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SHORT COMMUNICATION

BORSUK'S THEOREM THROUGH COMPLEMENTARY PIVOTING

Imre BÁRÁNY

OTSZK, KTT, Budapest, Hungary

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In this short note a simple and constructive proof is given for Borsuk's theorem on antipodal points. This is done through a special application of the complementary pivoting algorithm.

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In this paper we give a new proof of Borsuk's theorem on antipodal points [2]. This proof may be of some interest because it is simple and constructive. The need for such a combinatorial proof emerged in connection with a surprising application of Borsuk's theorem in graph theory [1, 5]. We shall make use of the so-called complementary pivoting algorithm (see e.g. [4, 6]). The reader is supposed to be familiar with this technique. We mention that our treatment is based mostly on [4].

Let x_i denote the i-th coordinate of $x \in \mathbf{R}^n$ for $i = 1, \dots, n$. Write $\|\cdot\|$ and $|\cdot|$ for the Euclidean resp. max norm. Put $S^n = \{x \in \mathbf{R}^{n+1} \colon \|x\| = 1\}$ and $C^n = \{x \in \mathbf{R}^{n+1} \colon |x| = 1\}$. If $\delta > 0$ and $A \subseteq \mathbf{R}^n$, then $\delta A = \{\delta x \in \mathbf{R}^n \colon x \in A\}$. A function $f : A \to \mathbf{R}^n$ is said to be odd if $x \in A$ implies $-x \in A$ and $f(-x) = -f(x)$ (here $A \subseteq \mathbf{R}^m$ for some m). We write $x < y$ for $x, y \in \mathbf{R}^n$ if x is lexicographically less than y. If K is a triangulation, then K^i denotes its i-dimensional simplices, in particular, K^0 is the set of vertices of K. Finally, e_i denotes the i-th basis vector of $\mathbf{R}^{n+1}$ for $i = 1, \dots, n+1$.

Theorem 1 (Borsuk [2]). *If $f : S^n \to \mathbf{R}^n$ is continuous and $n \geq 1$, then there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

It is clear that this theorem is equivalent to the following one.

Theorem 2. *If $f : C^n \to \mathbf{R}^n$ is an odd continuous map and $n \geq 1$, then there exists a point $x \in C^n$ with $f(x) = 0$.*

We shall prove this second theorem. Now we need some preparations.

First we shall define a special triangulation, L, of $\mathbf{R}^{n+1}$ as follows (L is the same as the triangulation K_1 of [6, page 29]). L^0 is the set of all integer lattice points of $\mathbf{R}^{n+1}$, and a set $\{y_1, y_2, \dots, y_{n+2}\} \subset L^0$ with $y_1 < y_2 < \cdots < y_{n+2}$ is the set of vertices of an $(n+1)$-simplex of L if there exists a permutation π of the numbers $1, 2, \dots, n+1$ such that for $i = 1, 2, \dots, n+1$

$$y_{i+1} = y_i + e_{\pi(i)}. \tag{1}$$

It is shown in [6] that L is indeed a triangulation of $\mathbf{R}^{n+1}$. Here we claim that L is symmetric with respect to the origin, i.e., $\sigma \in L$ implies $-\sigma \in L$. The proof of this fact is quite easy and is, therefore, omitted.

Now if $t \in \mathbf{R}^1$, then $[t]$ denotes the vector $(1, t, \dots, t^{n-1}) \in \mathbf{R}^n$. Let $0 < t_1 < t_2 < t_3 < \cdots < t_{2^{n+1}} < 1$ and for $u \in L^0$ let $m(u)$ be the integer for which

$$\begin{aligned} &m(u) \in \{1, 2, 3, \dots, 2^{n+1}\}, \\ &m(u) \equiv \sum_{i=0}^{n} 2^i u_{i+1} \bmod 2^{n+1}. \end{aligned} \tag{2}$$

Clearly, $m(u)$ is well-defined. Now let $h : (L^0 \setminus \{0\}) \to \mathbf{R}^n$ be defined in the following way

$$h(u) = \begin{cases} [t_{m(u)}] & \text{if } 0 < u,\ u \in L, \\ -[t_{m(u)}] & \text{if } u < 0,\ u \in L. \end{cases}$$

It is evident that h is odd. We shall need one more property of h: if $u_1, \dots, u_n \in L^0 \setminus \{0\}$ are the vertices of any $\sigma \in L^{n-1}$, then

$$\det[h(u_1), \dots, h(u_n)] \neq 0. \tag{3}$$

Indeed, if $v_1, \dots, v_{n+2}$ are the vertices of an $(n+1)$-simplex of L, then it is easy to check by (1) and (2) that $m(v_1), \dots, m(v_{n+2})$ are pairwise different integers. Then, a fortiori, $m(u_1), \dots, m(u_n)$ are again pairwise different and

$$\det[h(u_1), \dots, h(u_n)] = (-1)^\gamma \det[[t_{m(u_1)}], \dots, [t_{m(u_n)}]],$$

where γ is the number of u_i's with $u_i < 0$. Clearly, this last determinant is not equal to zero. This proves (3).

Proof of Theorem 2. In what follows a complementary pivoting routine will take place on a (vector labelled) finite triangulation of the set $H = H_k = \{x \in \mathbf{R}^{n+1}: 1 - 1/k \leq |x| \leq 1\}$ where $k \geq 2$ is an integer. This triangulation is defined to be $K = K_k = \{(1/k)\sigma: \sigma \in L \text{ and } (1/k)\sigma \subset H\}$. It is easy to check that K is a triangulation of H, K is finite and symmetric with respect to the origin, further, $K^0 \subset \partial H$ and diam $\sigma \leq 1/k$ for every $\sigma \in K$ (diam is meant in the max norm).

Clearly, $\partial H = B \cup C$ where $B = (1 - 1/k)C_n$ and $C = C_n$. Choose a vector $v \in \mathbf{R}^n$ such that $|v| < 1 - 1/k$ and $(v_1, \dots, v_n, 1 - 1/k) \in \operatorname{relint} \sigma_0$ for some $\sigma_0 \in K^n$, of course, $\sigma_0 \subset B$. Now we define a map $g : B \to \mathbf{R}^n$ by

$$g(x) = g(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n) - \frac{k}{k-1} x_{n+1}(v_1, \dots, v_n).$$

Clearly, g is odd and $g(x) = 0$ if and only if $x = \pm(v_1, \dots, v_n, 1 - 1/k)$.

Now let us define the vector labelling $l : K^0 \to \mathbf{R}^n$ by

$$l(x) = l_\epsilon(x) = \begin{cases} f(x) + \epsilon h(kx) & \text{if } x \in K^0 \cap C, \\ g(x) + \epsilon h(kx) & \text{if } x \in K^0 \cap B, \end{cases}$$

where $\epsilon > 0$. Extend this labelling rule to a piecewise linear $l : H \to \mathbf{R}^n$ map. l is odd. Now we claim that there exists a positive $\delta \le 1/k$ such that for $0 < \epsilon < \delta$ we have

(i) there are exactly two solutions, x_0 and $-x_0$, of the equation $l(x) = 0$ satisfying $x \in B$, and one of them, say x_0, lies in relint σ_0;

(ii) $0 \in l(\sigma)$, $\sigma \in K$ implies $\sigma \in K^n \cup K^{n+1}$.

Indeed, $|l_\epsilon(x) - g(x)| < \epsilon$ for every $x \in B$. Clearly, for some $\eta > 0$ $|g(x)| \ge \eta$ if $x \in B \setminus (\text{relint } \sigma_0 \cup \text{relint } -\sigma_0) = D$ whence $|l_\epsilon(x)| \ge \eta - \epsilon$ for every $x \in D$. Thus $l_\epsilon(x) = 0$ has no solution with $x \in D$ if $\epsilon < \eta$. Further, g and l_ϵ are linear maps on σ_0 and g has exactly one zero in σ_0. So if g and l_ϵ are sufficiently near, i.e., $\epsilon < \eta'$ for some $\eta' > 0$, then l_ϵ, too, has exactly one zero in σ_0, x_0. As we have seen x_0 cannot be on the relative boundary of σ_0 if $\epsilon < \eta$. So for $0 < \epsilon < \min(\eta, \eta')$ (i) holds true.

Suppose now that $0 \in l_\epsilon(\sigma)$ for some $\sigma \in K^{n-1}$. This means that for some $\alpha_i \ge 0$, $i = 1, \dots, n$,

$$\sum_{i=1}^{n} \alpha_i l_\epsilon(u_i) = 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1, \tag{4}$$

where $u_1, \dots, u_n$ are the vertices of σ. Writing $l_\epsilon(u_i) = a_i + \epsilon h(ku_i)$ (here either $a_i = f(u_i)$ or $a_i = g(u_i)$) we have from (4),

$$P(\epsilon) = \det[a_1 + \epsilon h(ku_1), \dots, a_n + \epsilon h(ku_n)] = 0.$$

$P(\epsilon)$ is a polynomial of ϵ and the coefficient of ϵ^n, $\det[h(ku_1), \dots, h(ku_n)]$ is different from zero by (3). This implies that $P(\epsilon) \ne 0$ for $\epsilon \in (0, \delta_\sigma)$ for some $\delta_\sigma > 0$, i.e., (4) cannot be true for $0 < \epsilon < \delta_\sigma$. This implies that for $0 < \epsilon < \delta$ with $\delta = \min(1/k, \eta, \eta', \min_{\sigma \in K^{n-1}} \delta_\sigma)$ (i) and (ii) hold true.

We mention that in the terminology of [4], the condition (ii) means that 0 is a regular value of the piecewise linear map l_ϵ. Now fix ϵ with $0 < \epsilon < \delta$.

Put $M = \{z \in H : l(z) = 0\}$. We define a graph G as follows. Its nodes are the points $x \in M$ with $x \in \sigma$ for some $\sigma \in K^n$ and two different nodes, x and y, form an edge of G iff $x, y \in \tau$ for some $\tau \in K^{n+1}$. The degree of a node of G is the number of edges adjacent to this node. We write $[u, v]$ for the line segment connecting $u \in \mathbf{R}^{n+1}$ and $v \in \mathbf{R}^{n+1}$. Due to the implication (ii) the following facts are true (for the proofs see [4] or [6]).

The degree of a node of G is 1 or 2 according to whether it is contained in ∂H or in int H. Further,

$$M = \{z \in H : z \in [x, y] \text{ for some edge } (x, y) \text{ of } G\}.$$

Together these facts imply that M is a 1-manifold (for the definition of 1-manifold, see [4]) and so it consists of a finite number of pairwise disjoint polygonal paths, and there are two types of these paths, the first type going from a boundary node to a boundary node without cycles and the second type being a a single cycle lying entirely in int H.

The main step in the proof of this facts is that for a node x of G with $x \in \sigma \subset \tau \in K^{n+1}$ and $\sigma \in K^n$ there exists exactly one node, x', of G with $x' \in \sigma' \in K^n$, $\sigma' \subset \tau$ for which (x, x') is an edge of G. To determine x' and σ' from x, σ and τ is easy, it is done through a linear programming pivot step (see [4] or [6]).

Our algorithm follows a path $l(x) = 0$.

Start with the triple (x_0, σ_0, τ_0) where $\tau_0 \in K^{n+1}$ is the only $(n+1)$-simplex containing σ_0.

Step j for $j = 0, 1, 2, \ldots$. For the triple (x_j, σ_j, τ_j) determine $x_{j+1} \in \tau_j$ as the only node of G adjacent to x_j and $\sigma_{j+1} \in K^n$ with $x_{j+1} \in \sigma_{j+1}$. If $x_{j+1} \in \partial H$, then stop, else determine $\tau_{j+1} \in K^{n+1}$ as the only $(n+1)$-simplex containing σ_{j+1} and different from τ_j. (The rules for this end are given in [6, p. 35]). Proceed to step $j+1$ with the triple $(x_{j+1}, \sigma_{j+1}, \tau_{j+1})$.

We know that this algorithm produces a path through the nodes $x_0, x_1, \ldots, x_p$ from the boundary node x_0 to the boundary node x_p $(p \geq 1)$. We claim that $x_p \in C$. Suppose, on the contrary, that $x_p \in B$, then, by property (i) of l, we must have $x_p = -x_0$. Starting now the algorithm with the triple $(-x_0, -\sigma_0, -\tau_0)$, we shall get the polygonal path through the points $-x_0, -x_1, \ldots, -x_p$ because l is odd. These two paths are not disjoint for $x_p = -x_0$ and so they coincide: $x_{p-1} = -x_1, x_{p-2} = -x_2, \ldots, x_0 = -x_p$. Let z be the "middle point" of the first path, i.e.,

$$z = \begin{cases} x_{p/2} & \text{if } p \text{ is even,} \\ \frac{1}{2}(x_{(p+1)/2} + x_{(p-1)/2} & \text{if } p \text{ is odd.} \end{cases}$$

It is easy to check that $z \in H$ and $z = -z$. Consequently $z = 0$ and $0 \in H$. This contradicts to the definition H for $x \in H$ implies $|x| \geq 1 - 1/k$.

As we have seen $x_p \in C$. Then condition (ii) implies that $\sigma_p \subset C$. Writing $y_1, \ldots, y_{n+1}$ for the vertices of σ_p we have for some $\alpha_i \geq 0$ $(i = 1, \ldots, n+1)$ that

$$x_p = \sum_{i=1}^{n+1} \alpha_i y_i, \qquad \sum_{i=1}^{n+1} \alpha_i = 1, \qquad \sum_{i=1}^{n+1} \alpha_i l(y_i) = 0.$$

This implies that

$$\left|\sum_{i=1}^{n+1} \alpha_i f(y_i)\right| = \left|\sum_{i=1}^{n+1} \alpha_i \epsilon h(k y_i)\right| \leq \epsilon \sum_{i=1}^{n+1} \alpha_i \leq 1/k. \tag{5}$$

For each $k = 2, 3, \ldots$ we have an n-simplex $\sigma_{p(k)} \subset C$ whose vertices satisfy (5). There is a subsequence of $\sigma_{p(k)}$ that converges to a point $y \in C$ because C is compact and $\operatorname{diam} \sigma_{p(k)}$ tends to zero. By continuity and (5) we must have $f(y) = 0$. And this is what we wanted to prove.

Remarks. This proof is not "quite constructive" because δ and so ϵ in the perturbation $\epsilon h(ku)$ is not determined constructively. One may hope that, as usual (see e.g., [4] or [6]), a lexicographic scheme could be used to produce a path between B and C. However, it is not difficult to find an example showing that this is not the case, i.e., an example when the lexicographic scheme produces a path between the two solutions $g(x) = 0$, $x \in B$. In connection with this we mention the following theorem which is similar to Browder's theorem (see [3]).

Theorem 3. *Suppose* $n \geq 1$ *and* $f : C^n \times [0, 1] \to \mathbf{R}^n$ *is a continuous map with* $f(-x, t) = -f(x, t)$ *for* $(x, t) \in C^n \times [0, 1]$. *Then there exists a connected set* $K \subset C^n \times [0, 1]$ *meeting both* $C^n \times \{0\}$ *and* $C^n \times \{1\}$ *such that* $f(x, t) = 0$ *for every* $(x, t) \in K$.

This theorem can be proved combining the ideas of the proof of Browder's theorem in [4, p. 129] and this paper. We omit details.

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References

[1] I. Bárány, "A short proof of Kneser's conjecture", *Journal of Combinatorial Theory* 25 (1978) 325–326.
[2] K. Borsuk, "Drei Sätze über die n-dimensionalische euklidische Sphäre", *Fundamenta Mathematicae* 20 (1933) 177–190.
[3] F.E. Browder, "On continuity of fixed points under deformations of continuous mappings", *Summa Brasiliensis Mathematicae* 4 (1960) 183–191.
[4] B.C. Eaves, "A short course in solving equations with PL homotopies" *SIAM-AMS Proceedings* 9 (1976) 73–143.
[5] L. Lovász, "Kneser's conjecture, chromatic number and homotopy", *Journal of Combinatorial Theory* 25 (1978) 319–324.
[6] M.J. Todd, *The computation of fixed points and applications* (Springer, Berlin, 1976).