Bálint Tóth
Insitute of Mathematics
Technical University Budapest

FUNCTIONAL ANALYSIS

Handwritten lecture notes based on chapters I, II, III, VI, VII and Suppl. of

Michael Read, Barry Simon: *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis* (Revised and enlarged edition), Academic Press, Inc. London, New York etc, 1980

4. Bounded linear oprators, part 1: * norm-, weak- and strong topologies on spaces of bounded linear operators * the adjoint operator * spectrum and resolvent * projections, unitary operators, positive oprators, polar decomposition *

VI $\boxed{\text{Bounded Operators}}$

$\mathcal{L}(X, Y) = \{ T : X \to Y : T \text{ linear \& Bounded} \}$ if $Y = X$ we write $\mathcal{L}(X)$

vector sp.

The $\boxed{\text{norm}}$ topology on $\mathcal{L}$: (or uniform topology)

$$\|T\| = \sup \frac{\|Tx\|_Y}{\|x\|_X}$$

$(\mathcal{L}, \|\cdot\|)$ is a Banach sp.

The $\boxed{\text{Strong operator top}}$: motivation

$x \in X \quad : \quad E_x : \mathcal{L}(X, Y) \to Y$

$$E_x(T) = Tx$$

the weakest topology with $\forall E_x$ Continuous

$$\mathcal{L} \ni T \mapsto Tx \in Y$$

neighbourhood base of $0 \in \mathcal{L}$

$H(x_1, \dots x_n; \varepsilon) = \{ S \mid \|Sx_i\| < \varepsilon \quad \text{for} \quad i = 1, \dots n \}$

The $\boxed{\text{weak operator topology}}$

$x \in X,\ \ell \in Y^*$

$E_{x,\ell} : \mathcal{L} \longrightarrow \mathbb{C} \ : \ E_{x\ell}(T) = \ell(Tx)$

neighbourhood base of $0 \in \mathcal{L}$:

$N(x_1, \dots, x_n; \ell_1, \dots, \ell_m; \varepsilon) = \{ S \in \mathcal{L} \mid (\forall i j) \ |\ell_j(Sx_i)| < \varepsilon \}$

$w\text{-top} \leq st\text{-top} \leq \|\cdot\|\text{-top}$ $\boxed{\text{problem 1}}$

Example on $\mathcal{L}(\ell_2)$

(1) $T_n(\xi_1, \xi_2, \dots) = (\frac{1}{n}\xi_1, \frac{1}{n}\xi_2, \dots)$

$T_n \xrightarrow{\|\cdot\|} 0.$ $\boxed{\text{Problem 2}}$

(2) $S_n(\xi_1, \xi_2, \dots) = (0, 0, \dots, 0, \xi_n, \xi_{n+1}, \dots)$

$S_n \xrightarrow{st} 0$ but $S_n \not\xrightarrow{\|\cdot\|} 0$

(3) $W_n(\xi_1, \xi_2, \dots) = (0, 0, \dots 0, \xi_1, \xi_2, \dots)$

$W_n \xrightarrow{w} 0$ but $W_n \not\xrightarrow{st} 0.$

Theorem: (i) $T_n \in \mathcal{L}(\mathcal{H})$: $(\forall x \in \mathcal{H})$ $T_n x$ cvg then $T_n \xrightarrow{st} T.$

(ii) $T_n \in \mathcal{L}(\mathcal{H})$ $(\forall x, y \in \mathcal{H})$ $(T_n x, y)$ cvg then $T_n \xrightarrow{w} T.$ $\boxed{\text{Problem 3}}$

only for sequences

Proof: uniform boundedness

More : $\boxed{\text{problem 5.}}$ $T_t\,\varphi(\cdot) = \varphi(\cdot + t)$ $\begin{matrix} L_2(\mathbb{R}, dx) \\ L_2(\mathbb{R}, e^{-x^2}dx) \end{matrix}$

$\boxed{\text{problem 6}}$ Continuity (?) of $\langle A, B\rangle \mapsto AB$ in various topologies

kernel : $\operatorname{Ker} T = \{ x \mid Tx = 0 \}$ closed

range : $\operatorname{Ran} T = \{ y \mid y = Tx \}$ not always closed

$\boxed{\text{problem 7}}$

Hilbert space adjoint

$$T^* \qquad (T^*x, y) = (x, Ty)$$

$$\left\{ \begin{aligned} (\operatorname{Ker} T)^\perp &= \overline{(\operatorname{Ran} T^*)} \\ \operatorname{Ker} T &= (\operatorname{Ran} T^*)^\perp \end{aligned} \right\}$$

The adjoint operator:

Def: X, Y Banach space, X^*, Y^* their duals

$$T \in \mathcal{L}(X,Y) \qquad T' \in \mathcal{L}(Y^*, X^*)$$

$$(T'\ell)(x) = \ell(Tx)$$

Theorem $T \mapsto T'$ is an isometric isomorphism of $\mathcal{L}(X,Y)$ into $\mathcal{L}(Y^*, X^*)$

Proof linearity straightforward

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\| \le 1} \|Tx\| \overset{\text{Hahn-Banach}}{=} \sup_{\|x\|\le 1} \left(\sup_{\|\ell\|\le 1} \ell(Tx) \right) =$$

$$= \sup_{\|\ell\|\le 1} \left[\sup_{\|x\|\le 1} (T'\ell)(x) \right] = \sup_{\|\ell\|\le 1} \|T'\ell\| = \|T'\|_{\mathcal{L}(Y^*, X^*)}$$

Example: $X = \ell_p = Y \qquad 1 \le p < \infty$

$X^* = \ell_q = Y^* \qquad 1 < q \le \infty$

$$T(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$$
$$T'(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots)$$

On $L_p(0,1)$: $(Tf)(x) = \int K(x,y) f(y)\,dy$, $(T'g)(x) = \int K(y,x) g(y)\,dy$

An example of $S \in \mathcal{L}(Y^*, X^*)$ which is not adjoint of any $T \in \mathcal{L}(X,Y)$: $X = \ell_1 = Y$; $X^* = \ell_\infty = Y^*$

$S(\xi_1, \xi_2, \dots) = S\xi = (1, 0, 0, \dots) \cdot \lambda(\xi)$ — λ extension of $\lim \xi$

Corollary of the closed graph thm.

Hellinger - Toeplitz theorem

$\mathcal{H}$; A everywhere defined linear op $\mathcal{H} \to \mathcal{H}$

$(\forall x,y \in \mathcal{H})$: $(x, Ay) = (Ax, y)$

then A is bounded

Proof we prove $\Gamma(A) \subset \mathcal{H} \oplus \mathcal{H}$ is closed

let $\langle x_n, Ax_n \rangle \in \Gamma(A)$

$\langle x_n, Ax_n \rangle \longrightarrow \langle x, y \rangle$ in $\mathcal{H} \oplus \mathcal{H}$

we have to prove $y = Ax$ where $x = \lim x_n$

$(\forall z \in \mathcal{H})$ $(z, y) = \lim (z, Ax_n) = \lim (Az, x_n) =$

$(Az, x) = (z, Ax)$

$\boxed{\text{problem } 13}$ of Banach spaces

more generally: let A, A^* be everywhere def. & $(x, Ay) = (A^* x, y)$ then A, A^* are bounded

see $\boxed{\text{problem } 18.}$

Importance: unbounded "selfadjoint" operators are not everywhere defined

(e.g Δ)

The Hilbert space adjoint:

$$C : \mathcal{H} \to \mathcal{H}^* \qquad Cy = (y, \cdot)$$

C: Conjugate linear isometry, by Riesz surjective

$$T^* = C^{-1} T' C$$

$$(x, Ty) = (Cx)(Ty) = (T'Cx)(y) = (C^{-1}T'Cx, y) = (T^*x, y)$$

$\to$ back to the old definition of Hilbert space adjoint

(a) $T \mapsto T^*$: isometric, conjugate linear isomorphism

Basic properties of the adjoint:

(b) $(TS)^* = S^* T^*$, (c) $(T^*)^* = T$

(d) $(T^{-1})^* = (T^*)^{-1}$ if $T^{-1} \in \mathcal{L}(\mathcal{H})$

(e) $T \mapsto T^*$ $\|\cdot\|$-cont, w-cont, but not st-cont (in ∞-dim)

(f) $\|T^*T\| = \|T\|^2$ $\leftarrow$ detailed

relies on $\|A\| = \sup_{\|x\|=1} (x, Ax)$ for $A = A^*$

Def selfadjointness $A^* = A$

orthogonal projection $P^2 = P = P^*$

insert: Hellinger-Toeplitz

(are indeed projections: connection with proj. th.)

$\boxed{\text{problem 9}}$: $\|A\| = \sup_{\|x\|=1} (x, Ax)$ for $A = A^*$

(from work on $\mathcal{L}(X,X) = \mathcal{L}(X)$, $\mathcal{L}(\mathcal{H})$)

$$\operatorname{Ker} A^* = (\operatorname{Ran} A)^{\perp}$$

$$\overline{\operatorname{Ran} A} = (\operatorname{Ker} A^*)^{\perp}$$

$$x \in \operatorname{Ker} A^* \iff A^* x = 0 \iff (\forall y \in \mathcal{H}) \ (A^* x, y) = 0$$

$$\iff (\forall y \in \mathcal{H}) \ (x, Ay) = 0 \iff x \in (\operatorname{Ran} A)^{\perp}$$

$$A = A^* \implies \sigma(A) \subseteq \mathbb{R}$$

$$\mu = \lambda + i\varepsilon$$

$$\|((\lambda + i\varepsilon) I - A)\varphi\|^2 = \cancel{\quad} \ \varepsilon^2 \|\varphi\|^2$$

$$\varepsilon^2 \|\varphi\|^2 + \|(\lambda - A)\varphi\|^2 \geq \varepsilon^2 \|\varphi\|^2$$

$$\overline{\operatorname{Ran} [(\lambda + i\varepsilon) - A]} = \operatorname{Ker} ((\lambda - i\varepsilon) - A) = \{0\}$$

The Spectrum

finite dim: spectrum = set of (algebraic) eigenvalues

$= \{\lambda \mid \det(\lambda - T) = 0\}$ with multiplicity

Def. resolvent set $\rho(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T$ has a $\boxed{\text{bdd}}$ inverse$\}$
(open)

(analytic) $R_\lambda(T) = (\lambda I - T)^{-1}$ resolvent operator

spectrum (compact) $\sigma(T) = \mathbb{C} \setminus \rho(T)$

problem 8a) eigenvalue, eigenvector $\quad Tx = \lambda x \quad x \neq 0$

point spectrum $\sigma_p(T) = \{\text{eigenvalues}\}$

(residual spectrum: λ is not an eigenvalue & $\mathrm{Ran}(\lambda I - T)$ is not dense)

$A = A^* \Rightarrow \sigma(A) \subset \mathbb{R}$

~~example~~ An instructive example: $\mathbb{C}^{2L+1} \ni \{x = (x_m)_{m=-L}^{L} \mid x_m \in \mathbb{C}\}$

$\boxed{\text{on } \mathbb{C}^{2L+1}}$ $\quad (Tx)_m = \frac{1}{2} x_{m+1} + \frac{1}{2} x_{m-1}$

with periodic b.c.

diagonalization: eigenvectors $x_m^{(k)} = \exp i 2\pi \frac{k}{2L+1} m$

$$T = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & \ddots & \\ 1 & & & 1 & 0 \end{pmatrix} \qquad k = -L, \dots, L$$

$\lambda^{(k)} = \cos 2\pi \frac{k}{L}$

$T x^{(k)} = \lambda^{(k)} x^{(k)}$

 as $L \to \infty$ "$\mathbb{C}^{2L+1} \to \ell_2(\mathbb{Z})$" in some sense

on $\ell_2(\mathbb{Z})$ $\quad (Tx)_m = \frac{1}{2} x_{m-1} + \frac{1}{2} x_{m+1}$

"eigenvalues" $\cos\theta$ $\qquad \theta \in [0, 2\pi)$ $[-\pi, \pi]$

"eigenvectors" $x_m^{(\theta)} = \exp i\theta m \notin \ell_2(\mathbb{Z})$

Truth: no eigenvalues at all ✓

spectrum $= [-1, 1] = \{\cos\theta \mid \theta \in [0, 2\pi)\}$

some reminiscence

proof:
$$x_m^{(\theta, L)} = \begin{cases} \frac{1}{\sqrt{2L+1}} e^{i\theta m} & |m| \leq L \\ 0 & |m| > L \end{cases}$$

$$\|x^{(\theta,L)}\| = 1, \quad \|(\cos\theta I - T)\, x^{(\theta,L)}\| \sim \frac{1}{\sqrt{L}}$$

$\Rightarrow \cos\theta I - T$ not invertible as bounded op

assume λ eigenvalue ; x eigenvector

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 2\lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} \qquad |\lambda| > 1 \text{ increases exponentially}$$

$$\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 2\lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} \qquad |\lambda| \leq 1 \text{ trigonometric}$$

Theorem (about resolvent) $T \in \mathcal{L}(X)$

(i) $\rho(T) \subset \mathbb{C}$ is open and $\lambda \mapsto R_\lambda(T)$ is analytic on $\rho(T)$ / components

(ii) $\lambda, \mu \in \rho(T)$, $[R_\lambda, R_\mu] = 0$ &

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\lambda(T) R_\mu(T) \quad \text{I.}$$

$$\lambda \in \rho(T) \cap \rho(S) \quad R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T-S) R_\lambda(S) \quad \text{II.}$$

(iii) $\emptyset \neq \sigma(T) \subset \{ z \mid |z| \le \|T\| \}$

Proof. (i) $\lambda_0 \in \rho$ let $|\lambda - \lambda_0| < \left(\| R_{\lambda_0} \| \right)^{-1}$

$$\tilde{R}_\lambda = R_{\lambda_0} \cdot \underbrace{\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n [R_{\lambda_0}]^n}_{\text{abs. convergent}} = \sum (\lambda_0 - \lambda)^n R_{\lambda_0}^{n}$$

formally: $\dfrac{1}{\lambda_0 - T} \cdot \sum_{n=0}^{\infty} \left(\dfrac{\lambda_0 - \lambda}{\lambda_0 - T} \right)^n =$

$$\frac{1}{\lambda_0 - T} \cdot \frac{1}{I - \frac{\lambda_0 - \lambda}{\lambda_0 - T}} = \frac{1}{\lambda_0 - T} \cdot \frac{1}{\frac{\lambda - T}{\lambda_0 - T}} = \frac{1}{\lambda - T}$$

$$(\lambda I - T) \tilde{R}_\lambda = \left[(\lambda_0 - T) - (\lambda_0 - \lambda) \right] \tilde{R}_\lambda =$$

$$\sum_{n=0}^{\infty} \left[(\lambda_0 - \lambda)^n (R_{\lambda_0})^n - (\lambda_0 - \lambda)^{n+1} (R_{\lambda_0})^{n+1} \right] = I$$

proves $\underline{\text{analyticity}}$ of $\lambda \mapsto R_\lambda$

subadditive convergence lemma:

$a_n \in \mathbb{R}$ sequence

$a_{n+m} \leq a_n + a_m$ subadditivity

then: $\exists \lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}$

Proof: fix N $\quad n = kN + r \quad k \in \mathbb{N}\cup\{0\}, \; r = 0, 1, \dots, N-1$

uniquely

$$\frac{a_n}{n} = \frac{a_{kN+r}}{kN+r} \leq \frac{k a_N + a_r}{kN + r} \leq \frac{a_N + \frac{a_r}{k}}{N + \frac{r}{k}}$$

$$\Rightarrow \overline{\lim_{n\to\infty}} \frac{a_n}{n} \leq \frac{a_N}{N} \quad (\forall N.)$$

$\Rightarrow$ Q.E.D.

application of all sorts.
e.g. # selfavoiding walks on $\mathbb{Z}^d$

(ii) $R_\lambda - R_\mu = R_\lambda (\mu I - T) R_\mu - R_\lambda(\lambda - T) R_\mu =$
$= R_\lambda(\mu I - \lambda I) R_\lambda = (\mu - \lambda) R_\mu R_\lambda$

(iii) $|\lambda| > \|T\|$

$$R_\lambda(T) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n \quad \text{is abs. cvg}$$

$$\|R_\lambda(T)\| \leq \frac{1}{|\lambda|} \cdot \frac{1}{1 - \|T\|/|\lambda|} = \frac{1}{|\lambda| - \|T\|}$$

Liouville's Theorem: entire & bdd $\to$ constant

Definition spectral radius: $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$

Theorem $T \in \mathcal{L}(X)$

(i) $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (the limit exists)

(ii) $X = \mathcal{H}$, $T = T^*$: $r(T) = \|T\|$

Proof (i) existence of the limit: subadditive convergence lemma problem 11

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$$R_\lambda = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n \quad \text{Laurent series}$$

abs. cvg for $|\lambda| > (r(T))$ — Hadamard's thm

pr. 10 spectral radius of $1/\lambda$ — by def — $\sum a_n z^n$ $\quad r_{cvg} = \overline{\lim} |a_n|^{1/n}$ $^{-1}$

look at the Laurent series $R_\lambda = \frac{1}{\lambda} \sum_{n=0}^{\infty} T^n \cdot \lambda^{-n}$

denote τ the rad. of conv of $\sum_{0}^{\infty} T^n z^{-n}$

Hadamard: $\tau = \left(\lim \|T^n\|^{1/n} \right)^{-1}$

claim $r(T) = \frac{1}{\tau}$

(1) $|\lambda| > \frac{1}{\tau} \Rightarrow \sum T^n \lambda^n$ is ab conv $\Rightarrow$
$\{|\lambda| > \frac{1}{\tau}\} \subset \varrho(T)$ i.e. $\boxed{\frac{1}{\tau} \geq r(T)}$

(2) since R_λ is analytic on $\{|\lambda| > r(T)\}$
$\Rightarrow \frac{1}{\tau} \leq r(T)$.

Spectrum of the adjoint: $T \in \mathcal{L}(X)$ $T' \in \mathcal{L}(X^*)$

Theorem: $\sigma(T') = \sigma(T)$, $\rho(T') = \rho(T)$

for $\lambda \in \rho(T)$: $R_\lambda(T') = [R_\lambda(T)]'$

Proof $\lambda \in \rho(T)$

$$(\lambda I - T')[R_\lambda(T)]' = I$$

since $\left[(\lambda - T')[R_\lambda(T)]' \ell\right](x) =$

$$\ell(R_\lambda(T)(\lambda - T)x) = \ell(x) \quad \forall \ell, x$$

thus $\rho(T) \subset \rho(T')$ and $R_\lambda(T') = [R_\lambda(T)]'$

$$\Updownarrow$$

$$\sigma(T') \subset \sigma(T)$$

$\sigma(T') = \sigma(T)$: $\quad \sigma(T'') \subset \sigma(T') \subset \sigma(T) \subset \sigma(T'')$

$T'' \in \mathcal{L}(X^{**})$ extension of T

spectrum of T'' can not be smaller than that of T (resolvent of T'' cannot be larger than that of T)

an instructive example:

$L(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots)$ acts on l_p $1 \le p \le \infty$

$R(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2 \dots)$ acts on l_q $1 \le q \le \infty$

$\|L\|_p = \|R\|_q = 1$

$\boxed{\text{spectrum of } L}$

$1 \le p < \infty$: $|\lambda| < 1$ eigenvalues

$L(\lambda, \lambda^2, \dots) = \lambda(\lambda, \lambda^2, \dots)$

$|\lambda| = 1$ not eigenvalues, not residu

Ran $(\lambda I - L)$ is dense, if not

$\exists\, \Lambda \in l_p^*$: $\Lambda[(\lambda I - L)x] = [(\lambda I - R)\Lambda](x) =$
$\underset{\circ}{\#}$

i.e. $(\lambda I - R)\Lambda = 0$
impossible

$p = \infty$ $|\lambda| \le 1$ eigenvalues

$\boxed{\text{Spectrum of } R}$

$1 \le p < \infty$ no eigenvalue

$|\lambda| < 1$: residual (Ran $(\lambda I - R)$ not dense

$\Lambda = (\lambda, \lambda^2, \dots) \in l_p^* = l_q$

$(\forall x \in l_p)$ $\Lambda((\lambda I - R)(x)) = 0$

$|\lambda| = 1$ $\quad 1 < p < \infty$: not residual

$\quad p = 1$: residual

$p = \infty$ $\quad |\lambda| < 1$ residual

$\quad |\lambda| = 1$ residual (lengthy)

Proposition : (a) $\lambda \in \sigma_{res}(T) \Rightarrow \lambda \in \sigma_{point}(T')$

(b) $\lambda \in \sigma_{point}(T) \to \lambda \in \sigma_{point}(T') \cup \sigma_{res}(T')$

$\boxed{\text{prob. 12}}$

Proof. (a) $\overline{Ran(\lambda - T)} \neq X \Rightarrow \left(\exists \underset{\neq 0}{\Lambda} \in X^*\right)$:

$Ran(\lambda - T) \subset Ker \Lambda$

$(\forall x) \quad \Lambda\left((\lambda - T)(x)\right) = \left[(\lambda - T')\Lambda\right](x) = 0$

$\Rightarrow (\lambda - T')\Lambda = 0$. qed

(b) $(\lambda - T)x = 0$, for $\ell \in \overline{Ran(\lambda - T')}$: $\ell(x) = 0$

but by Hahn - Banach $\exists\, m \in X^*$: $m(x) \neq 0$

$\to \overline{Ran(\lambda - T')} \neq X^*$

Corollary of (a) $A = A^* \to \sigma_{res}(A) = \emptyset$

$R_\lambda = (\lambda - T)^{-1}$ Laurent series $\frac{1}{\lambda} \sum_{n=0}^{\infty} T^n \lambda^{-n}$

$S_z = z \sum_{n=0}^{\infty} T^n z^n$ radius of cvg of S_z : Δ

Hadamard's thm ~~analytic on $\{|z|<\Delta\}$~~

$\Delta = \left(\limsup \|T^n\|^{1/n} \right)^{-1}$

$z \mapsto S_z$ is analytic on $\{|z| < \Delta\}$ and is singular somewhere on $\{|z| = \Delta\}$

$r(T) \leq \limsup \|T^n\|^{1/n}$ ✓

assume $r(T) < \limsup \|T^n\|^{1/n}$

$\Rightarrow$ $\lambda \mapsto R_\lambda$ analytic on $\{|\lambda| > r\}$

$\Rightarrow$ $z \mapsto S_z$ analytic on $\{|z| < \frac{1}{r}\}$

Contradiction with Hadamard

Positive operators & polar decomposition :

operators on $\mathcal{H}$ have some of the properties of complex numbers!

- algebraic operation : multiplication, unit element / inverse
- power series etc.

$A = A^*$ ~ real elem
unitary ~ $|z| = 1$
$z \mapsto \bar{z}$ ~ $A \mapsto A^*$

$$z = \exp i(\arg z) \cdot |z|$$

sort of unitary . . . positive

Def : positive operator : $B \in \mathcal{L}(\mathcal{H})$ $\quad$ $\mathcal{H}$ complex Hilbert space

$(\forall x \in \mathcal{H}) \quad (x, Bx) \geq 0$

$B \geq C \iff B - C$ is positive

Positive $\Rightarrow$ self adjoint $\quad (x, Ax)$ real

$$4(x, Ay) = \left[\left((x+y), A(x+y)\right) - \left((x-y), A(x-y)\right)\right]$$
$$- i\left[\left((x+iy), A(x+iy)\right) - \left((x-iy), A(x-iy)\right)\right]$$
$$= \left[\left(A(x+y), (x+y)\right) - \left(A(x-y), (x-y)\right)\right]$$
$$- i\left[\left(A(x+iy), x+iy\right) - \left(A(x-iy), (x-iy)\right)\right] = 4(Ax, y)$$

works only in the complex case

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{R}^2$ is positive but not self adjoint
on $\mathbb{C}^2$ is not positive (but unitary)

$A \in \mathcal{L}(\mathcal{H})$ $\qquad A^*A$ is positive !

$|z| = \sqrt{\bar{z} z}$: " $|A| = \sqrt{A^*A}$ " ?

Lemma $\quad \sqrt{1-z} = \sum_{n=0}^{\infty} c_n z^n \qquad |z| < 1$

The expansion is absolutely convergent for $|z| \leq 1$!

Proof $\quad c_0 = 1 \qquad c_n = \frac{1}{n!} \frac{d^n}{dz^n} \sqrt{1-z} \Big|_{z=0}$

$$\frac{d}{dz}(1-z)^{1/2} = -\frac{1}{2}(1-z)^{-\frac{1}{2}}$$

$$\frac{d^2}{dz^2} = -\frac{1}{2}\cdot\frac{1}{2}\cdot(1-z)^{-3/2}$$

$$\frac{d^n}{dz^n}(1-z)^{1/2} = -\frac{1}{2}\frac{2(n-1)!}{2^{2(n-1)}(n-1)!}(1-z)^{-\frac{2n-1}{2}}$$

$$\frac{d^{n+1}}{dz^{n+1}}(1-z)^{1/2} = -\frac{1}{2}\frac{(2(n-1))!}{2^{2(n-1)}(n-1)!}\cdot\frac{2n-1}{2}\cdot\frac{2n}{2\cdot n}(1-z)^{-\frac{2(n+1)-1}{2}} =$$

$$= -\frac{1}{2}\frac{(2n)!}{2^{2n}\,n!}(1-z)^{-\frac{2(n+1)-1}{2}}$$

$\forall n \geq 1$: $c_n < 0$!!

$$\sum_1^H |c_n| = 1 - \sum_0^H c_n =$$

$$1 - \lim_{x \nearrow 1} \sum_0^H c_n x^n \leq 1 - \lim_{x \nearrow 1} \sum_0^\infty c_n x^n =$$

$$= 1 - \lim_{x \nearrow 1} \sqrt{1-x} = 1$$

Theorem (square root lemma). $A \in \mathcal{L}(\mathcal{H})$ $A \geq 0$

$\exists! \; B \in \mathcal{L}(\mathcal{H}) : B \geq 0 \;\&\; B^2 = A$

and $[C, A] = 0 \iff [C, B] = 0$.

Proof assume $\|A\| \leq 1$

$$\text{"} \sqrt{A} = \sqrt{I - (I-A)} = \sum_0^\infty c_n (I-A)^n \text{"}$$

$\|(I-A)\| = \sup_{\|x\| \leq 1} (x, (I-A)x) \leq 1$ ⟹ !

the Power series converge absolutely!

$$B = \sum_{n=0}^\infty c_n (I-A)^n$$

$B^2 = A$: by rearrangement of the power series

$$B^2 = \sum_{n=0}^\infty (I-A)^n \left(\sum_{k=0}^n c_k \cdot c_{n-k} \right) = (I-A)^0 - (I-A) = A$$

$[C, A] = 0 \Rightarrow [C, B] = 0$ straightforward positivity : $(0 \leq) I - A \leq I$

$\|\varphi\| \leq 1 \qquad 0 \leq (\varphi, (I-A)^n \varphi) \leq 1$ ✓

$(\varphi, B\varphi) = 1 + \sum_{n \geq 1} c_n \; (\varphi, (I-A)^n \varphi) \geq$

$1 + \sum_{n \geq 1} c_n = 0.$

uniqueness : let B' : $B' \geq 0$ & $B'^2 = A$

$[B', A] = [B', (B')^2] = 0 \Rightarrow [B', B] = 0$

$\underset{\geq 0}{(B-B') B (B-B')} + \underset{\geq 0}{(B-B') B' (B-B')} = 0$

$\Rightarrow (B-B') B (B-B') - (B-B') B' (B-B') = (B-B')^3 = 0$

$\Rightarrow (B-B')^4 = 0 \Rightarrow B - B' = 0$

Definition $|A| = \sqrt{A^* A}$.

Warning: $|AB| \neq |A| \cdot |B|$ (e.g. take P, Q orth. pr.

$|A| \neq |A^*|$ $\qquad A = R : |A| = I$

$B = L \quad |L| = P$ on $(0, -)$

$|A+B| \nleq |A| + |B|$. problem 16

Given : $A = U|A|$ ← unitary

not exactly : ex : Right shift

Def unitary : U invertible , $U^* = U^{-1}$

isometry $\|Ux\| = \|x\|$

partial isometry $\|Ux\| = \|x\|$ on $\mathrm{Ker}\, U^\perp$

$\mathcal{H} = \mathrm{Ker}\, U \oplus (\mathrm{Ker}\, U)^\perp$

$\mathcal{H} = (\mathrm{Ran}\, U)^\perp \oplus \mathrm{Ran}\, U$

U isometry $(\mathrm{Ker}\, U)^\perp \mapsto \mathrm{Ran}\, U$

U^* isometry $\underset{(\mathrm{Ker}\, U^*)^\perp}{\mathrm{Ran}\, U} \mapsto \underset{\mathrm{Ran}\, U^*}{(\mathrm{Ker}\, U)^\perp}$

Prop $\begin{pmatrix} U^*U = P_i & \text{on } (\mathrm{Ker}\, U)^\perp \\ UU^* = P_f & \text{on } \mathrm{Ran}\, U \end{pmatrix}$ & UU^*, U^*U proj

⇑ U partial is

problem 18

Theorem (Polar decomposition) $A \in \mathcal{L}(\mathcal{H})$

$\exists\, U$ partial isometry : $A = U|A|$

$\mathrm{Ker}\, U = \mathrm{Ker}\, A$ -> unique

$\mathrm{Ran}\, U = \overline{\mathrm{Ran}\, A}$

Proof. $U : \mathrm{Ran}|A| \to \mathrm{Ran}\, A$: $U(|A|\varphi) = A\varphi$

$\||A|\varphi\|^2 = (\varphi, |A|^2\varphi) = (\varphi, A^*A\varphi) = \|A\varphi\|^2$ => well defined isometry between $\mathrm{Ran}|A| \to \mathrm{Ran}\, A$

extend to $\overline{\mathrm{Ran}\,|A|} = \overline{(\mathrm{Ran}\,A^*)}$

extend to $(\mathrm{Ran}\,|A|)^\perp = \mathrm{Ker}\,|A| = \mathrm{Ker}\,A$ as 0