Every point is critical

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Abstract

We show that, for any compact Alexandrov surface $S$ (without boundary) and any point $y$ in $S$, there exists a point $x$ in $S$ for which $y$ is a critical point. Moreover, we prove that uniqueness characterizes the surfaces homeomorphic to the sphere among smooth orientable surfaces.

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1. Introduction

In this paper, by surface we always mean a compact 2-dimensional Alexandrov space with curvature bounded below and without boundary, as defined by Burago, Gromov and Perelman in [1]. It is known that our surfaces are topological manifolds (see [1, Section 11]). Let $\mathcal{A}$ be the space of all surfaces.
For any surface $S$, denote by $\rho$ its metric, and by $\rho_x$ the distance function from $x$, given by $\rho_x(y) = \rho(x, y)$. A point $y \in S$ is called critical with respect to $\rho_x$ (or to $x$), if for any direction $\tau$ of $S$ at $y$ there exists a segment (i.e., a shortest path) from $y$ to $x$ whose direction at $y$ makes an angle not greater than $\pi/2$ with $\tau$. For the definition of the set of directions at an arbitrary point of an Alexandrov surface, see again [1]. A geodesic is a curve which is locally a segment. We recall that geodesics of $S$ do not bifurcate [1].

The survey [2] by K. Grove presents the principles, as well as applications, of the critical point theory for distance functions.

Every point on a surface admits a critical point. It suffices, indeed, to take a point farthest from it. Conversely, is it true that every point is a critical point of some other point? Certainly, not every point on every surface is a farthest point from some other point: On an ellipsoid of revolution with an NS-axis much longer than the other two, no point of the equator is farthest from any other point. Concerning the set of all critical points, however, the answer is affirmative, as Theorem 1 shows.

For the set-valued function associating to each point of a surface the set of all farthest points on the surface, the relationship between being single-valued and being surjective is investigated as Theorem 2 shows.

Theorem 2 characterizes the smooth orientable surfaces homeomorphic to the sphere.

For any point $x$ in $S$, denote by $Q_x$ the set of all critical points with respect to $x$, and by $Q_x^{-1}$ the set of all points $y \in S$ with $x \in Q_y$. Let $M_x$, $F_x$ be the sets of all relative, respectively absolute, maxima, of $\rho_x$. For properties of $Q_x$ and its subsets $M_x$ and $F_x$ in Alexandrov spaces, see [3,11], and the survey [9].

A forthcoming paper, [4], will provide for orientable surfaces an upper bound for $\text{card}Q_y^{-1}$ depending on the genus, and use it to estimate the cardinality of diametrically opposite sets on $S$. The case of points $y$ in orientable Alexandrov surfaces, which are common maxima of several distance functions, is treated in [10].

We denote by $T_y$, the space of directions at $y \in S$; the length $\lambda T_y$ of $T_y$ satisfies $\lambda T_y \leq 2\pi$ [1]. If $\lambda T_y < 2\pi$ then $y$ is called a conical point of $S$. A surface without conical points is called smooth.

There might exist a direction $\tau \in T_y$ such that no segment starts at $y$ in direction $\tau$. On most convex surfaces, the set of such directions $\tau$, called singular, is even residual in $T_y$, for each $y$ (see Theorem 2 in [12]). However, the set of non-singular directions is always dense in $T_y$. For those $\tau$, for which there is a geodesic $\Gamma$ with direction $\tau$ at $y$, a so-called cut point $c(\tau)$ is associated, defined by the requirement that the arc $yc(\tau) \subset \Gamma$ is a segment which cannot be extended further beyond $c(\tau)$ (remaining a segment). This is well-defined, because in an Alexandrov space of curvature bounded below segments (and geodesics) do not bifurcate. The set of cut points in all non-singular directions at $y$ is the cut locus $C(y)$ of the point $y$.

Recall that a tree in $S$ is a set $T \subset S$ any two points of which can be joined by a unique Jordan arc included in $T$. A set $L \subset S$ is a local tree if each of its points $x$ has a neighbourhood $V$ in $S$ such that the connected component $K_x(V)$ of $L \cap V$ containing $x$ is a tree. The degree of a point $x$ of a local tree is the cardinality of the set of components of $K_x(V) \setminus \{x\}$ if the neighbourhood $V$ of $x$ is chosen such that $K_x(V)$ be a tree. A point of the local tree $L$ is called an extremity of $L$ if it has degree 1, and a ramification point of $L$ if it has degree at least 3.

It is known that $C(y)$, if it is not a single point, is a local tree (see [8, Theorem A, p. 534]), even a tree if $S$ is homeomorphic to the sphere (which is easily seen). Theorem 4 in [14] and Theorem 1 in [12] yield the existence of surfaces $S$ on which the set of all extremities of any cut
locus is residual in $S$. It is known, however, for any surface $S$ and point $y \in S$, that $C(y)$ has an at most countable set $C_3(y)$ of ramification points (see [8, Theorem A, p. 534]).

If $S$ is not a topological sphere, the cyclic part of $C(y)$ is the minimal (with respect to inclusion) subset $C^{cp}(y)$ of $C(y)$, whose removal from $S$ produces a topological (open) disc. It was introduced by some of us in [5]. Let $C_3^{cp}(y)$ be the set of points of degree at least 3 in $C^{cp}(y)$.

It is possible that, for some point $x \in C(y)$, there exists a whole nondegenerate arc $A \subset T_y$ such that $c(\tau) = x$ for all $\tau \in A$, providing a pencil of segments from $y$ to $x$. Let $C^*(y)$ denote the set of all such points in $C(y)$. Also, let $C^e(y)$ be the set of extremities of $C(y)$.

The points in $Cl(y) = C(y) \setminus (C_3(y) \cup C^*(y) \cup C^e(y))$ are called cleave points.

2. Main result

In order to prove the main result of the paper we need the following lemmas.

**Lemma 1.** Let $y$ belong to a surface $S$ not homeomorphic to the sphere. Then $C^{cp}(y)$ is a local tree with no extremities and with finitely many ramification points, each having finite degree in $C^{cp}(y)$.

This essentially follows from Myers’ early investigation of the topological properties of the cut locus [7], and the observation that $C(y) \setminus C^{cp}(y)$ is a (countable) union of disjoint trees.

**Lemma 2.** If $y \in S$, then all points of $C(y)$ but at most countably many are cleave points or extremities. Moreover, for any cleave point $x$, there are exactly two distinct directions $\tau_1(x), \tau_2(x) \in T_y$ with $c(\tau_1(x)) = c(\tau_2(x)) = x$.

The first part follows from the already mentioned at most countability of $C_3(y)$ and from the obvious countability of $C^*(y)$. The second part is straightforward.

**Lemma 3.** Let $S \in A$, $y \in S$, and $x \in C(y)$. Assume that there are two segments $\sigma_1, \sigma_2$ from $y$ to $x$ such that $S \setminus (\sigma_1 \cup \sigma_2)$ be disconnected, at least one of its components being a topological open disc $\Delta$. If the angle between $\sigma_1$ and $\sigma_2$ at $y$ toward $\Delta$ is larger than $\pi$, then $y \in Q_x$ or there exists a point in $C(y) \cap \Delta$ with respect to which $y$ is critical.

**Proof.** We will identify in this paper $T_y$ with a circle of centre $0$ and length $\lambda T_y \leq 2\pi$ in $R^2$. Let $D \subset R^2$ be the compact disc bounded by $T_y$. We want to find $x \in C(y)$, for which $y \in Q_x$. This is equivalent to $0 \in \text{conve}^{-1}(x)$, by the definition of a critical point.

Let $A^* \subset T_y$ be the arc of all directions toward the closure of $\Delta$.

If $c(\tau) = x$ for all $\tau \in A^*$, then $y \in Q_x$. In the contrary case, if all components of $A^* \setminus c^{-1}(x)$ have length at most $\pi$, then again $y \in Q_x$. Suppose one component, $A$, has length $\lambda A > \pi$. For any non-singular $\tau \in A$, $c^{-1}(c(\tau)) \subset A$, because no segment starting at $y$ in a direction belonging to $A$ meets again any segment starting at $y$ in a direction not belonging to $A$ (since segments do not bifurcate). This follows from the Jordan curve theorem and the fact that $\Delta$ is homeomorphic to an open disc. Let $B_\tau$ be the shortest subarc of $A$ including $c^{-1}(c(\tau))$ (possibly reduced to $\{\tau\}$).

Choose a non-singular direction $\tau_0$ in the interior of $A$ at distance (measured on $T_y$) less than $(\lambda T_y - \lambda A)/2$ from the mid-point of $A$. Then either

(i) $0 \in \text{conve}^{-1}(c(\tau_0))$, or
(ii) $B_{\tau_0} = \{\tau_0\}$ (and $c(\tau_0)$ is an extremity of $C(y)$), or else
(iii) $0 < \lambda B_{\tau_0} < \lambda T_y/2$. 


Put \( x' = c(\tau_0) \).

In the first case (i), \( y \in Q_{x'} \). In the last two cases, there is a single Jordan arc \( J \subset C(y) \) from \( x \) to \( x' \). The multivalued mapping \( z \mapsto c^{-1}(z) \) defined on \( J \) is upper semicontinuous. For this reason, if \( z \in J \setminus C_3(y) \) is close to \( x \) and \( \tau \in c^{-1}(z) \), then \( \lambda B_\tau > \pi \geq \lambda T_y/2 \). For the same reason, if \( z \in J \setminus C_3(y) \) is close to \( x' \) and \( \tau \in c^{-1}(z) \), then \( \lambda B_\tau < \lambda T_y/2 \). Once again due to the upper semicontinuity of the above mapping, we have the intermediate value theorem in the form that, in between, there exists a point \( z_0 \in J \) for which \( 0 \in \text{conve}^{-1}(z_0) \).

**Theorem 1.** Every point on every surface is critical with respect to some point of the surface.

**Proof.** Let \( S \in A \) and \( y \in S \). We keep the notation from the proof of Lemma 3.

**Case 1.** \( S \) is homeomorphic to the sphere.

If \( C(y) \) is a single point, the conclusion is true. Suppose \( C(y) \) is not a point, but remember it is a tree.

Choose a point \( x \in C(y) \) different from an extremity of \( C(y) \). If all components of \( T_y \setminus c^{-1}(x) \) have length at most \( \pi \), then \( y \in Q_x \). If one component has length larger than \( \pi \), then the conclusion follows from Lemma 3.

**Case 2.** \( S \) is homeomorphic to the projective plane.

In this case \( C^{cp}(y) \) is a closed Jordan curve on \( S \). If \( x \in C^{cp}(y) \) is a ramification point of \( C(y) \), then at least three segments join \( x \) to \( y \), at least two of which come locally from the same side of \( C^{cp}(y) \) and thus bound a topological disc.

For any \( x \in C^{cp}(y) \) and \( \alpha \in c^{-1}(x) \), let \( I(\alpha) = \alpha_- \alpha_+ \subset T_y \) be the maximal arc containing \( \alpha \) such that, for each non-singular \( \tau \in I(\alpha) \), either \( c(\tau) \notin C^{cp}(y) \) or \( c(\tau) = x \). (The indices \( -,+ \) are taken according to a certain orientation of \( T_y \). We will say that \( \alpha_- \) is the left endpoint of the arc \( I(\alpha) \), while \( \alpha_+ \) is its right endpoint.) In other words, \( I(\alpha) \) is the maximal arc \( A' \subset T_y \) containing \( \alpha \), so that \( c(A'') \cap C^{cp}(y) \) be a single point, where \( A'' \) is the set of all non-singular directions in \( A' \). Of course, \( I(\alpha) \) may well be reduced to the singleton \( \{\alpha\} \). Then \( \alpha_- = \alpha_+ = \alpha \). This certainly happens if \( x \) is a cleave point, hence at all but at most countably many points of \( C^{cp}(y) \). If \( \alpha_- \neq \alpha_+ \), then there is a pencil of segments from \( x \) to \( y \), or \( x \) is a ramification point of \( C(y) \), or both.

If the arc \( I(\alpha) \) has length more than \( \pi \), either \( x \in Q_{x'}^{-1} \) in the case of the pencil, or we find a point in \( Q_{x'}^{-1} \) on a branch (i.e. component of \( C(y) \setminus C^{cp}(y) \)) of \( C(y) \) lying between the segments from \( y \) to \( x \) in directions \( \alpha_- \) and \( \alpha_+ \), by Lemma 3.

Assume now that \( \lambda I(\alpha) \leq \pi \) for all \( \alpha \in c^{-1}(x) \) and \( x \in C^{cp}(y) \).

For any \( x \in C^{cp}(y) \cap Cl(y) \), there are precisely two directions \( \tau_1(x) \) and \( \tau_2(x) \) with \( c(\tau_1) = c(\tau_2) = x \), by Lemma 2. When \( x \) describes \( C^{cp}(y) \cap Cl(y) \), moving from a position \( x_0 \) along \( C^{cp}(y) \) until it reaches again \( x_0 \), then \( \tau_1(x) \) moves from \( \tau_1(x_0) \) to \( \tau_2(x_0) \), jumping over arcs like \( I(\alpha) \), and \( \tau_2(x) \) moves from \( \tau_2(x_0) \) to \( \tau_1(x_0) \), also jumping over similar arcs.

If \( 0 \) was, say, on the right hand side of the line-segment \( \tau_1(x)\tau_2(x) \) when \( x \) started its motion, \( 0 \) ends up being on its left side. Therefore, for some position \( x \), either \( 0 \) is on \( \tau_1(x)\tau_2(x) \) or, more generally, \( 0 \in \text{conv}(I(\alpha) \cup I(\beta)) \) for some subarcs \( I(\alpha) \), \( I(\beta) \) of \( T_y \) with \( c(\alpha) = c(\beta) = x \). In both cases \( y \in Q_x \).

**Case 3.** \( S \) is neither homeomorphic to the sphere, nor to the projective plane.

In the proof of this case, to simplify notation, we shall write \( C, C^{cp}, Cl, C_3^{cp} \) instead of \( C(y), C^{cp}(y), Cl(y), C_3^{cp}(y) \), respectively, as the point \( y \) remains the same all the way. See Figs. 1 and 2.
Fig. 1. Cut locus $C(y)$ of a point $y$ on a torus. Here, $C_3(y) = \{b, b_1, c\}$, and $C^{cp}_3(y) = \{b\}$. The cyclic part $C^{cp}_3(y)$ consists of two cycles. The degree of $b$ in $C(y)$ is 7, and in $C^{cp}_3(y)$ is 4.

Fig. 2. This is $T_y$ corresponding to Fig. 1. There are four distinct arcs $I(\alpha)$ with $c(\alpha) = b$, denoted here simply by $b$, three of which are non-degenerate. There are two distinct arcs $I(\alpha)$ with $c(\alpha) = b_1$, denoted here by $b_1$, one degenerate and one not.

Take a point $x \in C^{cp} \setminus Cl$, and a direction $\alpha \in c^{-1}(x)$. (We have $C^{cp} \not= \emptyset$, as we are not in Case 1.) Notice that $C^{cp} \setminus Cl \supset C^{cp}_3$.

Consider first the case that $x \not\in C^{cp}_3$. We recall that $I(\alpha) = \alpha_- \alpha_+ \subset T_y$ is the maximal arc containing $\alpha$ such that, for each non-singular $\tau \in I(\alpha)$, either $c(\tau) \not\in C^{cp}$ or $c(\tau) = x$. If $I(\alpha)$ is not reduced to the singleton $\{\alpha\}$, we connect $\alpha_-$ to $\alpha_+$ by a line segment $\overline{\alpha_- \alpha_+}$ in $D$. (There must be some $\alpha \in c^{-1}(x)$ with $\alpha_- \not= \alpha_+$, since $x$ is not a cleave point.)
Consider now the case \( x \in C_3^{cp} \). (If \( C_3^{cp} = \emptyset \) then \( C^{cp} \) is a cycle and \( S \) is homeomorphic to the projective plane, which we now exclude.)

For each such \( x \) we have finitely many arcs like \( I(\alpha) \), perhaps degenerate, as many as the degree of \( x \) in \( C^{cp} \). For each arc \( I(\alpha) \), we connect \( \alpha_- \) (resp. \( \alpha_+ \)) to one endpoint of another arc \( I(\beta) \) with \( \beta \in c^{-1}(x) \), \( \beta \neq \alpha \) (resp. \( I(\gamma) \) with \( \gamma \in c^{-1}(x) \), \( \gamma \neq \alpha \)) in the following way.

By symmetry, it suffices to explain this for \( \alpha_- \). Let \( z \in C^{cp} \cap Cl \) converge to \( x \) in such a way that \( \tau_1(z) \) converges to \( \alpha_- \) from the left. Then \( \tau_2(z) \) tends to some point \( f(\alpha_-) \) in \( c^{-1}(x) \). It is not hard to see that this point \( f(\alpha_-) \) is an endpoint of another arc \( I(\beta) \), moreover \( \tau_2(z) \) tends to \( \beta_- \) from the left if the endpoint \( f(\alpha_-) \) is \( \beta_- \), and to \( \beta_+ \) from the right if \( f(\alpha_-) = \beta_+ \). We define this function \( f \) for \( \alpha_+ \) and for all other endpoints of arcs like \( I(\alpha) \), with \( c(\alpha) = x \) (while \( x \) remains fixed).

Connect \( \alpha_- \) to \( f(\alpha_-) \) by a line-segment \( \alpha_- f(\alpha_-) \) and \( \alpha_+ \) to \( f(\alpha_+) \) by a line-segment \( \alpha_+ f(\alpha_+) \).

When the same process is applied to the endpoint \( f(\alpha_-) \) of \( I(\beta) \), we obtain \( f(f(\alpha_-)) = \alpha_- \), so its connection with the endpoint \( \alpha_- \) of \( I(\alpha) \) is confirmed. This is due to the fact that for \( z \in C^{cp} \cap Cl \) the map \( \tau_1(z) \mapsto \tau_2(z) \) is an involution.

This shows that the union of the line-segments \( \alpha_- f(\alpha_-), \alpha_+ f(\alpha_+) \) and their analogs, for fixed \( x \in C_3^{cp} \), forms a cycle (closed polygonal line in \( D \)). Now we do this for all (finitely many) \( x \in C_3^{cp} \). So, we obtain a finite set of cycles. Their vertices on \( T_y \) are endpoints of (perhaps degenerate) arcs like \( I(\alpha) \).

Let now \( I(\alpha), I(\beta) \) be two arcs on \( T_y \) with \( c(\alpha) = x_1, c(\beta) = x_2 \) and \( x_1, x_2 \in C_3^{cp} \), (where \( x_1 \) and \( x_2 \) are different or not) so that \( \alpha_+, \beta_- \) are consecutive on \( T_y \). Then, as it is easy to check, \( f(\alpha_+) \) and \( f(\beta_-) \) are also consecutive on \( T_y \). Consider the cycle made up by the arcs \( \alpha_+ \beta_- \) and \( f(\alpha_+) f(\beta_-) \), and the line-segments \( \beta_- f(\beta_-), \alpha_+ f(\alpha_+) \), and all analogous cycles, in addition to the previous ones.

Moreover, consider the cycle formed by the arc \( \alpha_- \alpha_+ \) and the line-segment \( \alpha_- \alpha_+ \), plus all analogous cycles, not only for \( c(\alpha) \notin C_3^{cp} \), but also for \( c(\alpha) \in C_3^{cp} \).

Let \( C_1, \ldots, C_n \) be all the cycles defined above. Each of them is either a closed polygonal line, or a 2-cycle formed by one subarc of \( T_y \) and one line-segment, or a 4-cycle formed by two subarcs of \( T_y \) and two line-segments.

When \( \tau \) runs along \( T_y \), \( c(\tau) \) describes various parts of \( C \).

When \( \tau \) runs along an arc \( I(\alpha), c(\tau) \) either is constant (in the case of a pencil of segments), or runs through a tree in \( C \), starting and ending at the ramification point \( c(\alpha) \) of \( C \). The presence of a cycle which is a closed polygonal line indicates that its vertices are directions of segments all ending at the same ramification point of \( C^{cp} \).

If \( 0 \in \bigcup_{j=1}^n C_j \), i.e. \( 0 \) lies not inside but on some cycle, then \( 0 \) belongs to one of the line-segments, as \( 0 \notin T_y \). Hence \( c^{-1}(x) \) contains, for some \( x \in C^{cp} \setminus Cl \), two diametrically opposite points of \( T_y \), and we are done.

If not, consider the odd–even winding number, or degree modulo 2 (as it is called in Milnor’s book [6, p. 20]), \( w(C_j) = w(0, C_j) \) of the cycles \( C_j \) with respect to \( 0 \). We have

\[
\sum_{i=1}^n w(C_i) = w \left( \sum_{i=1}^n C_i \right) = w(T_y) = 1,
\]
because each line-segment is used exactly twice and each arc in $T_\gamma$ exactly once. This shows that $w(C_i) = 1$ for some cycle $C_i$.

Assume this cycle $C_i$ is of the form $\alpha_+ \beta_- f(\beta_-) f(\alpha_+)$, where $c(\alpha) = x_1$ and $c(\beta) = x_2$. We let $\tau$ move on the arc connecting $x_1$ to $x_2$ in $C^\text{cp}$, avoiding non-cleave points, so $c^{-1}(\tau) = \{\tau_1(z), \tau_2(z)\}$ is well-defined. Remember that $c^{-1}$ is upper semi-continuous everywhere.

If $\alpha_+ \beta_-$ and $f(\beta_-) f(\alpha_+)$ are of the same orientation on $T_\gamma$, then the proof parallels that of Case 1: $\tau_1(z) \in \alpha_+ \beta_-$ and $\tau_2(z) \in f(\alpha_+ f(\beta_-))$ move in contrary directions.

If $\alpha_+ \beta_-$ and $f(\beta_-) f(\alpha_+)$ are of opposite orientations on $T_\gamma$, then $\tau_1(z)$ and $\tau_2(z)$ move in the same direction, but $0$ lies on different sides of the line-segment $\tau_1(z) \tau_2(z)$ when $\tau$ is close to $x_1$ and when it is close to $x_2$; this and the argument of Case 2 yield the conclusion.

Assume next that $C_i$ is a cycle $\alpha_- \alpha_+ \cup \alpha_+ \alpha_-$. Then the conclusion follows from Lemma 3.

Finally, if $C_i$ is one of the other cycles (with all edges line-segments), then $w(C_i) = 1$ means that $0$ is surrounded by $C_i$, which is impossible if $0 \notin \text{conv} C_i$. By construction, $\text{conv} C_i = \text{conv} c^{-1}(x)$ for some $x \in C_i^\text{cp}$.

The proof is complete. $\square$

A characterization of the sphere. The following lemma shows that in general one cannot hope for a better lower bound. It extends Theorem 3 in [11] and admits a similar proof, which will therefore be omitted.

**Lemma 4.** Assume $S \in \mathcal{A}$, $y \in S$ is not conical, and $x \in \mathcal{Q}_y^{-1}$ is such that the union $U$ of two segments from $x$ to $y$ separates $S$. If a component $S'$ of $S \setminus U$ contains no segment from $x$ to $y$ then $\mathcal{Q}_y^{-1} \cap S' = \emptyset$. In particular, if the union of any two segments from $x$ to $y$ separates $S$ then $\mathcal{Q}_y^{-1} = \{x\}$.

**Theorem 2.** A smooth orientable surface $S$ is homeomorphic to the sphere if and only if each point in $S$ is critical with respect to precisely one other point of $S$.

**Proof.** If $S$ is homeomorphic to the sphere, then $\text{card} \mathcal{Q}_y^{-1} \geq 1$ by Theorem 1. Using now Lemma 4, we obtain $\text{card} \mathcal{Q}_y^{-1} = 1$.

Next, we show that every orientable surface of positive genus contains a point $y$ with $\text{card} \mathcal{Q}_y^{-1} > 1$.

To see this, let $\Omega$ denote a shortest simple closed curve which does not separate $S$. Then $\Omega$ is a closed geodesic. Moreover, for any of its points $z$, $\Omega$ is the union of two segments of length $\lambda \Omega / 2$ starting at $z$ and ending at $z\Omega$. Consider the family $\mathcal{C}$ of all simple closed not contractible curves $C$ which cut $\Omega$ at precisely one point, such that $\Omega$ separates $C$ locally at $\Omega \cap C$. Then clearly $\mathcal{C} \neq \emptyset$, by the choice of $\Omega$. Let $\Omega'$ be a shortest curve in $\mathcal{C}$; it is a closed geodesic too. Moreover, by the definition of $\mathcal{C}$ and by the choice of $\Omega'$, the latter is the union of two segments starting at $\{y\} = \Omega \cap \Omega'$ and ending at $y\Omega'$. It follows that $\mathcal{Q}_y^{-1}$ contains at least two points, $y\Omega$ and $y\Omega'$.

**Open problem.** Every smooth orientable surface of positive genus possesses points $x$, $y$ such that $y$ is critical with respect to $x$ and two segments from $y$ to $x$ have opposite directions at $y$ (see the proof of Theorem 2). Is the same true for all smooth surfaces homeomorphic to the sphere? Or, at least, if $A_0$ denotes the space of all Alexandrov surfaces homeomorphic to the sphere, endowed with the Hausdorff–Gromov metric, is there a dense set in $A_0$ with the above property? For convex surfaces, this problem was raised in [13], and is still open.
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