Balanced partitions of vector sequences

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Abstract

Let $d, r \in \mathbb{N}$ and $\| \cdot \|$ be any norm on $\mathbb{R}^d$. Let $B$ denote the unit ball with respect to this norm. We show that any sequence $v_1, v_2, \ldots$ of vectors in $B$ can be partitioned into $r$ subsequences $V_1, \ldots, V_r$ in a balanced manner with respect to the partial sums: For all $n \in \mathbb{N}$, $\ell \leq r$, we have $\left\| \sum_{i \leq k} v_i \right\| \leq \frac{1}{r} \sum_{i \leq k} v_i \| \leq 2.0005d$. A similar bound holds for partitioning sequences of vector sets. Both results extend an earlier one of Bárány and Grinberg [I. Bárány, V.S. Grinberg, On some combinatorial questions in finite-dimensional spaces, Linear Algebra Appl. 41 (1981) 1–9] to partitions in arbitrarily many classes.

Keywords: Discrepancy; Balanced partition; Vector balancing game

1. Introduction

Let $d, N \in \mathbb{N}$. We use the short-hand $[N] := \{1, \ldots, N\}$. Let $\| \cdot \|$ be any norm on $\mathbb{R}^d$ and $B = \{v \in \mathbb{R}^d : \|v\| \leq 1\}$ its unit ball. In this paper, we give extensions of the Bárány–Grinberg theorem to partitions into more than two classes. In its most general version, this theorem states the following [1].

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Theorem 1. Let \( V_1, \ldots, V_N \subseteq B \) such that \( 0 \in \text{conv}(V_i) \) for all \( i \in [N] \). Then there are \( v_i \in V_i \) such that for all \( n \in [N] \),

\[
\left| \sum_{i \in [n]} v_i \right| \leq 2d.
\]

The most interesting special case of Theorem 1 is that all \( V_i \) are of the form \( V_i = \{v_i, -v_i\} \), cf. [2] as well. In this case, Theorem 1 yields that for any sequence \( v_1, \ldots, v_N \) of vectors in \( B \) there are signs \( \varepsilon_i \in \{-1, 1\} \) such that \( \| \sum_{i \in [n]} \varepsilon_i v_i \| \leq 2d \) for all \( n \in [N] \). In other words, there is a partition \( [N] = I_1 \cup I_2 \) such that \( \| \sum_{i \in I_j \cap [n]} v_i - \frac{1}{2} \sum_{i \in [n]} v_i \| \leq d \) for all \( n \in [N] \) and \( j \in [2] \). This partitioning version of the Bárány–Grinberg theorem was extended to partitions into \( r \) classes with error bound \((r - 1)d\) in [3]. In the following section, we show that the factor \((r - 1)\) can be replaced by a constant.

In the third section of this paper, we show that if the stronger condition \( \sum_{i \in V_i} v = 0 \) (instead of \( 0 \in \text{conv}(V_i) \)) holds for all \( i \in [N] \), then for each \( i \in [N] \) there are \( r \) distinct vectors \( v_{i\ell} \in V_i \), \( \ell \in [r] \), such that \( \| \sum_{i \in [n]} v_{i\ell} \| \leq 5d \) holds for all \( n \in [N] \) and \( \ell \in [r] \), where \( r \leq \max(||V_i|| i \in [N]) \).

It is worth mentioning here that the results hold for all norms in \( \mathbb{R}^d \). This is due to the fact that proofs use linear dependences among some vectors, with the norm playing very little role. But most likely, much better bounds are valid for particular norms. For instance, it is conjectured that for \( r = 2 \) and Euclidean norm the best bound is of order \( \sqrt{d} \). This was proved by Spencer [4] when \( N = O(d) \), but the general case when \( N \) is arbitrary is open.

In the proofs of both results below we invoke the recursive method of [3], which states, roughly speaking, that if one can guarantee the existence of a 2-partition with good bound on its discrepancy, then one can guarantee the existence of an \( r \)-partition with a slightly weaker bound on its discrepancy. Precisely, we have the following:

Theorem 2. Let \( r \geq 2 \) be an integer. Let \( v_1, \ldots, v_n \) be a sequence of vectors and \( \mathcal{E} \) be a set of subsets of \([n]\). Assume that for all integers \( 1 \leq r_1 < r_0 \leq r \) and all \( V_0 \subseteq [n] \) there is a \( V_1 \subseteq V_0 \) such that for all \( E \in \mathcal{E} \),

\[
\left| \sum_{i \in V_1 \cap E} v_i - \frac{r_1}{r_0} \sum_{i \in V_0 \cap E} v_i \right| \leq K.
\]

Then there is a partition \( V = V_1 \cup \cdots \cup V_r \) such that for all \( \ell \in [r] \) and \( E \in \mathcal{E} \) we have

\[
\left| \sum_{i \in V_\ell \cap E} v_i - \frac{1}{r} \sum_{i \in V \cap E} v_i \right| \leq C(r)K,
\]

where \( C(r) \) is an absolute constant satisfying \( C(r) \leq 2.0005 \) for all \( r \in \mathbb{N} \).

Note that the assumption of the theorem is equivalent to saying that for all integer \( 1 \leq r' \), \( r'' < r_0 \leq r \) with \( r' + r'' = r_0 \) and all \( V_0 \subseteq [n] \) there is a partition \( V' \cup V'' = V_0 \) such that for all \( E \in \mathcal{E} \),

\[
\left| \sum_{i \in V' \cap E} v_i - \frac{r'}{r_0} \sum_{i \in V_0 \cap E} v_i \right| \leq K \quad \text{and} \quad \left| \sum_{i \in V'' \cap E} v_i - \frac{r''}{r_0} \sum_{i \in V_0 \cap E} v_i \right| \leq K.
\]
This implies the theorem via Theorem 2.

**Proof.** Assume for all \( \alpha(v) \) are linearly dependent, so there are details can be found in [3].

The proof of Theorem 2 starts with setting \( I. \) Bárány, B. Doerr / Linear Algebra and its Applications 414 (2006) 464–469

\[ \sum_{k} V \in \mathbb{R} \]

\[ X \in \mathbb{R} \]

\[ C(r)dB. \]


Theorem 3. For every sequence \( V \subset B, \) and for every integer \( r \geq 2, \) there is a partition of \( V \) into \( r \) subsequences \( X_1, \ldots, X_r \) such that for all \( k \) and \( j \)

\[ \sum_{k} X_j \in \frac{1}{r} \sum_{k} V + C(r)dB. \]

**Proof.** Assume \( r_0 = r_1 + r_2 \) (with positive integers \( r_1, r_2 \)). We are going to construct a partition of \( V \) into subsequences \( Y_1 \) and \( Y_2 \) such that for each \( k \) and \( j = 1, 2, \)

\[ \sum_{k} Y_j \in \frac{r_j}{r_0} \sum_{k} V + dB. \]

This implies the theorem via Theorem 2.

For the construction of \( Y_1, Y_2 \) we use a modified version of the method of “floating variables” as given in [1]. Define \( V_k = \{v_1, v_2, \ldots, v_{k+d}\}, k = 0, 1, 2, \ldots \) We are going to construct mappings \( \beta_k : V_k \to \{-r_1, r_2\} \) and subsets \( W_k \subset V_k \) with the following properties (for all \( k \)):

(i) \( \sum_{V_k} \beta_k(v) = 0, \)

(ii) \( \beta_k(v) \in \{-r_1, r_2\} \) whenever \( v \in W_k, \)

(iii) \( |W_k| = k \) and \( W_k \subset W_{k+1}. \)

The construction is by induction on \( k. \) For \( k = 0, W_0 = \emptyset \) and \( \beta_0 = 0 \) clearly suffice. Now assume that \( \beta_k \) and \( W_k \) have been constructed and satisfy (i) to (iii). The \( d + 1 \) vectors in \( V_{k+1} \setminus W_k \) are linearly dependent, so there are \( \alpha(v) \in \mathbb{R} \) not all zero such that

\[ \sum_{V_{k+1} \setminus W_k} \alpha(v) = 0. \]

Putting \( \beta_k(v_{k+d+1}) = 0, \) we have

\[ \sum_{W_k} \beta_k(v) + \sum_{V_{k+1} \setminus W_k} (\beta_k(v) + t\alpha(v))v = 0 \]

for all \( t \in \mathbb{R}. \) For \( t = 0 \) all coefficients lie in \( [-r_1, r_2]. \) Hence for a suitable \( t = t^*, \) all coefficients still belong to \( [-r_1, r_2], \) and \( \beta_k(v) + t\alpha(v) \in [-r_1, r_2] \) for some \( v = v^* \in V_{k+1} \setminus W_k. \)

Set now \( W_{k+1} = W_k \cup \{v^*\} \) and \( \beta_{k+1}(v) = \beta_k(v), \) if \( v \in W_k, \) and \( \beta_{k+1}(v) = \beta_k(v) + t^*\alpha(v), \) if \( v \in V_{k+1} \setminus W_k. \) Now \( W_{k+1} \) and \( \beta_{k+1} \) satisfy the requirements. Moreover, \( \beta_{k+1}(v) = \beta_k(v) \) for all \( v \in W_k. \)

We now define the subsequences \( Y_1 \) and \( Y_2. \) Put \( v_i \) into \( Y_1 \) if \( v_i \in W_k \) and \( \beta_k(v_i) = r_2 \) for some \( k, \) and put \( v_i \) into \( Y_2 \) if \( v_i \in W_k \) and \( \beta_k(v_i) = -r_1 \) for some \( k. \) As \( \beta_k(v) = \beta_{k+1}(v) \) once \( v \in W_k,
this definition is correct for all vectors that appear in some $W_k$. The remaining (at most $d$) vectors can be put into $Y_1$ or $Y_2$ in any way. Set $\gamma(v) = r_2$, if $v \in Y_1$, and $\gamma(v) = -r_1$, if $v \in Y_2$.

Clearly, $r_2 \sum_k Y_1 - r_1 \sum_k Y_2 \in r_0dB$ for all $k \leq d$. For $k > d$ we have,

$$r_2 \sum_k Y_1 - r_1 \sum_k Y_2 = \sum_{V_h} \gamma(v)v = \sum_{V_h} \gamma(v)v - \sum_{V_h} \beta_h(v)v$$

$$= \sum_{V_h}(\gamma(v) - \beta_h(v))v = \sum_{V_h \setminus W_h} (\gamma(v) - \beta_h(v))v.$$

The last sum contains at most $d$ non-zero terms, each having norm at most $r_0$. Thus

$$r_2 \sum_k Y_1 - r_1 \sum_k Y_2 \in r_0dB$$

for every $k$. Adding this to the trivial equation $r_1 \sum_k Y_1 + r_1 \sum_k Y_2 = r_1 \sum_k V$ (expressing that $Y_1, Y_2$ form a partition of $V$), we obtain

$$\sum_k Y_1 \in \frac{r_1}{r_0} \sum_k V + dB$$

for every $k$. □

3. Vector selection

Let now $V_1, \ldots, V_N$ be a sequence of finite subsets of $B$ such that $|V_i| \geq r$ for all $i \in [N]$. An $r$-selection of $(V_i)$ is a mapping $\chi : [N] \times [r] \to \mathbb{R}^d$ such that $\chi(i, [r])$ is an $r$-element subset of $V_i$ for all $i \in [N]$. For such a $\chi$, we define its discrepancy with respect to $(V_i)$ by

$$\text{disc}(\chi, (V_i)_{i \in [N]}) = \max_{n \in [N]} \max_{\ell \in [r]} \left\| \sum_{i \in [n]} \left( \chi(i, \ell) - \frac{1}{|V_i|} \sum_{v \in V_i} v \right) \right\|.$$

**Theorem 4.** There is an $r$-selection with discrepancy at most $5d$.

We mention that this theorem also holds for infinite sequences of finite subsets of $B$.

To prove the theorem, we apply the following lemma twice.

**Lemma 5.** Let $r \in \mathbb{N}$, $r \geq 2$. Let $V_1, \ldots, V_N \subseteq B$ such that $|V_i| \geq r$ for all $i \in [N]$. Then for all $k \in [r]$ there are $U_i \subseteq V_i$ such that $|U_i| = k$ for all $i \in [N]$ and $\max_{n \in [N]} \| \sum_{i \in [n]} (\sum_{v \in U_i} v - \frac{k}{|V_i|} \sum_{v \in V_i} v) \| \leq 2d$.

**Proof.** We give an algorithm for the construction of the sets $U_i$. For each $i \in [N], v \in V_i$ put $x_{iv} = \frac{k}{|V_i|}$. We iteratively change these numbers to zeros and ones in such a way that $U_i := \{ v \in V_i \mid x_{iv} = 1 \}$ gives the desired solution. For the start let $n = 1$. What we do is the following: View those $x_{iv}$ such that $x_{iv} \notin \{0,1\}$ and $i \leq n$ as variables. If there is exactly one solution to the linear system

$$\sum_{i \in [N]} \sum_{v \in V_i} \left( x_{iv} - \frac{k}{|V_i|} \right) v = 0, \quad (1)$$

$$\sum_{v \in |V_i|} x_{iv} = k, \quad i \in [N], \quad (2)$$

$$x_{iv} \in [0,1], \quad i \in [N], \quad v \in V_i,$$
then increase \( n \) by one and try again. Otherwise our existing solution may be changed in such a way that at least one more variable \( x_{iv} \) becomes 0 or 1. If \( n \) reaches \( N \) and no solution can be found, then stop and change the remaining non-integral values of \( x_{iv} \) to 0 or 1 in such a way that (2) is still fulfilled.

Assume that in some step of this iteration no solution can be found. Then there are at least as many constraints containing variables as there are variables. Let \( q \) be the number of constraints of type (2) that contain a variable. Then the total number of constraints containing variables is at most \( d + q \), and the number of variables is at least \( 2q \). Hence \( q \leq d \) holds if no non-trivial solution can be found, and at most \( q + d \leq 2d \) of the \( x_{iv} \), \( i \leq n \), are not in \( \{0, 1\} \). Denote the set of these pairs \((i, v)\) by \( I \). Since the remaining \( x_{iv}, i \leq n \), are not changed anymore, our final solution \( \bar{x} \) satisfies

\[
\sum_{i \in [n]} \sum_{v \in V_i} \left( \bar{x}_{iv} - \frac{k}{|V_i|} \right) v = \sum_{i \in [n]} \sum_{v \in V_i} \left( x_{iv} - \frac{k}{|V_i|} \right) v + \sum_{(i, v) \in I} \left( \bar{x}_{iv} - x_{iv} \right) v
\]

\[
= \sum_{(i, v) \in I} \left( \bar{x}_{iv} - x_{iv} \right) v.
\]

Since \( |I| \leq 2d \), we conclude \( \| \sum_{i \in [n]} \sum_{v \in V_i} \left( \bar{x}_{iv} - \frac{k}{|V_i|} \right) v \| \leq 2d \) for all \( n \in [N] \). Since \( \bar{x}_{iv} \in \{0, 1\} \), putting \( U_i := \{ v \in V_i | \bar{x}_{iv} = 1 \} \) gives the desired solution. \( \square \)

**Proof of the theorem.** Let us assume first that \( |V_i| = r \) for all \( i \in [N] \). Then, by the above lemma, for all integers \( r_1, r_2 \) such that \( r = r_1 + r_2 \) there are \( U_i^{(1)} \cup U_i^{(2)} = V_i \) such that \( |U_i^{(j)}| = r_j \) and \( \| \sum_{i \in [n]} \left( \sum_{v \in U_i^{(j)}} v - \frac{r_j}{r} \sum_{v \in V_i} v \right) \| \leq 2d \). Hence from Theorem 2, we obtain an \( r \)-selection (actually an \( r \)-partition) of \((V_i)\) such that

\[
\left\| \sum_{i \in [n]} \left( \chi(i, \ell) - \frac{1}{r} \sum_{v \in V_i} v \right) \right\| \leq 2C(r)d
\]

for all \( n \in [N], \ell \in [r] \).

If \( |V_i| > r \) for some \( i \), apply the Lemma 5 (with \( k = r \)) to obtain \( \bar{V}_i \subseteq V_i \) such that \( |\bar{V}_i| = r \) and \( \| \sum_{i \in [n]} \left( \sum_{v \in \bar{V}_i} v - \frac{r}{|V_i|} \sum_{v \in V_i} v \right) \| \leq 2d \). By the above, there is an \( r \)-selection for \( (\bar{V}_i) \) such that

\[
\left\| \sum_{i \in [n]} \left( \chi(i, \ell) - \frac{1}{r} \sum_{v \in \bar{V}_i} v \right) \right\| \leq 2C(r)d
\]

for all \( n \in [N], \ell \in [r] \). Note that, trivially, \( \chi \) is also an \( r \)-selection for \( (V_i) \). It satisfies

\[
\left\| \sum_{i \in [n]} \left( \chi(i, \ell) - \frac{1}{|V_i|} \sum_{v \in V_i} v \right) \right\|
\]

\[
\leq \left\| \sum_{i \in [n]} \left( \chi(i, \ell) - \frac{1}{r} \sum_{v \in \bar{V}_i} v \right) \right\| + \left\| \sum_{i \in [n]} \left( \frac{1}{r} \sum_{v \in \bar{V}_i} v - \frac{1}{|V_i|} \sum_{v \in V_i} v \right) \right\|
\]

\[
\leq 2C(r)d + \frac{1}{2} 2d
\]
for all \( n \in [N] \) and \( \ell \in [r] \). By noting that \( C(2) = 1 \) and \( C(r) \leq 2.0005 \) for all \( r \in \mathbb{N} \), we obtain the constant of 5.

We may remark that a closer inspection of \( C(r) \) for small \( r \) yields better constants. For example, easy calculations by hand or Lemma 3.5 in [3] show that \( C(r) + \frac{1}{r} \leq 2.1 \) for \( r \leq 10 \) (for \( r = 7 \) observe that \( C(7) \leq \max\{\frac{1}{3} + C(3), \frac{1}{4} + C(4)\} \)). Hence the bound \( C(r) \leq 2.0005 \) implies \( C(r) + \frac{1}{r} \leq 2.1 \) for all \( r \in \mathbb{N} \), leading to a constant of 4.2 instead of 5.

The following is an immediate consequence of Theorem 4.

**Corollary 6.** Let \( r, N \in \mathbb{N} \). For \( i \in [N] \) let \( V_i \subseteq B \) such that \( \sum_{v \in V_i} v = 0 \) and \( |V_i| \geq k \). Then there is a \( k \)-selection of \((V_i)\) such that

\[
\left\| \sum_{i \in [n]} \chi(i, \ell) \right\| \leq 5d
\]

for all \( n \in [N] \) and \( \ell \in [r] \).

This answers a question of Emo Welzl concerning multiclass extensions of Theorem 1 posed at the Oberwolfach Seminar on “Discrepancy Theory and its Applications” in March 2004. It is clear that the stronger assumption \( \sum_{v \in V_i} v = 0 \) is necessary. Already for \( d = 1 \) and \( r = 2 \), the sequence \( V_i = \{-\frac{1}{2}, 1\} \) shows that \( 0 \in \text{conv}(V_i) \) does not suffice.

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**References**


