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Computing the Volume is Difficult*

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Abstract. For every polynomial time algorithm which gives an upper bound $\overline{\text{vol}}(K)$ and a lower bound $\underline{\text{vol}}(K)$ for the volume of a convex set $K \subset R^d$, the ratio $\overline{\text{vol}}(K)/\underline{\text{vol}}(K)$ is at least $(cd/\log d)^d$ for some convex set $K \subset R^d$.

1. Introduction

The problem addressed in this paper is the behavior of algorithms that compute the volume of convex sets. We prove a negative result. For any polynomial time algorithm which gives a lower bound $\underline{\text{vol}}(K)$ and an upper bound $\overline{\text{vol}}(K)$ for the volume of a convex set $K \subset R^d$, the ratio $\overline{\text{vol}}(K)/\underline{\text{vol}}(K)$ is at least $(cd/\log d)^d$ for some convex body $K \subset R^d$ where c is a constant independent of d.

Our model of a convex set coincides with that of Lovász [9] and Grötschel *et al.* [7]. In this model a convex set $K \subset R^d$ is black box that answers questions of the following type. Given a point $x \in Q^d$, is $x \in K$? In this case we say that the black box (or the convex set) is given by a membership oracle. The convex set K may be given by a separation oracle as well. This is again a black box which, given a point $x \in Q^d$, decides whether $x \in K$ and if it is not, the box then gives a hyperplane separating x and K.

A moment's meditation shows that one needs some further information on the convex set given by the black box. So the black box will have to wear an additional guarantee: the convex set described by this box is contained in RB^d and contains rB^d, where B^d is the Euclidean unit ball around the origin and

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$R > r > 0$. In this case we say that the oracle describing the convex body is well guaranteed. For technical reasons we assume that $R = 2^{l_1}$ and $r = 2^{-l_2}$ where l_1 and l_2 are nonnegative integers; then the input size of the oracle is $d + (1 + l_1) + (1 + l_2)$. So we assume that convex sets are given by a separation oracle which is well guaranteed. Using a version of the ellipsoid method, Lovász [9] gave an algorithm that determines a lower bound $\underline{\mathrm{vol}}(K)$ and an upper bound $\overline{\mathrm{vol}}(K)$ for the volume of the convex set K. This algorithm is polynomial in the input size of the oracle and has the following property:

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq d^{d/2}(d+1)^d.$$

Moreover, if the convex set described by the oracle is centrally symmetric, then the result is better:

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq d^d.$$

Both estimations seem to be very poor, but the following result of Elekes [5] (see Lovász [9]) shows that any polynomial time algorithm must leave a huge gap between $\overline{\mathrm{vol}}(K)$ and $\underline{\mathrm{vol}}(K)$. He proved, in fact, that there is no polynomial time algorithm which would compute a lower bound and an upper bound for $\mathrm{vol}(K)$ with

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq 1.999^d.$$

Lovász [8] thought that even $(\overline{\mathrm{vol}}(K)/\underline{\mathrm{vol}}(K))^{1/d}$ cannot be bounded. We prove this in a stronger form in Theorem 1.

Theorem 1. *There is no polynomial time algorithm which would compute a lower and an upper bound for* $\mathrm{vol}(K)$ *with*

$$\frac{\overline{\mathrm{vol}}(K)}{\underline{\mathrm{vol}}(K)} \leq \left(c \frac{d}{\log d}\right)^d,$$

where the constant c does not depend on d.

Theorem 1 shows that Lovász' algorithm is very close to being optimal when the oracle contains centrally symmetric convex bodies only.

Let $V(d, n)$ denote the maximum volume of the convex hull of n points from B^d. Theorem 1 will follow from Theorem 2.

Theorem 2. *If $n = d^a$, then, for sufficiently large d,*

$$\frac{V(d, n)}{\mathrm{vol}(B^d)} \le \left(\frac{2ae \log d}{d}\right)^{d/2}.$$

The estimation in Theorem 2 is fairly good. This can be seen from Theorem 3.

Theorem 3. *If $n = d^a$, then, for sufficiently large d,*

$$\frac{V(d, n)}{\mathrm{vol}(B^d)} \ge \left(\frac{(2a-3) \log d}{d}\right)^{d/2}.$$

Theorems 2 and 3 may be written as

$$c_1\left(\frac{a \log d}{d}\right)^{1/2} < \left(\frac{V(d, d^a)}{\mathrm{vol}(B^d)}\right)^{1/d} < c_2\left(\frac{a \log d}{d}\right)^{1/2},$$

and these inequalities are the approximation of the ball by polytopes with "few" vertices. We have some other results in this direction which will be published in a forthcoming paper [1].

We will use a beautiful new result of Bourgain and Milman [2] which we now describe. Let $\mathcal{K}$ be the family of all centrally symmetric (with respect to the origin), convex, compact, d-dimensional bodies in R^d. The polar, K^*, of $K \in \mathcal{K}$ is defined as

$$K^* = \{x \in R^d : \langle x, y\rangle \le 1, \forall y \in K\},$$

where $\langle x, y\rangle$ denotes the scalar product. An old conjecture says that for all $K \in \mathcal{K}$

$$\mathrm{vol}(K)\,\mathrm{vol}(K^*) \ge 4^d / d!.$$

Bourgain and Milman [2] proved this in a slightly weaker form: for all $K \in \mathcal{K}$

$$\mathrm{vol}(K)\,\mathrm{vol}(K^*) \ge c_0^d / d!,$$

where $c_0 > 0$ is a universal constant.

We will see from the proofs that the constant c in Theorem 1 can be taken for $c_0(4\pi ae)^{-1}$ when the algorithm considered tests the membership on $n = d^a$ points.

In the last section we give some results about the complexity of computing the width of a convex body.

2. Proof of Theorem 1

We use a well-guaranteed separation oracle with some additional properties. The first is that the oracle discloses (as a first step, say) that $\varepsilon_i e_i \in K$ and $K \subset \{x \in R^d : \langle x, \varepsilon_i e_i \rangle \le 1\}$ for each $\varepsilon_i \in \{-1, 1\}$ and $i = 1, \dots, d$ where $e_1, \dots, e_d$ form an orthonormal basis in R^d. This property simply means that K is contained in the cube of side length 2 and contains the cross polytope of diameter 2. In accordance with this $l_1 = \lceil \frac{1}{2} \log d \rceil$ and $l_2 = \lfloor -\frac{1}{2} \log d \rfloor$. Thus the input size of the oracle is $d + 1 + l_1 + 1 + l_2 < 2d$ if d is large enough.

We need some notation. For $x \in R^d$ $(x \ne 0)$ define $x^0 = x / \|x\|$ and $H^+(x^0) = \{z \in R^d : \langle z, x^0 \rangle \le 1\}$ and $H^-(x^0) = \{z \in R^d : \langle z, x^0 \rangle \ge -1\}$. The second additional property of the oracle is that for the question "is $x \in K$" it answers "$x^0 \in K$ and $-x^0 \in K$ and $K \subset H^+(x^0)$ and $K \subset H^-(x^0)$." So the oracle gives the endpoints of the line segment $\{\lambda x : \lambda \in R\} \cap K$ and also the supporting hyperplanes at the endpoints. We mention that this information (with any prescribed precision) can be obtained from a separation oracle in polynomial time. So our oracle is just a little stronger than a usual separation oracle on centrally symmetric convex bodies.

Now we begin the proof. Assume that we have an algorithm that gives an upper bound and a lower bound for the volume of a convex body given by the above separation oracle. Let us run this algorithm with $K = B^d$ first, the points whose membership has been asked by the algorithm are $x_1, x_2, \dots, x_n$ with $n = d^a$ $(a > 1)$. Assume the algorithm produced $\overline{\mathrm{vol}}(B^d)$ and $\underline{\mathrm{vol}}(B^d)$.

Now set $C = \mathrm{conv}\{\pm e_1, \dots, \pm e_d, \pm x_1^0, \dots, \pm x_n^0\}$. It is clear that when running the algorithm with C or with C^* (the polar of C), the questions and the answers are the same as with B^d, so

$$\overline{\mathrm{vol}}(B^d) = \overline{\mathrm{vol}}(C^*) \ge \mathrm{vol}(C^*)$$

and

$$\underline{\mathrm{vol}}(B^d) = \underline{\mathrm{vol}}(C) \le \mathrm{vol}(C).$$

Then

$$\frac{\overline{\mathrm{vol}}(B^d)}{\underline{\mathrm{vol}}(B^d)} \ge \frac{\mathrm{vol}(C^*)}{\mathrm{vol}(C)} = \mathrm{vol}(C^*)\,\mathrm{vol}(C) \left(\frac{1}{\mathrm{vol}(C)} \right)^2.$$

From the result of Bourgain and Milman [2] we infer

$$\frac{\overline{\mathrm{vol}}(B^d)}{\underline{\mathrm{vol}}(B^d)} \ge \frac{c_0^d}{d!} \left(\frac{\mathrm{vol}(B^d)}{\mathrm{vol}(C)} \right)^2 \left(\frac{1}{\mathrm{vol}(B^d)} \right)^2.$$

Now the number of vertices of C is $2(n + d) \approx d^a$, so from Theorem 2 we have

$$\frac{\overline{\mathrm{vol}}(B^d)}{\underline{\mathrm{vol}}(B^d)} \ge \left(\frac{c_0 d}{4\pi e a \log d} \right)^d. \qquad \square$$

Remark. It may seem strange that the volume of the unit ball (when it is given by a separation oracle) cannot be determined within a large factor. However, this is not so surprising when one thinks of the fact that among all convex bodies the ellipsoids admit the worst approximation by polytopes. (See Macbeath [10] for an exact statement.)

3. Proof of Theorem 2

Some preparation is needed. Given a convex set $C \subset R^d$ with $L = \text{aff}(C)$, define $L^\perp$ as the maximal subspace of R^d orthogonal to L. Further, for $\rho > 0$ let

$$C^\rho := C + (L^\perp \cap \rho B^d),$$

i.e., C^ρ is the set of points $x \in R^d$ such that if x' is the nearest point to x in C, then $\|x - x'\| \le \rho$ and $x - x'$ is orthogonal to L. Define $\rho(d, 1) = 1$, $\rho(d, d) = d^{-1}$ and for $1 < k < d$

$$\rho(d, k) = \left(\frac{d-k+1}{d(k-1)}\right)^{1/2}.$$

We need a lemma which says that any point of a simplex in B^d is "near" and "orthogonal" to some $(k-1)$-face of the simplex.

Lemma. *Given a simplex F in B^d and $k \in \{1, 2, \ldots, d\}$ and a point $x \in F$, there is a $(k-1)$-face F_k of F with $x \in F_k^{\rho(d,k)}$.*

Proof. An easy calculation shows that the statement of the lemma is true when $k = 1$. The case $k = d$ is equivalent to the following well-known fact (see Fejes Tóth [6]). The ratio of the radii of the circumscribed and inscribed balls of a simplex in R^d is at least d. We prove the lemma using this fact for the cases $k = 2, 3, \ldots, d-1$. Rename x as x_{d+1} and F as F_{d+1}. By the above fact there is a facet F_d such that if x_d denotes the projection of x_{d+1} to F_d, then $\|x_{d+1} - x_d\| \le d^{-1}$ and $x_{d+1} - x_d$ is orthogonal to $\text{aff}(F_d) = H_d$. Now F_d lies in $H_d \cap B^d$, so F_d lies in B^{d-1} if we choose the origin in H_d properly. On applying the same argument to $F_d \subset B^{d-1}$ and x_d we get a point x_{d-1} in a facet F_{d-1} of F_d such that $\|x_d - x_{d-1}\| \le 1/(d-1)$ and $x_d - x_{d-1}$ is orthogonal to $\text{aff}(F_{d-1}) = H_{d-1}$. And so on. We stop with $x_k \in F_k$. The vectors $x_{j+1} - x_j$ $(j = d, \ldots, k)$ are pairwise orthogonal and all of them are orthogonal to F_k. Consequently, $x_{d+1} - x_k$ is orthogonal to F_k. By Pythagoras' theorem, $\|x_{d+1} - x_k\|^2 = \|x_{d+1} - x_d\|^2 + \|x_d - x_{d-1}\|^2 + \cdots + \|x_{k+1} - x_k\|^2 \le 1/d^2 + 1/(d-1)^2 + \cdots + 1/k^2 < 1/(d(d-1)) + 1/((d-1)(d-2)) + \cdots + 1/(k(k-1)) = (d-k+1)/(d(k-1))$, as claimed. □

Remark. It is very likely that the smallest value of $\rho(d, k)$ for which the lemma holds is $((d-k+1)/(dk))^{1/2}$. This is the value of $\rho(d, k)$ when F is a regular simplex with its vertices in S^d. However, for our purposes the $\rho(d, k)$ from the lemma will do and we could gain nothing in Theorem 2 with the best value of ρ.

Now we prove Theorem 2. Let $x_1, \ldots, x_n \in B^d$. By Carathéodory's theorem (see Danzer *et al.* [4]) every point $x \in \operatorname{conv}\{x_1, \ldots, x_n\}$ belongs to some simplex with vertices from $\{x_1, \ldots, x_n\}$, i.e., $x \in \operatorname{conv}\{x_{i_0}, \ldots, x_{i_d}\} = F$ for some indices $1 \le i_0 < i_1 < \cdots < i_d \le n$. By the lemma, F has a $(k-1)$-dimensional face F_k with $x \in F_k^{\rho(d,k)}$. This implies that $\operatorname{conv}\{x_1, \ldots, x_n\} \subseteq \bigcup \{C^{\rho(d,k)}: C = \operatorname{conv}\{x_{j_1}, \ldots, x_{j_k}\}\}$ where the union is taken over all k-tuples from $\{x_1, \ldots, x_n\}$. This shows that

$$\operatorname{vol}(\operatorname{conv}\{x_1, \ldots, x_n\}) \le \binom{n}{k} \max\{\operatorname{vol}(C^{\rho(d,k)}): C = \operatorname{conv}\{a_1, \ldots, a_k\},\ a_1, \ldots, a_k \in B^d\}.$$

It is now easy to see that

$$\begin{aligned}
&\max\{\operatorname{vol}(C^{\rho(d,k)}): C = \operatorname{conv}\{a_1, \ldots, a_k\} \subseteq B^d\} \\
&\quad = \max\{\operatorname{vol}_{k-1}(\operatorname{conv}\{a_1, \ldots, a_k\}): a_1, \ldots, a_k \in B^d\} \\
&\qquad \times \operatorname{vol}_{d-k+1}(B^{d-k+1})[\rho(d,k)]^{d-k+1} \\
&\quad = \left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{k^{1/2}\pi^{(d-k+1)/2}}{(k-1)!\,\Gamma((d-k+1)/2+1)} [\rho(d,k)]^{d-k+1}.
\end{aligned}$$

This implies that

$$V(d,n) \le \binom{n}{k} \left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{k^{1/2}\pi^{(d-k+1)/2}}{(k-1)!\,\Gamma((d-k+1)/2+1)} [\rho(d,k)]^{d-k+1}.$$

This holds for every $k = 1, 2, \ldots, d$. Now we choose $k = d(2\log n)^{-1} = d(2\alpha \log d)^{-1}$. This gives, after a tiresome calculation,

$$\frac{V(d,n)}{\operatorname{vol}(B^d)} < \frac{e^{d(1/2 - 1/a + \varepsilon)} 2^{d/2} (a \log d)^{d/2}}{d^{d/2}}$$

for every $\varepsilon > 0$ if d is large enough. □

4. Proof of Theorem 3

We would like to compute the expected volume of the convex hull of n points chosen uniformly and independently from S^d. Unfortunately there is no known formula for this. We use instead an integral formula due to Buchta *et al.* [3] which gives the expected surface area $E(d,n)$ of the convex hull of n points chosen uniformly and independently from S^d:

$$\begin{aligned}
E(d,n) = &\binom{n}{d} \frac{d w_{d-1}}{(d-1)^{d-1}} \left(\frac{w_{d-1}}{w_d}\right)^{d-1} \\
&\times \int_{-1}^{1} \left(\frac{w_{d-1}}{w_d} \int_p^1 (1-q^2)^{(d-3)/2}\, dq\right)^{n-d} (1-p^2)^{(d^2-d-2)/2}\, dp,
\end{aligned}$$

where $w_d = \text{Area}(S^d)$ denotes the surface area of S^d. In order to use this formula we choose $n-d$ points $x_1, \ldots, x_{n-d}$ uniformly and independently from S^d. Then we take d points $y_1, \ldots, y_d \in S^d$ in such a way that $x_1, y_1, \ldots, y_d$ form the vertices of a regular simplex. Denote by $L_1, \ldots, L_m$ the facets of $C = \text{conv}\{x_1, \ldots, x_{n-d}, y_1, \ldots, y_d\}$. C contains $d^{-1}B^d$ hence

$$\text{vol}(C) \geq d^{-2} \sum_{i=1}^{m} \text{vol}_{d-1}(L_i) = d^{-2}\,\text{Area}(C).$$

Moreover, $C \supset C_0 = \text{conv}\{x_1, \ldots, x_{n-d}\}$. Thus

$$\text{vol}(C) \geq d^{-2}\,\text{Area}(C) \geq d^{-2}\,\text{Area}(C_0).$$

This clearly implies that

$$V(d, n) \geq d^{-2}E(d, n-d).$$

After a lengthy computation (the details can be found in Bárány and Füredi [1]) we get that for d large enough

$$\frac{V(d,n)}{\text{vol}(B^d)} \geq \left(\frac{2(a-1)\log d}{d}\right)^{d/2} (\log d)^{-d/\log d}. \qquad \square$$

5. The Error in Computing the Width

Lovász [9] gives a polynomial time algorithm which computes a lower bound $\underline{w}(K)$ and an upper bound $\bar{w}(K)$ for the width $w(K)$ of a convex body $K \subset R^d$ with $\bar{w}(K)/\underline{w}(K) \leq d^{1/2}(d+1)$. The convex sets are again given by a well-guaranteed separation oracle. Elekes [5] proved that there is no polynomial time algorithm which would compute $\bar{w}(K)$ and $\underline{w}(K)$ with $\bar{w}(K)/\underline{w}(K) \leq 2$. We improve on this result.

Theorem 4. *There is no polynomial time algorithm which would compute an upper bound $\bar{w}(K)$ and a lower bound $\underline{w}(K)$ for the width of convex bodies $K \subset R^d$ with*

$$\bar{w}(K)/\underline{w}(K) \leq (d/(c\log d))^{1/2}.$$

Proof. We consider the same model as in the proof of Theorem 1. Then

$$\bar{w}(B^d) = \bar{w}(C^*) \geq w(C^*) = 2$$

and

$$\underline{w}(B^d) = \underline{w}(C) \leq w(C).$$

So the theorem will follow if we can show that

$$w(C) \leq 2(2a(\log d)/d)^{1/2}, \tag{1}$$

when $C \subset B^d$ is a centrally symmetric polytope with $n = 2d^a$ vertices, because then

$$\frac{\bar{w}(B^d)}{\underline{w}(B^d)} = \frac{w(C^*)}{w(C)} \geq \frac{2}{2((2a \log d)/d)^{1/2}} \geq \left(\frac{d}{2a \log d}\right)^{1/2}.$$

To see this one finds a spherical cap $S \subset S^d$ with

$$S \cap \{\pm e_1, \ldots, \pm e_d, \pm x_1^0, \ldots, \pm x_n^0\} = \varnothing$$

and

$$\text{dist}(0, \text{conv } S) = (2a \log d/d)^{1/2}.$$

This can be shown by a simple averaging argument.

Another way to see that (1) holds with the slightly weaker constant $2ae$ (instead of $2a$) is to use Theorem 2. It follows from there that C cannot contain the ball rB^d with $r > (2ae(\log d)/d)^{1/2}$. So there is a point z on the boundary of C with $\|z\| \leq (2ae(\log d)/d)^{1/2}$. Taking supporting hyperplanes to C at z and at $-z$ we get (1) with the weaker constant. □

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