ON A TOPOLOGICAL GENERALIZATION OF A THEOREM OF TVERBERG

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1. Introduction

Let Δ\(^j\) denote the \(j\)-dimensional simplex. The support of the point \(x \in \Delta^j\) is the minimal face of \(\Delta^j\) containing \(x\). A face of \(\Delta^j\) is understood to be closed.

The well-known theorem of Radon [3] can be formulated as follows.

**Proposition 1.** For any linear map \(f: \Delta^{n+1} \to \mathbb{R}^n\) there exist two disjoint faces \(\Delta^i\) and \(\Delta^j\) of \(\Delta^{n+1}\) whose images \(f(\Delta^i)\) and \(f(\Delta^j)\) are not disjoint.

This proposition is generalised in [1].

**Proposition 2.** The statement of Proposition 1 holds for any continuous map \(f: \Delta^{n+1} \to \mathbb{R}^n\).

Proposition 2 is a simple corollary of the following two statements.

**Statement A.** There exists a continuous map \(g: S^n \to \Delta^{n+1}\) such that for every \(x \in S^n\) the supports of \(g(x)\) and \(g(-x)\) are disjoint.

**Statement B (Borsuk’s and Ulam’s antipodal theorem [2]).** For any continuous map \(h: S^n \to \mathbb{R}^n\) there exists \(x \in S^n\) with \(h(x) = h(-x)\).

To see that the Statements A and B together imply Proposition 2 suppose that \(f: \Delta^{n+1} \to \mathbb{R}^n\) does not satisfy Proposition 2. Then the composition \(f \circ g: S^n \to \mathbb{R}^n\) would not satisfy Statement B and this would be a contradiction.

Another generalization of Proposition 1 is proved in [5].

**Proposition 3.** For any linear map \(f: \Delta^N \to \mathbb{R}^n\), where \(N = (p-1)(n+1)\), there exist \(p\) pairwise disjoint faces \(\Delta^i, \ldots, \Delta^p \subseteq \Delta^N\) such that \(f(\Delta^i) \cap \ldots \cap f(\Delta^p)\) is nonempty.

The aim of this paper is to prove the following.

**Theorem.** Suppose \(p\) is prime, \(n \geq 1\), \(N = (p-1)(n+1)\) and \(f: \Delta^N \to \mathbb{R}^n\) is a continuous map. Then there exist \(p\) pairwise disjoint faces \(\Delta^i, \ldots, \Delta^p\) of \(\Delta^N\) such that \(f(\Delta^i) \cap \ldots \cap f(\Delta^p)\) is nonempty.

We mention that if it were not for the restriction that \(p\) be prime, then this theorem would be a common generalization of Propositions 2 and 3. We do not know whether the Theorem holds for any \(p\) or not.

Received 17 May, 1979.

[J. LONDON MATH. SOC. (2), 23 (1981), 158–164]
2. The scheme of the proof

We shall deal with the odd primes only (for \( p = 2 \) see [1]). The idea of the proof of the Theorem is the same as in Proposition 2 with only the change that in both Statements A and B, instead of the sphere \( S^n \), we shall take a CW-complex \( X = X_{n,p} \), and, instead of the antipodal map, we shall have the cyclic group \( Z_p \) acting freely on \( X \). The action of its generator is denoted by \( \omega \).

**Definition.** Let us take \( p \) disjoint copies of the \( n(p-1) \)-dimensional disc and identify their boundaries. This is the CW-complex \( X_{n,p} \). The identified boundary, \( S^{n(p-1)-1} \), is embedded into \( X_{n,p} \) via

\[
i : S^{n(p-1)-1} \to X_{n,p}.
\]

Suppose the cyclic group \( Z_p \) acts freely on the sphere \( S^{n(p-1)-1} \), and let \( \omega \) denote the action of its generator. This map \( \omega \) can be extended from \( S^{n(p-1)-1} \) to \( X_{n,p} \) as follows. If \((y, r, q)\) denotes the point of \( X_{n,p} \) from the \( q \)-th disc with radius \( r \) and \( S^{n(p-1)-1} \) coordinate \( y \), then put

\[
\omega(y, r, q) = (\omega y, r, q + 1),
\]

where \( q + 1 \) is reduced modulo \( p \). Clearly, this map \( \omega \) defines a free \( Z_p \) action on \( X_{n,p} \).

Note that on the odd dimensional sphere \( S^k \) there always exists a free \( Z_p \) action. So here we only need \( p \) to be odd. In Section 4 we shall specify \( \omega \).

We remark further that \( X_{n,p} \) is defined for every \( n, p \geq 1 \). It is clear that \( \dim X_{n,p} = n(p-1) \) and \( X_{n,p} \) is \([n(p-1)-1]\)-connected.

We shall prove the following two statements.

**Statement A'.** There exists a continuous map \( g : X \to \Delta^N \) such that for every \( x \in X \) the supports of the points \( g(x), g(\omega x), \ldots, g(\omega^{p-1} x) \) are pairwise disjoint.

**Statement B'.** For the map \( \omega \) defined in Section 4 and for any continuous map \( h : X \to \mathbb{R}^n \) there exists an \( x \in X \) such that \( h(x) = h(\omega x) = \ldots = h(\omega^{p-1} x) \).

Clearly, the Theorem follows from Statements A' and B'.

3. The proof of Statement A'

We shall prove this statement for every odd \( p \). We define the CW-complex \( Y_{n,p} \) as

\[
Y_{n,p} = \{(y_1, \ldots, y_p) : y_1, \ldots, y_p \in \Delta^N, \text{ and the supports of } y_1, \ldots, y_p \text{ are pairwise disjoint}\}.
\]

Clearly, there exists a free \( Z_p \) action on \( Y_{n,p} \); its generator maps \((y_1, \ldots, y_p) \in Y_{n,p} \) into \((y_2, \ldots, y_p, y_1) \in Y_{n,p} \).

Now the existence of the map \( g : X \to \Delta^N \) of Statement A' is equivalent to the
existence of a $Z_p$ equivariant map $G : X \to Y_{N,p}$. The existence of such a map follows from homotopy theory if $\dim X - 1$ is not greater than the connectedness of $Y_{N,p}$. Indeed, given a $Z_p$ equivariant cell subdivision of the space $X$ one can construct an equivariant map $G : X \to Y_{N,p}$ by induction on the dimension of the cells in the following way.

**Step 0.** Choose a 0-cell from each orbit of 0-cells (that is vertices). Define the map $G$ on these vertices arbitrarily and extend this map to a $Z_p$ equivariant map of all vertices.

**Step k.** Suppose that $G$ has been defined on the $(k-1)$-skeleton of $X$. Choose a cell from each orbit of $k$-cells. The map $G$ is defined on the boundary of these cells. By the $(k-1)$-connectedness of $Y$ the map $G$ can be extended to the $k$-cells chosen from each orbit. Now define $G$ on the other $k$-cells to be $Z_p$ equivariant.

So in order to prove Statement $A'$ it suffices to prove the following.

**Lemma 1.** For all natural numbers $N$ and $p$ with $N \geq p + 1$,

$$\pi_j(Y_{N,p}) = 0 \text{ for } 1 \leq j \leq N - p.$$  

**Proof.** (For this elementary proof we are indebted to the referee. Our original proof used the Leray spectral sequence.) Let $0, \ldots, N$ denote the vertices of $\Delta^N$. The Cartesian power $(\Delta^N)^p$ has a natural structure as a cell complex, a typical cell being $\sigma_1 \times \cdots \times \sigma_p$, with each $\sigma_i$ a face of $\Delta^N$. The cell is also described by the $p$-tuple $(A_1, \ldots, A_p)$, where $A_i$ is the set of vertices of $\sigma_i$. Those $p$-tuples where the $A_i$ are pairwise disjoint form a subcomplex (isomorphic to) $Y_{N,p}$.

In view of Hurewicz's theorem it suffices to prove that $\pi_i(Y_{N,p}) = 0$ (this would imply that $H_1(Y_{N,p}) = 0$) and that $H_2(Y_{N,p}) = \cdots = H_{N-p}(Y_{N,p}) = 0$. The case where $p = 1$ is trivial because $Y_{N,1}$ is the same as $\Delta^N$. When $p$ is greater than one it is convenient to consider, for $i = 0, \ldots, N$, that subcomplex $Y_{N,p,i}$ of $Y_{N,p}$ which one gets by requiring $A_i$ to be a subset of $\{i, i+1, \ldots, N\}$. Thus

$$Y_{N,p} = Y_{N,p,0} \supseteq Y_{N,p,1} \supseteq \cdots \supseteq Y_{N,p,N}.$$  

We shall show that the groups in question vanish for every $Y_{N,p,i}$ and so in particular for $Y_{N,p}$.

The proof is by double induction, on $p$ and $N-i$. We assume that our assertion holds for $Y_{N',p,i}$, whenever either $2 \leq p' < p$ or $p' = p$ and $N' - i' < N - i$.

If $i = N$, then $Y_{N,p,i} = Y_{N,p,N}$ is homeomorphic to $Y_{N-1,p-1}$, and so we can assume that $i < N$.

In order to prove that $\pi_i(Y_{N,p,i}) = 0$ it suffices to show that any given closed edge-path in the 1-skeleton of $Y_{N,p,i}$ can be deformed by homotopies in $Y_{N,p,i}$ until it lies in $Y_{N,p,i+1}$. Let $u_1, \ldots, u_m = u_1$ be the vertices of the given path. Thus each $u_k$ is described by a $p$-vector with distinct components from $0, \ldots, N$ (each $A_i$ is a singleton), and $u_{k+1}$ differs from $u_k$ in at most one component.

The deformation will be done in four steps, each step consisting of several small deformations on short subpaths.

In the first step one "separates" those neighbouring pairs $u_k, u_{k+1}$ for which $i+1$
occurs as second, third, ..., or $p$-th component in both $u_k$ and $u_{k+1}$. Clearly we can assume that $u_k \neq u_{k+1}$. Let $u_k = (\ldots, x_i, \ldots, i+1, \ldots)$ and $u_{k+1} = (\ldots, y_i, \ldots, i+1, \ldots)$, say with $x \neq y$, where only the changing component and the component equal to $i+1$ are indicated. Put $u'_k = (\ldots, x_i, \ldots, z, \ldots)$ and $u'_{k+1} = (\ldots, y_i, \ldots, z, \ldots)$ where $z$ is chosen among those elements of $\{0, \ldots, N\}$ which are not components of $u_k$ or $u_{k+1}$.

In view of the assumption that $N \geq p + 1$, such a number $z$ exists. It is easy to see that the deformation of the subpath $u_k u_{k+1}$ into $u_k u'_k u'_{k+1} u_{k+1}$ is a homotopy over the 2-cell $(\ldots, \{x, y\}, \ldots, \{z, i+1\}, \ldots)$ in $Y_{N, p, i}$. Let $v_1, \ldots, v_1$ be the path obtained by the first step.

In the second step one deletes each $v_k$ which has $i+1$ among its last $p-1$ components. In this case, as a result of the first step, $v_{k-1} = (\ldots, x_i, \ldots)$, $v_k = (\ldots, i+1, \ldots)$ and $v_{k+1} = (\ldots, y_i, \ldots)$, and $x \neq i+1, y \neq i+1$. It is clear that the deletion of $v_k$ is a homotopy over the 2-cell $(\ldots, \{x, y, i+1\}, \ldots)$ in $Y_{N, p, i}$ (or over a 1-cell if $x = y$). Let $w_1, \ldots, w_1$ be the path obtained.

The third step is similar to the first one and consists of insertion of a pair of vertices of the form $(i+1, \ldots)$ between every pair $w_k = (i, \ldots)$, $w_{k+1} = (i, \ldots)$. In the fourth step, which is similar to the second one, all vertices of the form $(i, \ldots)$ are deleted. This gives a path in $Y_{N, p, i+1}$ as desired.

It remains to prove that $H_j(Y_{N, p, i}) = 0$ when $2 \leq j < N - p$. We compute the homology using the given cell complex. The boundary operator $\partial$ is defined on cells by

$$\partial(\sigma) = \partial(A_1, \ldots, A_p) = \sum (-1)^{e(x)} (A_1, \ldots, A_{r(x)} \setminus \{x\}, \ldots, A_p).$$

Here the sum is taken over those $x$ in $A_1 \cup \ldots \cup A_p$ which belong to an $A_r = A_{r(x)}$ with $|A_r| \geq 2$, and $e(x)$ is defined by

$$e(x) = |A_1| + \ldots + |A_{r(x)} - 1| + |\{y : y \in A_{r(x)} \text{ and } y < x\}|.$$

Now the cells are of four different types according to whether

1. $i \in A_1$ and $|A_1| > 2$,
2. $i \in A_1$ and $|A_1| = 2$,
3. $\{i\} = A_1$,
4. $A_1 \subset \{i+1, \ldots, N\}$.

Now put $\sigma^- = (A_1 \setminus \{i\}, A_2, \ldots, A_p)$ if $\sigma$ is of the first or the second type, $\sigma^x = (\{i\}, A_2, \ldots, A_p)$ if $\sigma$ is of the second type and $\sigma^+ = (A_1 \cup \{i\}, A_2, \ldots, A_p)$ if $\sigma$ is of the fourth type and $i \notin A_2 \cup \ldots \cup A_p$. Let $C$ be a $j$-chain in $Y_{N, p, i}$. Now $C = C_1 + C_2 + C_3 + C_4$ where $C_k$ is the sum of those cells of $C$ which are of the $h$-th type ($h = 1, 2, 3, 4$), and clearly

$$\partial C = (C_1 + C_2)^- - (\partial((C_1 + C_2)^-))^+ - C_3^+ + \partial C_3 + \partial C_4.$$

Assume now that $\partial C = 0$. We must prove that $C$ bounds in $Y_{N, p, i}$ and we start by observing that $(\partial((C_1 + C_2)^-))^+ = 0$, so that $\partial((C_1 + C_2)^-) = 0$. Now $(C_1 + C_2)^-$ is a
(j − 1)-chain in Y_{N,p,i+1}, and it is even isomorphic to a (j − 1)-chain in Y_{N−1,p,i+1}, as the vertex i does not appear in it. Thus, by the hypothesis of induction, (C_1 + C_2)^− = ∂D, where D is a j-chain in Y_{N,p,i+1}, not involving the vertex i. (Here we have made use of the fact that N > p + 1 for we are finished with the case where N = p + 1.) This means that

$$\partial(D^+) = D - (C_1 + C_2) + \text{terms of the third type},$$

and so $C + \partial(D^+)$ has only terms of the third and fourth types. Put $C + \partial(D^+) = C_3 + C_4$. Then $\partial C_3 = \partial C_4 = 0$, as $\partial C = 0$. But $C_3$ bounds in $Y_{N,p,i}$, because $H_j(Y_{N−1,p−1}) = 0$, and $C_4$ bounds in $Y_{N,p,i}$ because $H_j(Y_{N,p,i+1}) = 0$. This finishes the proof.

4. Proof of Statement B'

First we shall specify the map $\omega : X_{n,p} \to X_{n,p}$. As we have seen, it is enough to specify $\omega : S^{n(p−1)−1} \to S^{n(p−1)−1}$. Now let $\theta : \prod_{1}^{p} R^n \to \prod_{1}^{p} R^n$ be defined by

$$\theta(v_1, ..., v_p) = (v_2, ..., v_p, v_1).$$

Put $D = \{ (v, v_1, ..., v) \in \prod_{1}^{p} R^n : v \in R^n \}$. Then $\theta$ acts freely on $\prod_{1}^{p} R^n \setminus D$ (this is the point where we need $p$ to be prime). So $\theta$ acts freely on the unit sphere of the orthogonal complement of $D$ (relative to $\prod_{1}^{p} R^n$), or, what is the same thing, on the sphere $S^{n(p−1)−1}$. Now we define $\omega$ as the restriction of the map $\theta$ to this sphere. It is clear that $S^{n(p−1)−1}$ is $\theta$-invariant and it is a $\theta$-equivariant deformation retract of the space $\prod_{1}^{p} R^n \setminus D$.

Now we prove Statement B' with this $\omega$. Suppose, on the contrary, that there exists a map $h : X \to R^n$ for which Statement B' does not hold. Then the image of the map $H : X \to \prod_{1}^{p} R^n$ defined by $H(x) = (h(x), h(\omega x), ..., h(\omega^{p−1} x))$ is disjoint from the diagonal $D$. It is obvious that $H$ is equivariant, that is $H \omega = \theta H$.

Further, the injection $i : S^{n(p−1)−1} \to X$ is $\omega$-equivariant and, in view of the $[n(p−1)−1]$-connectedness of $X$, homotopic to zero. Thus the diagram

$$
\begin{array}{ccc}
S^{n(p−1)−1} & \xrightarrow{i} & X \\
\text{retr.} & & \text{retr.} \\
\downarrow{\omega} & & \downarrow{\omega} \\
S^{n(p−1)−1} & \xrightarrow{i} & X \\
\end{array}
$$

$$
\begin{array}{ccc}
& H & \\
\downarrow{\theta} & \downarrow{\theta} & \\
& \prod_{1}^{p} R^n \setminus D & \rightarrow S^{n(p−1)−1} \\
\end{array}
$$

\[\omega \downarrow \quad \omega \downarrow \quad \omega \downarrow \quad \omega \downarrow\]
is commutative. Then the composition of the horizontal maps, \( \zeta: S^{n(p-1)-1} \to S^{n(p-1)-1} \), is equivariant and homotopic to zero. This implies that \( \zeta \) must have degree zero. But the following lemma will show that \( \deg \zeta \equiv 1 \mod p \), thus providing a contradiction.

**Lemma 2.** Suppose that \( k \geq 1, p \geq 2 \) and we are given a free \( \mathbb{Z}_p \) action on the sphere \( S^k \). Then an arbitrary equivariant map \( \alpha: S^k \to S^k \) has degree 1 modulo \( p \).

**Remark.** Note that here we do not need \( p \) to be prime.

Lemma 2 is proved in [4; Theorem 8.3, p.42]. Here we present a simple proof.

**Proof.** Write \( \theta \) for the action of the generator of \( \mathbb{Z}_p \) and choose a \( \theta \)-invariant cell subdivision on the sphere \( S^k \). Since \( \pi_j(S^k) = 0 \) for \( j < k \) the restrictions of the map \( \alpha \) and the identity map \( S^k \to S^k \) to the \( (k-1) \)-skeleton of \( S^k \) are equivariantly homotopic. Hence one can assume that \( \theta \) coincides with the identity on the \( (k-1) \)-skeleton. (To see this more precisely one can use an argument from homotopy theory which is similar to the one in the proof of Statement A'.)

Let us consider the map \( F: \bigcup_{j=1}^{p} S^k_j \to S^k \) of the disjoint union of \( p \) copies of \( k \)-spheres \( S^k_1, \ldots, S^k_p \) into the sphere \( S^k \) defined by the formula

\[
F(x) = \begin{cases} 
\alpha(x) & \text{if } x \in S^k_1, \\
x & \text{otherwise}.
\end{cases}
\]

It is clear that \( \deg F = \deg \alpha + p - 1 \).

The \( \theta \)-invariant cell subdivision of \( S^k \) obviously has the property that the orbit of an arbitrary cell \( \sigma \) consists of \( p \) cells. (These are \( \sigma, \theta(\sigma), \ldots, \theta^{p-1}(\sigma) \).)

Now let \( \beta: S^k \to S^k \) be a continuous map which coincides with \( \alpha \) on one of the \( k \)-cells of each orbit and with the identity on the others. Define the map

\[
G: \bigcup_{j=1}^{p} S^k_j \to S^k
\]

by

\[
G(x) = \beta \circ \theta^j(x) \quad \text{if } x \in S^k_j \quad (j = 1, \ldots, p).
\]

Then on the one hand \( \deg F = \deg G \) and on the other hand \( \deg G \equiv 0 \mod p \). This proves the lemma.

**References**


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