quad$ %

Theorem (almost) (I.8) $\qquad$ $f : [-\pi,\pi] \to \mathbb{C}$ periodic, diff'able

then $\sum_{-M}^{M} b_n e_n(x) \longrightarrow f(x)$ pointwise as $M \to \infty$

Theorem II 8: $f : [-\pi, \pi] \to \mathbb{C}$ periodic
cont. diff.

$$f_N(x) = \sum_{-N}^{N} c_n \frac{1}{\sqrt{2\pi}} e^{inx}$$

$$f_N(x) \xrightarrow{\text{unif}} f(x)$$

Lemma (Riemann, Lebesgue)

φ_n o-n System in $L_2(-\pi,\pi)$
uniformly bdd $\sup_{-\pi\le x\le\pi} |\varphi_n(x)| \le K < \infty$.

$f \in L_1(-\pi,\pi)$

Then: $\int_{-\pi}^{\pi} f(x)\,\overline{\varphi_n}(x)\,dx \longrightarrow 0$

Proof of the Lemma: $L_2 \subset L_1$ densely

for $f \in L_2$ the Lemma is true by Bessel's ineq.

for $b < \infty$

$$f_b(x) = \begin{cases} f(x) & \text{if } |f(x)| \le b \\ 0 & \text{if } |f(x)| > b \end{cases}$$

1 step choose b so that $\int_{-\pi}^{\pi} |f - f_b|\,dx < \frac{\varepsilon}{2K}$

2. step choose n so that $(\varphi_n, f_b) < \frac{\varepsilon}{2}$

then

$$\left|\int \overline{\varphi_n} f\,dx\right| = \left|\int \overline{\varphi_n}(f - f_b)\,dx + \int \overline{\varphi_n} f_b\,dx\right| \le \varepsilon.$$

[redacted] of the theorem:

$$f_N(x) = \sum_{n=-N}^{N} c_n e_n(x) =$$

$$\sum_{n=-N}^{N} \frac{1}{\sqrt{2\pi}} \cdot \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{inx} =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{in(y-x)} f(y)\, dy$$

$$\sum_{n=-N}^{N} e^{-in(y-x)} = e^{iN(y-x)} \sum_{n=0}^{2N} e^{-in(y-x)} =$$

$$e^{iN(y-x)} \frac{1-e^{-i(2N+1)(y-x)}}{1-e^{-i(y-x)}} = \frac{e^{i(N+\frac{1}{2})(y-x)} - e^{-i(N+\frac{1}{2})(y-x)}}{e^{\frac{i}{2}(y-x)} - e^{-\frac{i}{2}(y-x)}}$$

$$= \frac{\sin\left(N+\frac{1}{2}\right)(y-x)}{\sin \frac{1}{2}(y-x)}$$

periodic of period 2π

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left(N+\frac{1}{2}\right)(y-x) \cdot \frac{f(y)}{\sin\frac{1}{2}(y-x)}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left(N+\frac{1}{2}\right)y \; \frac{f(x+y)}{\sin\frac{1}{2}y}\, dy$$

bdd.

$$f_N(x) - f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \sin\left(N+\frac{1}{2}\right)y \; \frac{f(x+y)-f(x)}{\sin\frac{1}{2}y}\, dy$$

II 9. f cont diff $\Rightarrow f_N \xrightarrow{L_2} f$

let $g \in L_2$; $(e_n, g) = 0 \ \forall n$

$\Rightarrow$ $(f, g) =$

$\lim_N (f_N, g) = 0$

as cont-diff are dense in L_2

$\Rightarrow$ $\boxed{g = 0.}$

Introduction to ergodic theory I.

Automorphisms of measure spaces : $(\Omega, \mathcal{F}, \mu, T)$

$(\Omega, \mathcal{F}, \mu)$ probability space

$T : \Omega \to \Omega$ invertible & measure preserving

$$(\forall F \in \mathcal{F}) : \ \mu(T^{-1}F) = \mu(F)$$

question : $\displaystyle \frac{1}{n} \cdot \sum_{k=0}^{n-1} f(T^k x)$

$$\int_\Omega f(x)\, d\mu(x)$$

how are they related (equal ?)

time averages $\overset{?}{=}$ phase space averages

motivation : stat. phys. (L. Boltzmann's "ergodic hypothesis"

applic : probability, number theory, etc.

Poincaré's recurrence theorem $\boxed{\text{problem IF}}$

$E \in \mathcal{F}$ $\quad F = \{x \in \Omega \mid \exists n_0 < \infty : \ n > n_0 \ T^n x \notin E\}$

$T^n x \in E$

$\mu(F) = 0$.

Remarks : long recurrence time

Lemma (a) U unitary: $Uf = f \iff U^*f = f$

(b) $(\operatorname{Ran} A)^\perp = \operatorname{Ker} A^*$, $\overline{\operatorname{Ran} A} = (\operatorname{Ker} A^*)^\perp$

Proof (a) U invertible: $Uf = f \iff U^{-1}f = f$

(b) $(x \in \operatorname{Ker} A^*) \iff (A^*x = 0) \iff$

$(\forall y \in \mathcal{H}: (A^*x, y) = 0) \iff (\forall y \in \mathcal{H}: (x, Ay) = 0)$

$\iff (x \in (\operatorname{Ran} A)^\perp)$ $\boxed{\text{see also problem 19}}$

Koopman's Lemma: $(\Omega, \mathcal{F}, \mu, T)$ automorphism

$\mathcal{H} = L_2(\Omega, \mu)$

$U \in \mathcal{L}(\mathcal{H})$: $[Uf](x) = f(Tx)$

U is unitary

Proof U is onto

$(Uf, Ug) = \int \bar{f}(Tx)\, g(Tx)\, d\mu(x) =$

$\int \bar{f}(x)\, g(x)\, d\mu(T^{-1}x) =$

$\int \bar{f}(x)\, g(x)\, d\mu(x) = (f, g)$

von Neumann's mean ergodic Theorem:

U unitary ; P proj onto $\mathrm{Ker}(U-I)$

$$\forall f \in \mathcal{H} : \quad \frac{1}{n} \sum_{0}^{n-1} U^k f \longrightarrow Pf$$

Proof $\mathcal{H} = \mathrm{Ker}(U-I) \oplus \left(\mathrm{Ker}(U-I)\right)^{\perp} = \mathrm{Ker}(U-I) \oplus \left(\mathrm{Ker}(U^{*}-\right.$

$$\mathrm{Ker}(U-I) \oplus \overline{\mathrm{Ran}(U-I)}$$

– $f \in \mathrm{Ker}(U-I)$; $Pf = f$, $U^k f = f$

$$\frac{1}{n} \cdot \sum_{0}^{n-1} U^k f = f = Pf$$

– $f \in \mathrm{Ran}(U-I)$ $\quad f = Ug - g \quad g \in \mathcal{H}$

$$Pf = 0$$

$$\frac{1}{n} \cdot \sum_{0}^{n-1} U^k f = \frac{1}{n}\left(U^n g - g\right) \longrightarrow 0$$

relevance to the question of ergodic theory:

$$\frac{1}{n} \cdot \sum_{0}^{n-1} f(T^n \cdot) \xrightarrow{L_2} \underbrace{Pf(\cdot)}$$

analogue for continuous time dynamics? $\boxed{\text{Probl. 18}}$

Definition: ergodicity: $(\Omega, \mathcal{F}, \mu, T)$ is ergodic iff
$$\left(T^{-1}F = F\right) \Rightarrow \mu(F) = 0 \text{ or } 1.$$

Examples ① $\Omega = S^1$, $\mathcal{F}$ Borel μ Lebesgue

$$T_\alpha x = x + \alpha \quad (\text{mod } 1)$$

- $\alpha \in \mathbb{Q} \Rightarrow$ not ergodic
- $\alpha \in \mathbb{R} \setminus \mathbb{Q} \to$ ergodic $\boxed{\text{Problem 20}}$

② $\Omega = S^1 \times S^1 = \mathbb{T}^2$ $\quad \alpha = (\alpha_1, \alpha_2)$

$$T_\alpha (x, y) = (\alpha_1 + x, \alpha_2 + y)$$

$\frac{\alpha_1}{\alpha_2} \in \mathbb{Q}$: not ergodic

$\frac{\alpha_1}{\alpha_2} \notin \mathbb{Q}$: ergodic

③ $\Omega = \{-1, 1\}^{\mathbb{Z}}$ $\quad \mu$ - Bernoulli

$$\mu(\omega_{i_1} = \epsilon_1, \ldots, \omega_{i_n} = \epsilon_n) = 2^{-n}$$

$$T(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) = (\ldots, \omega_0, \omega_1, \omega_2, \ldots)$$

is ergodic

WLLN : Consequence of mean erg Th.

SLLN : individual erg Th.

Prop. ergodicity $\Leftrightarrow$ [$Uf = f$ iff $f = \text{const}$]

$(\Rightarrow)$ assume $f \neq \text{const}$, $Uf = f$

$$\exists\, a : \mu\big(\underbrace{\{x \mid f(x) < a\}}_{\text{invariant}}\big) \neq 0, 1$$

$(\Leftarrow)$ not ergodic $\Rightarrow$ $\exists\, \underset{\text{invariant}}{A} : \mu(A) \neq 0, 1$

take $f = \chi_A$

$$\boxed{\text{Consequence: } T \text{ ergodic} \quad \frac{1}{n}\sum_{0}^{n-1} f(T^k\cdot) \xrightarrow{L_2} \int f\, d\mu}$$

Individual (Birkhoff - Khinchin) ergodic Theorem

$(\Omega, \mathcal{F}, \mu, T)$ auto. $f \in L_1(\Omega, \mu)$

$\exists\, f^* \in L_1$, f^* invariant :

- $\frac{1}{n}\sum_{0}^{n-1} f(T^k x) \xrightarrow{a.e.} f^*(x)$

If T is ergodic $f^*(x) = \int f\, d\mu$. = const

Example. 4. Billiards.
3 bs: LLH.

more about ergodic theory: after the spectral theorem.