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# Note

## A Short Proof of Kneser's Conjecture

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In the paper a short proof is given for Kneser's conjecture. The proof is based on Borsuk's theorem and on a theorem of Gale.

Recently, Lovász has given a proof for Kneser's conjecture [4]. He used Borsuk's theorem. This fact gave the author the idea of the proof we present here.

THEOREM (Kneser's conjecture [3]). *If the n-tuples of a set of $2n + k$ elements are partitioned into $k + 1$ classes, then one of the classes contains two disjoint n-tuples.*

For the graph-theoretic formulation of this theorem and other comments on it see Lovász's paper [4].

We will need two theorems. Put $S_k = \{x \in R^{k+1}: \| x \| = 1\}$ and $H(a) = \{x \in S_k: \langle a, x\rangle > 0\}$ if $a \in S_k$.

BORSUK'S THEOREM. *If $S_k$ is the union of $k + 1$ sets which are open in $S_k$, then one of these sets contains antipodal points.*

GALE'S THEOREM. *If n and k are nonnegative integers, then there is a set $V \subset S_k$ with $2n + k$ elements such that $| H(a) \cap V | \geqslant n$ for each $a \in S_k$.*

In the original form of Borsuk's theorem the $k + 1$ sets are supposed to be closed in $S_k$ (see [1]). It is an easy exercise to show from the original form that the above statement is true. For a proof of Gale's theorem see [2]. It is perhaps worth mentioning that this theorem is the dual (in the sense of Gale diagrams) of the fact that there exist in $R^d$ $[d/2]$-neighborly polytopes with any number of vertices.



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*Proof of the theorem.* Identify the $2n + k$ points with the set $V$ of Gale's theorem. Suppose its $n$-tuples are split into $k + 1$ classes, i.e., they have a $(k + 1)$-coloring. This defines a $(k + 1)$-coloring of $S_k$ in the following way: if $a \in S_k$ then the point $a$ gets the color of each $n$-tuple in $H(a) \cap V$. By Gale's theorem, every $a \in S_k$ has got at least one color. It is easy to show that the points of the same color form an open set in $S_k$. Now apply Borsuk's theorem. Then there is a point $a \in S_k$ such that $a$ and $-a$ are of the same color. The point $a$ got this color from an $n$-tuple in $H(a) \cap V$ and the point $-a$ got this color from an $n$-tuple in $H(-a) \cap V$. Clearly, these two $n$-tuples are disjoint and of the same color.

*Remark* 1. In the proof we used only those $n$-tuples whose cone hull is a pointed cone (i.e., it does not contain $a$ and $-a$ at the same time if $a \neq 0$). This means, e.g., that the conditions of the theorem can be weakened for $n \geqslant k + 1$.

*Remark* 2. The method of the proof gives the following theorem: If $G(V, E)$ is a hypergraph and there is a map $f: V \to S_k$ such that for each $a \in S_k$ $H(a)$ contains an edge of $G$, then in any $(k + 1)$-coloring of the edges of $G$ there are two disjoint edges of the same color.

The author could neither prove nor disprove the converse of this theorem. It would be interesting to know if it is true or not.

## References

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