NEW EXOTIC FOUR-MANIFOLDS WITH $\mathbb{Z}/2\mathbb{Z}$ FUNDAMENTAL GROUP

MÁRTON BEKE, LÁSZLÓ KOLTAI, SARAH ZAMPA

Abstract. We extend a construction of Stipsicz-Szabó ([13]) of infinitely many irreducible exotic smooth structures of some closed four-manifolds with even b_2^+ and fundamental group $\mathbb{Z}/2\mathbb{Z}$. We use the double node surgery and rational blow down constructions of Fintushel-Stern ([6, 4]) on some elliptic fibrations equipped with a free involution. The construction is done in an equivariant manner and the factor manifolds are distinguished by the Seiberg-Witten invariants of their universal covers.

1. Introduction

Two smooth manifolds are said to be *exotic* if they are homeomorphic but not diffeomorphic to each other. In this case, we call the two smooth structures *exotic*, and a manifold invariant is usually used to distinguish smooth structures, *i.e.*, to show non-diffeomorphism of the two manifolds. In the following, we use the Seiberg-Witten invariants [15] given by the function

$$SW_X : H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

where X is a smooth, closed, oriented four-manifold with $b_2^+ > 1$. This is encoded in a formal series $\mathcal{SW}_X := \sum_{\alpha \in H^2(X;\mathbb{Z})} SW_X(\alpha)e^{\alpha}$. Note, that the number of *"basic classes"*, *i.e.* classes α with $SW_X(\alpha) \neq 0$, is finite, and so the sum is also finite. Indeed, if two manifolds are diffeomorphic, then their Seiberg-Witten invariants have to agree (up to automorphism). On the other hand, the homeomorphism of the two manifolds will arise from the generalization of Freedman's classification theorem to manifolds with fundamental group $\mathbb{Z}/2\mathbb{Z}$ (see [8]).

In this paper, we will expand the construction given in [13, Theorem 4.6] by exploiting the extra two $E(n)$ pieces of the fibration described below. The following theorem relies on the rational blow down construction, knot and double node surgery constructions of Fintushel-Stern (see [4, 5, 6] respectively).

Theorem 1.1. *The manifold* $Z_1 \# 2n\,\mathbb{CP}^2\,\# l\overline{\mathbb{CP}^2}$ *where* $l \in \{5n+6, 5n+12, \ldots, 8n\}$ *if* $n = 2q \not\equiv 6 \pmod 8$ *and* $l \in \{5n+9, 5n+15, \ldots, 8n\}$ *if* $1 < n = 2q+1 \not\equiv 5 \pmod 8$*, admits infinitely many irreducible smooth structures.*

Here, Z_1 denotes the quotient of $S^2 \times S^2$ with the fixed point free involution ι, which applies the antipodal map on the two S^2 components. We conjecture that the same theorem is valid in the remaining $n \equiv 5, 6 \pmod 8$ cases as well.

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2. Preliminaries

2.1. **Topological results.** We will make use of the homeomorphism classification of oriented smooth manifolds with $\pi_1 = \mathbb{Z}/2\mathbb{Z}$, to this end consider X and its two-fold universal cover $\tilde{X}$. There are 3 possibilities with regards to the spinness of these two manifolds:

I) neither are spin
II) both are spin
III) X is not spin, but $\tilde{X}$ is.

This is called the *w_2-type* of the manifold X. Note, that these are the only possibilities, since spinness is equivalent with having $w_2 = 0$, and naturality of this class means, that if X is spin, then so is $\tilde{X}$. Now we can state

Theorem 2.1 ([8, Theorem C]). *Let X_1, X_2 be two closed oriented smooth 4-manifolds with fundamental group $\mathbb{Z}/2\mathbb{Z}$. The manifolds are homeomorphic if and only if their Euler characteristic, signature and w_2-type agree.*

2.2. **Knot surgery.** Consider a knot $K \subset S^3$, and a simply connected smooth manifold X with $b_2^+ > 1$, and with an embedded homologically essential torus T (in the following we denote the n-torus T^n) of self-intersection 0 and simply connected complement. The manifold $X_K := (X \setminus \nu T) \cup_\phi (S^3 \setminus \nu K) \times S^1$ is called a knot surgery of X using K, where $\phi : T^3 \to T^3$ is chosen so that the longitude of K is identified with the normal circle of T. Note that this does not determine ϕ completely: we pick and fix such a function.

The following theorem shows the importance of this construction:

Theorem 2.2 ([5]). *With the setup as above $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(e^{2[T]})$ holds, where Δ_K is the symmetric Alexander polynomial of K.*

In the special case of elliptic fibrations over the sphere, which we will be using, more can be done ([6]). Let K be any genus one knot, and pick a minimal genus Seifert surface Σ and a non-separating loop Γ inside Σ satisfying:

(1) Γ bounds a disc which intersects K in two points.
(2) Γ has linking number $+1$ in S^3 with its pushoff on Σ.

Definition 2.3. Consider a neighborhood of a smooth fiber F in an elliptic fibration containing exactly two nodal fibers with the same monodromy, and with all other fibers smooth. Such a neighborhood is called a *double node neighborhood.*

With K as previously and T a smooth fiber in a double node neighborhood, we do knot surgery with the map ϕ picked so that the meridian of K is identified with the vanishing cycle of the singular fibers (both are the same, since they have the same monodromy).

In the local picture a section of the elliptic fibration looks like a disc; the result of knot surgery is to remove a smaller disc from the section and replace it with the Seifert surface of K, thus, we obtain a punctured torus afterwards. Inside this torus sits the loop Γ, which by (1) bounds a twice punctured disc D' with boundary $\partial D' = \Gamma \cup m_1 \cup m_2$, where the m_i are meridians of K. By the choice of ϕ the meridians m_i bound disjoint vanishing discs D_i (corresponding to the two nodal singularities), hence Γ bounds a disc $D = D' \cup D_1 \cup D_2$. With respect to the framing of Γ given by the push-off on the surface Σ, the relative self-intersection of D is -1.

Remove now an annular neighborhood of Γ inside Σ, and close the resulting surface with two copies B_1 and B_2 of the disc D described above. These two discs intersect each other in one point, so the capped off surface is an immersed disc with a double point. To get the sign of the double point notice that B_1 and B_2 intersect with negative sign, but we have to change the orientation of B_2 in order to get an oriented surface, so the immersed disc has a positive double point. Thus in a double node neighborhood we can replace the genus introduced by the knot surgery with a positive double point if we use knots satisfying the conditions above.

2.3. **Rational blow down.** The final ingredient is the rational blow down ([4, 10]). Denote by C_p a linear plumbing (consecutive terms intersect transversely at a single point) of $p-1$ spheres with self-intersections $-(p+2), -2, \dots, -2$. The boundary of this linear plumbing is the lens space $\partial C_p = L(p^2, p-1)$. The main observation is that there is a manifold B_p with the same boundary, and with $H_*(B_p;\mathbb{Q}) = H_*(\{pt\};\mathbb{Q})$ ([7, Figure 8.42.]). If $C_p \subset X$ for some simply connected 4-manifold X, we call $X_p := (X \setminus \nu C_p) \cup B_p$ the rational blow down of X.

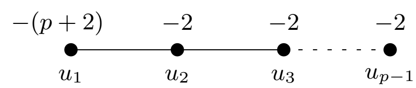

FIGURE 1. The configuration C_p

In special circumstances it is possible to follow the Seiberg-Witten invariants after a rational blow down. It is true in general, that if X_p is simply connected, then the Seiberg-Witten invariants of X completely determine those of X_p, but the computation is not always simple. A more manageable case which we will need is the following:

Definition 2.4. C_p is *Seiberg-Witten tautly embedded* if $|\alpha(u_1)| \leq p$ (where $u_1 \subset C_p$ is the sphere with self-intersection $-p-2$), and $\alpha(u_i) = 0$ (for the -2 spheres of the configuration) is satisfied for all basic classes α.

To obtain the Seiberg-Witten invariants after the blow down, one needs to find extensions of certain cohomology classes from $X \setminus \nu C_p$ to X_p, and vice versa. If an extension exists, and the expected dimension of the moduli space (defined by the right hand side of Equation (1)) stays non-negative, then the value of the invariant is unchanged after the blow down [4, Theorem 8.2]. In our case, the expected dimension of the moduli space is always zero.

In the tautly embedded case this extension process simplifies further, and one only has to consider the basic classes which evaluate maximally on u_1:

Theorem 2.5. *Let X be a simply connected 4-manifold with $b_2^+ \geq 2$, and C_p tautly embedded into it. If α' is a basic class of X_p, and $\alpha|_{X \setminus \nu C_p} = \alpha'|_{X_p \setminus B_p}$, then $|\alpha(u_1)| = p$ where $u_1 \subset C_p$ is the sphere with $[u_1]^2 = -p-2$.*

Finally the standard blow up formula also needs to be mentioned, as we will be relying on it in the following.

Theorem 2.6 ([3, Theorem 1.4])**.** *Let X be a 4-manifold of Seiberg-Witten simple type with $b_2^+ > 1$, and $X' = X \# \overline{\mathbb{CP}^2}$ its blow up, then $SW_X(\alpha) = SW_{X'}(\alpha \pm E)$, with the (Poincaré dual of the) new exceptional sphere denoted $E \in H^2(X \# \overline{\mathbb{CP}^2};\mathbb{Z})$.*

Remark 2.7. For the reader unfamiliar with Seiberg-Witten invariants, the *simple type* condition, i.e. that for every basic class α

$$\frac{\alpha^2 - (3\sigma(X) + 2\chi(X))}{4} = 0 \tag{1}$$

can be safely ignored, since $E(n)$ is of simple type for all $n > 1$ and simple type manifolds stay simple type under knot surgeries and rational blow downs. For an exposition, see e.g. [4, Section 8].

2.4. **An involution on** $E(2n+1)$**.** We briefly recall the construction of the extended involution, still denoted by $\iota : E(1) \to E(1)$ from [13, Section 3]. Consider $S^2 \times S^2$ and the antipodal map on both factors, denoted by $p \times p$. Now take two fibers and two constant sections, which are *not* mapped into each other by the respective p's, call the union of these 4 spheres C_0, and $C_1 = (p \times p)(C_0)$ is another set of 4 spheres.

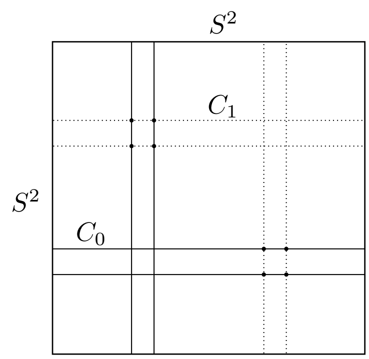

FIGURE 2. $S^2 \times S^2$ and the generators of the pencil C_0, C_1.

Note, that the involution $p \times p$ extends if we blow up $S^2 \times S^2$ at pairs of these intersection points, thus getting a well defined fibration $S^2 \times S^2 \# 8\overline{\mathbb{CP}^2} \to S^2$, still equipped with an involution, which we still call ι. This fibration is an elliptic fibration $E(1)$.

We extend this involution further by taking a smooth fiber and its pair under ι. Taking a fiber sum with a copy of $E(n)$ along this fiber, and its pair, the involution extends by exchanging the two copies of $E(n)$ in the manifold $E(2n+1) = E(n)\#_f E(1)\#_f E(n)$.

3. Construction

Now putting the above results together, we will do a number of double node surgeries on some elliptic surface $E(2n+1)$ using the twist knots K_m and K_1 depicted on Figure 4. Here, m denotes the parameter of our infinite exotic families on the fixed topological type, and n is the coefficient in the main theorem which determines the topological type. After the double node surgeries, we blow up the double points, and modify the fibration to produce a configuration C_p which will be rationally blown down.

More precisely, consider the fibration $E(2n+1)$ described in Subsection 2.4. Choose a section $s \in \Gamma(E(2n+1))$, and consider the image $\iota(s)$ of that section under the involution map ι (they have self-intersection $-2n-1$). Choose two smooth fibers (again equivariantly) of our fibration, and do knot surgery along them using the twist knot K_m.

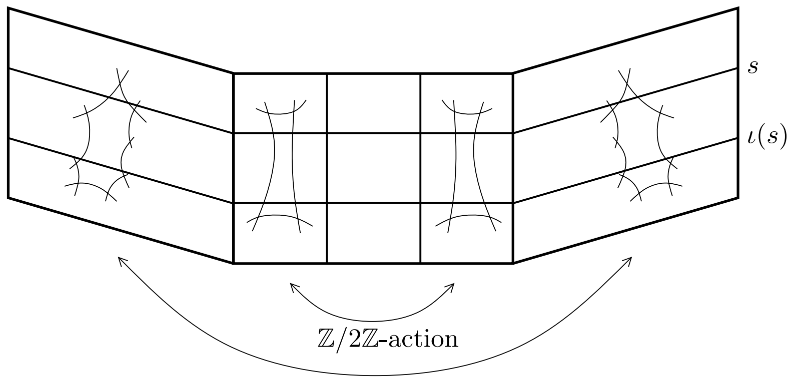

FIGURE 3. The elliptic surface $E(2n+1)$ seen as the fiber sum of $E(1)$ with two copies of $E(n)$.

Since $g_3(K_m) = 1$, the sections become genus two surfaces after the surgery, and since

$$1 = (ab)^6 = (a^3b)^3 = a^9(b^{a^6}b^{a^3}b) \tag{2}$$

in the monodromy group (see [14] for further details) we can assume that there are two I_4 fibers in our $E(1)$, positioned symmetrically with respect to ι.

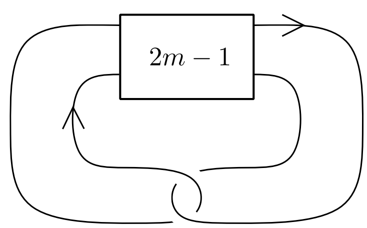

FIGURE 4. The twist knot K_m, where the box represents $(2m-1)$-many half twists. The genus 1 Seifert surface is obtained from the immersed disc bounded by the two parallel strands.

According to [6] we can use these fibers to exchange the genera for two positive double points on both fibers and both sections. Blowing up the four double points, the sections now have self-intersection $-2n-9$.

Next, we do $2k$ additional double node surgeries along the trefoil knot (we choose the left-handed trefoil K_1) positioned symmetrically with respect to ι in the two $E(n)$ parts of our $E(2n+1)$ (note that a number of other knots would work, but we choose the trefoil to keep the computations of the Seiberg-Witten invariants simple). As before, $g_3(K_1) = 1$ and its Alexander polynomial is $\Delta_{K_1} = t - 1 + t^{-1}$, allowing us to apply additional double node surgeries to produce double points on the spheres and blow them up. The existence of the necessary I_4 fibers for this construction follows from the monodromy factorisation in the next paragraph. This lowers the self-intersection of both sections to $-2n-9-8k$.

Next we produce a singular I_{8n-2} fiber, where the dual intersection graph of the fiber and the two sections looks like Figure 5.

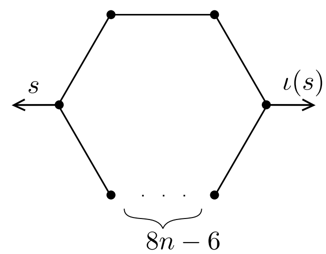

FIGURE 5. Dual graph of the fiber with the two sections mapped to each-other.

This can be achieved using a result of Korkmaz-Ozbagci ([9]). From this, we compute the following factorisation in the mapping class group of the twice punctured torus, where β denotes a right handed Dehn twist along the longitude, and $\overline{\beta}$ is its inverse (see Figure 6):

$$\delta_1\delta_2 = (\alpha_1\alpha_2\beta)^4 = (\alpha_1^3\alpha_2^{\overline{\beta}}\alpha_1\beta)^2 = \alpha_1^8\alpha_2^{\overline{\beta}\alpha_1^5}\beta^{\alpha_1^4}\alpha_2^{\overline{\beta}\alpha_1}\beta.$$

As we can see, this decomposition does not contain any α_2. In the fibration corresponding to this decomposition, the two sections would intersect the same (-2)-curve of the singular fiber. In order to separate the two sections we also need the decomposition $\delta_1\delta_2 = \alpha_1^6\alpha_2^3\beta^{\alpha_1^4\alpha_2^2}\beta^{\alpha_1^2\alpha_2}\beta$.

Using the two relations, we get that

$$\delta_1^n\delta_2^n = \alpha_1^{8n-2}\alpha_2^3\big((4n-1)\text{-many right handed Dehn twists}\big) \tag{3}$$

which implies that in the space $E(n)$ we can indeed achieve the configuration of Figure 5.

Finally, we perform two rational blow downs, one on this configuration and one on its pair under ι. The fibers were chosen to be longer than needed for a $C_{2n+7+8k}$; two additional spheres of the singular fiber obtained from this choice are used to make the configuration disjoint from its pair under ι, and to guarantee that the resulting manifold is simply connected. The fundamental group of the complement is generated by a normal circle of one of the endpoints of the linear plumbing $C_{2n+7+8k}$, the longer chain guarantees a bounding disc to this generator in the complement inside $E(2n+1)$, and by Van Kampen's theorem we get simple connectivity (the complement of a section and a fiber in $E(2n+1)$ is also simply connected [7, Section 3.1]).

Doing this procedure once requires an I_4 for the double node surgery, and the blowups force the configuration C_p to be longer by 8, so we use a^{12} from the

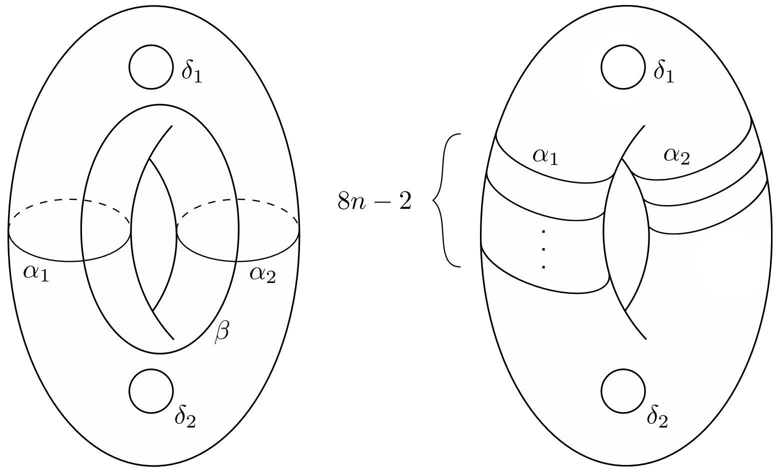

FIGURE 6. Generators of the mapping class group of the twice punctured torus and the configuration produced by the factorisation.

monodromy in total (from now on, a is a right handed Dehn twist in the mapping class group of the torus of which α_1 and α_2 are lifts). By Equation (3), we can produce a^{8n-2} in the monodromy factorization of $E(n)$. Therefore, we have $8n-2 = 2n+7+12k$, i.e. $k = \lfloor \frac{n}{2} - \frac{3}{4} \rfloor$ gives the maximum number of times we can perform the surgery. Note that this means the coefficient $8n-6k$ is equal to $5n+6$ if n is even, and to $5n+9$ if n is odd.

Remark 3.1. This choice leaves some singular fibers to work with, by using the factorisation of Equation (2) for the double node surgeries, the results can be improved by about 1 extra knot surgery for every fifth value of n, but the formulas become much more convoluted.

Now, we are ready to prove the main result:

Theorem 3.2. *There are infinitely many irreducible smooth structures on* $Z_1 \# 2n\,\mathbb{CP}^2 \# (8n-6k)\overline{\mathbb{CP}^2}$ *for* $n \in \mathbb{N}$ *and* $0 \le k \le \lfloor \frac{n}{2} - \frac{3}{4} \rfloor$ *where* $4 \nmid n-k$.

Remark 3.3. In order to show that our manifolds are non-spin, we use Rohlin's theorem (a spin four-manifold has signature divisible by 16, [11]). By the calculation below this means that we have to exclude those cases where $n \equiv 5$ or $6 \pmod 8$ (*i.e.*, when $n-k$ is divisible by 4). Note that the construction remains valid in these cases, and we suspect that the manifolds are non-spin as well.

Proof. The Seiberg-Witten invariant of $E(2n+1)$ is

$$\mathcal{SW}_{E(2n+1)} = (e^f - e^{-f})^{2n-1}$$

where $f \in H^2(E(2n+1);\mathbb{Z})$ is the fiber class ([4] [1, Example 1]). By [5], we get that after both surgeries with K_m the invariants get multiplied by $me^{2f} - (2m-1) + me^{-2f}$, and by $e^{2f} - 1 + e^{-2f}$ after each additional K_1-surgery. Since $E(2n+1)$ is of simple type, the potential basic classes become $(2n-1-2r)f$, where $r \in \{-2k-2, \ldots, 2n+2k+1\}$ (and this new manifold, homeomorphic to $E(2n+1)$ is still of simple type).

After the $4k+4$ blow ups the basic classes are of the form $(2n-1-2r)f + \sum_{i=1}^{4k+4} \pm E_i$ with r as before. The sections get blown up at double points, so the sphere is represented by $[s] - \sum_{i=1}^{2k+2} 2E_i$ and its pair by $[\iota(s)] - \sum_{i=2k+3}^{4k+4} 2E_i$. Note that the construction is done in an equivariant manner on two disjoint sections.

We compute:

$$\Big\langle (2n-1-2r)f + \sum_{i=1}^{4k+4} \pm E_i, [s] - \sum_{i=1}^{2k+2} E_i \Big\rangle = 2n-1-2r+2a,$$

where a represents the number of E_i's with negative sign in the basic class, so it is any even number satisfying $|a| \leq 2k+2$ and $\langle \cdot, \cdot \rangle$ is the intersection pairing. Thus the value above is at least $2n-1-4n-2-4k-4k-4 = -2n-8k-7$ and at most $2n-1+4k+4+4k+4 = 2n+8k+7$ meaning that both configurations are tautly embedded (see Definition 2.4), since the (-2)-spheres of the configuration are in a fiber, so f evaluates on them as 0.

We apply [7, Theorem 8.5.18.] to get that the only basic classes which extend to the rational blow down are $\alpha = (2n-1+4k+4)f + \sum_{i=1}^{4k+4} E_i$ and its negative, since this is the only class which evaluates maximally on both sections. This class corresponds to the leading coefficient of the invariant before the rational blow down. The leading coefficient of the product

$$(t-t^{-1})^{2n-1}(mt^2 - (2m-1) + mt^{-2})^2(t^2 - 1 + t^{-2})^{2k}$$

is m^2, thus these manifolds are all smoothly distinct.

Topologically, $\chi(E(2n+1)) = 24n+12$, which we change by $4k+4-4n-12-16k$ with the blow ups and the rational blow downs to obtain $20n-12k+4$. Furthermore, $b_2^+(E(2n+1)) = 4n+1$, $b_2^-(E(2n+1)) = 20n+9$, we only add and remove negative definite submanifolds, so the signature becomes $4n+1-(20n+9+4k+4-4n-12-16k) = -12n+12k$. This number is not divisible by 16 by assumption, thus our manifolds are not spin *i.e.*, type I. Factorisation by ι halves both σ and χ, so by the homeomorphism classification of $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ smooth manifolds the factor is homeomorphic to $Z_1 \# 2n\,\mathbb{CP}^2 \# (8n-6k)\overline{\mathbb{CP}^2}$, but as detected by the Seiberg-Witten invariant of its universal cover, they are not diffeomorphic.

Finally by the above calculation and Equation (1), we see that $\alpha^2 = \frac{3\sigma+2\chi}{4} = 4n+12k+8$ for our basic class α. Irreducibility follows from this, since in a reducible manifold there are two basic classes, the difference of which has square -4 by [12, Lemma 2.3], but in our case this number is always positive. $\square$

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