Yang-Mills theory seminar

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1 Első előadás

Stipsicz-Szabó jegyzetet használjuk/olvasandó. Postnikov: Lectures in Geometry 1-5. A negyedik kell. Háttér:

- M difható sokaság, ∂M peremes sokaság, zárt sokaság=kompakt peremtelen
- $S^n, \overline{B}^n$
- differenciálformák $\Lambda^k T^* M$ nyalábok, $\Omega^k(M) = C^\infty(M; \Lambda^k T^* M)$
- külső deriválás d lineáris, gradált leibniz és a négyzete nulla
- $\omega|_U = \sum_I' \omega_U^I dx^I$ jelöli lokálisan a k-formákat
- $\mathrm{d}\,\omega = \sum_I' \sum_1^m \frac{\partial \omega_U^I}{\partial x^i} dx^i \wedge dx^I$
- de Rahm kohomológia $H^k(M)$

Theorem 1.1 (Hodge)**.** *M kompakt, akkor* $\dim H^k(M) < \infty$.

Theorem 1.2. *M zárt, akkor $H^k(M) \equiv (H^{m-k}(M))^*$.*
Ha M összefüggő, akkor $H^0(M) = \mathbb{R}$.

$[\omega] \in H^k(M)$-el jelöljük a kohomológiaosztályokat. Legyen $\omega, \omega' \in [\omega]$ tisztességes k-forma reprezentánsok. Ezek zártak lesznek, és létzni fog egy ϕ $k-1$ forma, hogy $\omega' = \omega + d\phi$. Ahhoz hogy nulla legyen valaki $d\phi = \omega$ kell, ehhez biztosan süzkséges, hogy zárt legyen, és elegendő az, hogy eltűnik a $H^k(M)$.

Theorem 1.3. *(M, g) irányított zárt Riemann sokaság, akkor $H^k(M) \equiv \ker \nabla_k$ kanonikusan.*

A laplace $*d*d + d*d*$ elvileg, ez a közönséges Laplace a függvényeken, és általánosodik.
Integrálás: egy M peremes sokaságon egy ω $m-1$ forma $\int_M \mathrm{d}\,\omega = \int_{\partial M} \omega$.

2 Second Lecture

2.1 Vector Bundles

Definition 2.1. Smooth vector bundle, finite dimensional and mostly over $\mathbb{C}$.

Definition 2.2. Morphism of vector bundles. Isomorphism of vector bundles. $Aut(E)$ is the gorup of invertible endomorphisms.
We have a SES $1 \to \mathcal{G}_E \to Aut(E) \to Diff(M) \to 1$, the kernel is called the gauge group. I.e. it is the group of bundle isomorphisms covering the identity.

G_E will denote the sheaf (as a space), the C^∞ sections of which give back $\mathcal{G}_E$. G_E is the group of fiberise linear transformations of E, i.e. $G_E|_x = Aut(E_x) = GL(n;\mathbb{C})$. This is indeed the case, all maps arise. Now for $g_1, g_2 \in \mathcal{G}_E = C^\infty(M, G_E)$ we an defined $g_1 g_2$ as the products of sections, as linear maps. $Lie(\mathcal{G}_E) = C^\infty(M, \mathfrak{g}_E)$, where $\mathfrak{g}_E$ is the Lie-algebra bundle of G_E, i.e. $Hom(E_x)$.
Heads up, if the group is disconnected, its Lie algebra is only defined over the identity component.

Definition 2.3. Section.

HW: Prove that if $E = \Lambda^k T^* M \otimes_{\mathbb{R}} \mathbb{C}$, then $1 \to \mathcal{G}_E \to Aut(E_k) \to Diff(M) \to 1$ splits (and maybe is a direct product?). (what do we know about pullbacks of forms under diffeomorphisms?)

2.2 Gluing construction

3 Third lecture

Remark 3.1. $\mathcal{G}_E$ is a Hilbert Lie group (over L^2_k). Also $Iso(E) = Aut(C^\infty(M, E))$.

3.1 The covariant derivative on E

Motivation: we have a nice theory of analysis for exterior forms. We have a complex, the differentiation operator is unique and so on. We want something similar with vector valued k-forms.

Definition 3.2. Let $E \to M$ be a complex vector bundle, $\Omega^k(M, E) := C^\infty(M, \Lambda^k T^* M \otimes_{\mathbb{C}} E)$. We call these E-valued k-forms.

We want a $d_\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ with as many properties of the ordinary derivative as possible. We sacrifice uniqueness in the process, there are many such operators. There is also no cohomology, d^2_∇ will be the curvature.
Recall the glueing construction. A vectorbundle is the same as the choice of an open cover, and maps on the elments of the cover with maps into $Aut(F)$ satisfying the cocycle conditions. Similarly a section is a set of local functions compatible with the gluing maps.
First we construct $d_\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$. We can try coordinatewise differentiation, this does not work. The Leibniz rule and the gluing rule conflict.
So we require ∇ to take s to another section of a vector bundle ∇s. It has to be a first order differential operator, i.e. $\nabla(fs) = df \otimes s + f \nabla s$.
The solution will be covariant differentiation. We can compute, that $\nabla|_U = d + A_U$, where $A_U \in \Omega^1(U, End\, E)$. How does the A_U transform under chart change.

Definition 3.3. $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ is a covariant derivative on E if for any locally trivialising open cover $\{U_\alpha\}$ with gluing functions $g_{\alpha\beta}$ we have $\nabla = \{\nabla_U\}$ where $\nabla_U = d + A_U$ with $A_U \in \Omega^1(M, E)$ and $A_U = g_{UV} A_V g_{UV}^{-1} + g_{UV} d g_{UV}^{-1}$.
Extend these to all forms with the Leibniz rule.

We denote $\nabla_s(X) := \nabla_X s$ where X is a smooth vector field over M. Now the axiomatic version of the previous definition

Definition 3.4. ∇ is a covariant derivative if

1. $\nabla(fs) = df \otimes s + f\nabla s$
2. complex linear
3. $\nabla_{fX+gY} s = f\nabla_X s + g\nabla Y s$

Theorem 3.5. *If $E \to M$ is a vector bundle and ∇ is a covariant derivative, then $\nabla + a$ is another covariant derivative on E for any $a \in \Omega^1(M, End\ E)$. Consequently $\mathcal{A}_E$ (the set ofcovariant derivatives) is an infinite dimensional affine space over $\Omega^1(M, End\ E)$.*

Proof. If ∇ is a covariant derivative and a is a 1-form, then its clear that $\nabla + a$ is also a covariant derivative. In the other direction, given two covariant derivatives, a trivialising neighborhood U and a section s we get $(\nabla' - \nabla)s = (A'_U - A_U)s$, and these glue together well, since we subtrack the inhomogeneous part. □

Definition 3.6. If ∇, ∇' are two connections on a vector bundle we call them *gauge* equivalent if $\exists g \in \mathcal{G}_E$ such that $\nabla' = g\nabla g^{-1}$.
$G_\nabla := \{g \in \mathcal{G}_E : g\nabla g^{-1} = \nabla\}$.

Homework 3.7. *If $\nabla' = \nabla + a$ and $\nabla' = g\nabla g^{-1}$, what does this mean for a?*

Definition 3.8. $\mathcal{B}_E := \mathcal{A}_E/\mathcal{G}_E$ ir the gauge orbit space.

Definition 3.9. Let E be a vector bundle with ∇ a covariant derivative, s a smooth section and γ a smooth curve in M. We call s parallel w.r.t. ∇ along γ if $\nabla_{\gamma'} s = 0$.
If $E = TM$ then γ is autoparallel if $\nabla_{\gamma'}\gamma' = 0$.

4 MISSING

5 Fifth lecture

Last week: parallel transpost equation $\nabla_{\dot\gamma} s = 0$. If γ is a small closed loop from x_0 to x_0, we get a $\pi_{\nabla,\gamma}$ $\mathbb{C}$ linear automorphism of the fiber over x_0. We saw, that this map is equal to $id - \frac{1}{2}F_\nabla\epsilon^2 + o(\epsilon^3)$, where F_∇ is the curvature tensor $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Remark 5.1. Máshogy is meg lehet mutatni, hogy F_∇ s-ben algebrai.

Proof. $[\nabla_X, \nabla_Y](fs) - \nabla_[X, Y](fs)$. Use the defining properties of the covariant derivative it is a standard check. □

This means $F_\nabla : \Omega^0(M, E) \to \Omega^2(M, E)$ is a map, since its clearly antisymmetric in its first two arguments, i.e. $F_\nabla \in \Omega^2(M, End\ E)$.

Proposition 5.2. $d_\nabla^2 = d_\nabla \circ \nabla = F_\nabla$

Proof. Let $(U, x^1, \ldots, x^m, e_1 \ldots, e_\ell)$ be a local coordinate system and trivialisation over U,where $m = dimX$ and $\ell = dimE$. Let $s|_U = x_j$.

$$d_\nabla \nabla e_j = d_\nabla(A^i_j e_i) = dA^i_j e_i - A^i_j \wedge \nabla e_j = (dA^k_j + A^k_i \wedge A^i_j)e_k =: F^k_j e_k$$

This means that on U $F_\nabla|_U = dA + A \wedge A$.

On the other hand $A^i_j = \Gamma^i_{kj} dx^k$ by definition. We compute the curvature tensor locally (only the section component, we omit the vector fields).

$$F^k_j = d(\Gamma^k_{lj} dx^l) + \Gamma^k_{li} dx^l \wedge \Gamma^i_{pj} dx^p = (\frac{\partial \Gamma^k_{lj}}{\partial x^p} - \Gamma^k_{li}\Gamma^i_{pj}) dx^p \wedge dx^l$$

Writing out the components, we notice the local form of the curvature tensor. □

Remark 5.3. d^2_∇ is tensorial in its argument as well. $d^2_\nabla(fs) = d_\nabla(dfs + f\nabla s) = d^2 fs - df \wedge \nabla s + df \wedge \nabla s + f d_\nabla \nabla s = f d^2_\nabla s$.

It is clear, that d_∇ induces operators $\Omega^k(M, End\ E) \to \Omega^k(M, End\ E)$ for each k canonically.

Homework 5.4. *What is the induced connection on the dual bundle? The induced form on $E \otimes F$ will have the form $d + A_E \otimes 1 + 1 \otimes A_F$, use this for $E \otimes E^* = End\ E$.*

Lemma 5.5 (Differential Binachi identity)**.** $d^{End\ E}_\nabla F_\nabla = 0$

Remark 5.6. This is not the same as saying $d^3_\nabla = 0$, which is not true in general.

Proof. By the homework it turns out, that

$$d^{End\ E}_\nabla F_\nabla|_U = d^{E \otimes E^*}_\nabla (d^E_\nabla)^2|_U = dF + A \wedge F - F \wedge A$$

Substituting the form we found for $F_\nabla|_U = dA + A \wedge A$, we get 0 after computing the derivatives.

□

Remark 5.7. $A^U = A_i dx^i$, and $A \wedge A = [A_i, A_j] dx^i \wedge dx^j$. With Lie-algebra valued forms $A \wedge A \neq 0$ in general!

5.1 Characteristic classes – Chern-Weil theory

Start: G a lie group, $\rho : G \to AutV$ irreducible complex representation, and P a princpial G-bundle. From this data we can produce a complex vector bundle $E \to M$ by the associated bundle construction. Let $\nabla = d_\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ G-kompatibilis konnexió.

Definition 5.8. $P : \Omega^k(M, E) \to \Omega^k(M)$ is a *G-evaluation* if $\Phi \in \Omega^k \ni p(\Phi)$ and $P(\rho(g)\Phi) = P(\Phi)$ for all $g \in \mathcal{G}_E$.

Example 5.9. The trace map will be the most important example for us.

Lemma 5.10. *If P is an invariant evaluation, then*

Proof. Define $D : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$, which would make the diagram commute, i.e. $\forall \Phi \in Omega^k(M, e)$: $dp(\Phi) = p(D\Phi)$. This forces D to be a first order differential operator. Since p is invariant, we get $p(D'\Phi) = dp(\Phi) = dp(g\Phi) = p(D(g\Phi))$. This means, that $gD\Phi = D'g\Phi$, so D and D' are gauge conjugate. Since it transforms as the connection, it means that this is d_∇. □

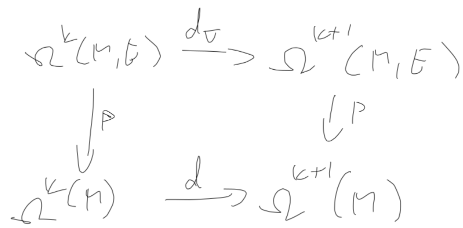

Figure 1:

Lemma 5.11 (Fundamental lemma)**.** *Let E be a complex vector bundle over M with structure group G. Let ∇ be a G-compatible covariant derivation on E. F_∇ denotes its curvature. Then $dtr(F_\nabla \wedge \cdots \wedge F_\nabla) = 0$, i.e. it represents a de-Rahm cohomology class on M.*
If ∇', ∇'' are two G-compatible covariant derivations on E, then $[tr(F_{\nabla'})] = [tr(F_{\nabla''})] \in H^{2k}(M)$, ergo this cohomology class is independent of the connection.

Proof. Use the previous lemma (on the endomorphism bundle of E, where F_∇ lives)

$$dtr(F_\nabla \wedge \cdots \wedge F_\nabla) = tr(d_\nabla F_\nabla \wedge \cdots \wedge F_\nabla) + \cdots + tr(F_\nabla \wedge \cdots \wedge d_\nabla F_\nabla) = 0 + \cdots + 0$$

by the Bianchi identity.
Let ∇ be a connection on E, $a \in \Omega^1(M, End\ E)$ and let $\nabla_t := \nabla + ta$ where $t \in (-\epsilon, \epsilon)$.

$$F_{\nabla_t} = d(A + ta) + (A + ta) \wedge (A + ta) = F_\nabla + td_\nabla a + t^2 a \wedge a$$

This means that $\frac{dF_{\nabla_t}}{dt}|_0 = d_\nabla a$, similarly one can come up with the fact that $\frac{dF_{\nabla_t}}{dt}|_{t=t_0} = d_{\nabla_{t_0}} a$. Let $\nabla =: \nabla', \nabla + a =: \nabla''$, ergo $\nabla' = \nabla_0, \nabla'' + \nabla_1$.

$$tr(F_{\nabla''} \wedge \dots) - tr(F_\nabla \wedge \dots) = \int_0^1 \frac{d}{dt} tr(F_{\nabla_t} \wedge \dots) dt = k \int_0^1 ktr(\frac{dF_{\nabla_t}}{dt} \wedge F_{\nabla_t} \wedge \dots) dt =$$

Using the lemma and the Bianchi identity we see

$$= k \int_0^1 tr(d_{\nabla_t} a \wedge F_{\nabla_t} \wedge \dots) dt = d\left(k \int_0^1 tr(a \wedge F_{\nabla_t} \dots) dt\right).$$

□

Definition 5.12. We call the $\int_0^1 tr(a \wedge F_{\nabla_t} \wedge \dots) dt \in \Omega^{2(k-1)+1}(M)$ form the Chern-Simons form.

Homework 5.13. *Compute the $k - 1 = 1$ Chern-Simons form: $\int_0^1 tr(a \wedge (F_\nabla + td_\nabla a + t^2 a \wedge a))dt$*

Another proof for the invariance of the homology class.

Lemma 5.14. *Let M be an oriented manifold and E a complex G-bundle equipped with ∇ and $N^{2k} \subset M$ a closed oriented submanifold. Consider the action*

$$s : \mathcal{A}_E \to \mathbb{R}\ \nabla \mapsto \int_N tr(F_\nabla \wedge \cdots \wedge F_\nabla)$$

then every ∇ is a critical point of s.

Proof. Compute the Euler-Lagrange equations for s. Let g be a Riemannian metric on M, then $s(\nabla) = \pm(F_\nabla, *(F_\nabla \wedge \dots \wedge F_\nabla))_{L_2(N,g|_N)}$, [*] since it is equal to $\int_M tr(F_\nabla \wedge *(*(F_\nabla \wedge \dots)))dt$.
Now pick $\nabla_t = \nabla + ta$. We computed before, that

$$s(\nabla_t) = (F_\nabla + td_\nabla a + t^2 a \wedge a, *((F_\nabla + td_\nabla a + t^2 a \wedge a) \wedge \dots \wedge F_\nabla + td_\nabla a + t^2 a \wedge a) =$$

$$= s(\nabla) + t(d_\nabla a, *(F_\nabla \wedge \dots \wedge F_\nabla)) + (k-1)t(F_\nabla, *(d_\nabla a \wedge F_\nabla \wedge \dots)) + o(t)$$

Now in the first order terms we use the Bianchi identity, and use "integration by parts", using the fact, that $\partial N = 0$ and $*^2 = \pm 1$.

$$(a, d_\nabla^* * (F_\nabla \wedge \dots)) \pm (k-1)(d_\nabla^* * F_\nabla, (a \wedge F_\nabla \wedge \dots))$$

Lemma 5.15. $d_\nabla^* = \pm * d_\nabla *$

Applying the Bianchi identity now completes the proof. □

Corollary 5.16. *The functional s is constant on the space of connections.*

Definition 5.17. E complex G-bundle over M, ∇ a connection, $F_\nabla \in \Omega^2(M, End_G\ E)$, then $\det(1 - \frac{t}{2\pi i} F_\nabla) = \sum t^j c_j(E) =: c(E)$. We call $c_j(E) \in H^{2j}(M)$ the jth Chern class of E.

Remark 5.18. $\det(1 + A) = \det e^{\log(1+A)} = e^{tr \log(1+A)} = e^{tr(1-A+A^2/2-\dots)}$ gives

$$\sum t^j c_j(E) = 1 - t\frac{trF_\nabla}{2\pi i} + t^2 \frac{tr(F_\nabla \wedge F_\nabla) - tr(F_\nabla) \wedge tr(F_\nabla))}{8\pi^2} - \frac{-2tr(F_\nabla^3) + 3tr(F_\nabla^2) \wedge tr(F_\nabla) - (trF_\nabla)^3}{48\pi^3 i}$$

Here we use $a^3 = a \wedge a \wedge a$ and similarly.

Let $\dim M = 4$ and E a rank 2 $SU(2)$ bundle. $c_0(E) = 1 \in H^0(M), c_1(E) = \frac{tr(F_\nabla)}{2\pi i} = 0 \in H^0(M)$ since the Lie algebra of $SU(2)$ consists of traceless matrices. $c_2(E) = \frac{tr(F_\nabla \wedge F_\nabla)}{8\pi^2} \in H^4(M) = \mathbb{R}$, this is nonzero generally.
We can also define $C_j(E, N^{2j}) = \int_{N^{2j}} c_j(E) \in \mathbb{R}$ is called a Chern number.

5.2 Typical behaviour of Chern classes

- Given $E \oplus F$,then $c(E \oplus F) = c(E)c(F)$.
- $f : M \to N$ differentiable, $E \to N$ a bundle with connection ∇, then there exists $f^*E \to M$ the pullback bundle, and $f^*\nabla$ the pullback connection. In this setup we have that $c(f^*E) = f^*(c(E))$.

Corollary 5.19. $c_j(E) \in Im(H^{2j}(M, \mathbb{Z}) \to H^{2j}(M))$, *i.e. the Chern numbers are whole numbers for any submanifold* N^{2j}.

[*] We can define $(\phi, \psi) := \int_M \phi \wedge *\psi$ to be the L_2 if we have differential forms $\Omega^k(M)$. Given a general vector bundle E, and given a pairing on E we can do the same construction for $\Omega^k(M, E)$

6 Sixth lecture

Existence of connections

Complex vector-bundle over a manifold $E \to M$, we defined what $\nabla : \Omega^0(M, e) \to \Omega^1(M, E)$ should satisfy. If there is at least one connection, then $\mathcal{A}_E$ is an affine space over $\Omega^1(M, End\ E)$. This is true, since given two connections their difference is $\in \Omega^1(M, End\ E)$.

Remark 6.1. If E is the trivial vector bundle of rank ℓ, then we can always define $\nabla_0 := d$ by choosing a trivialisation.

Fact 6.2 (Algebraic topology)**.** *If X is a compact topological space and $E \to X$ isa complex vector bunle over it, then there is another vector bundle $F \to X$ such that $F \oplus E$ is trivial.*

Now let M be a compact manifold and E any vector bundle over it. Using the previous topological lemma we find F such that $E \oplus F$ is trivial, let p denote the projection to E. We let $\nabla := d$ over this trivial bundle. It is homework to show, that restricting and projecting this connection to E satisfies the properties required. As discussed, every other connection will have the form $\nabla^E + a$.

Last week: we derived several important formulas for the covariant derivative

- $\nabla|_U = d + A^U$
- $F_\nabla|_U = dA^U + A^U \wedge A^U$
- $F_\nabla = (d_\nabla^E)^2$
- Bianchi identity: $d_\nabla^{End\ E} F_\nabla = d_\nabla^{End\ E}$
- we introduced $c_k(E) \in H^{2k}_{dR}(M)$ (for an $SU(2)$ bundle over a 4-manifold only c_2 is nontrivial)

Actually rank 2 complex vector $SU(2)$ vectorbundles are classified by their second Chern class. For this we use obstruction theory to classify bundles. Classifying $E \to M$ is the same as classifying $P_{SU(2)} \to M$. In general P_G is trivial iff it admits a continous section, so we need to understand the existence of sections of P_G over M. We will start with a section on a cell, and try to extend it cell by cell. By general obstruction theory we get an obstruction class $\in H^{k+1}(M, \pi_g(G))$. We know that $SU(2) = S^3$, so we know the first 3 homotopy groups. From this we get extendability to the 3 skeleton automatically. Thus if the obstruction class $c_2(E)$ vanishes, our bundle is trivial. In the other direction we need to construct bundles with any given $H^4(M, \mathbb{Z})$ element.

Fact 6.3. *Given a G compact Lie group, $\pi_2(G) = 0$.*

Fact 6.4. *Given a simple compact Lie group $\pi_3(G) = \mathbb{Z}$.*

Let us try the same thing with $SO(3) = SU(2)/\mathbb{Z}_2$. We get two obstructions $w_2 \in H^2(M, \mathbb{Z}_2)$ and $p_1 \in H^4(M, \mathbb{Z})$.
This does not help us in every case. Try to classify $SU(2)$ bundles over S^5. Clearly all obstruction classes vanish, but on the other hand we know, that there are $\pi_4(SU(2))$ many $SU(2)$ bundles over S^5, giving the gluing map on the equator.

Example 6.5. The trivial guy is of course $S^5 \times S^3 \to S^5$, and suprisingly $SU(3) \to S^5$ is the other one, where the map is projection to the first column. These spaces are distinguished by π_5.

6.1 Riemannian geometry (in m dimensions)

From now on $E = TM$, a real bundle.

Definition 6.6. $g \in C^\infty(M, S^2T^*M)$ is a *Riemannian metric* on M if $g_x : T_x \times T_x \to \mathbb{R}$ is a nondegenerate symmetric positie definite bilinear form at each $x \in M$.

Remark 6.7. Such a G always exists since $GL_n(\mathbb{R}) \sim O(n)$. Also $g_x(X, X) \geq 0$ and is equal to zero iff $X_x = 0$ at a point.

Definition 6.8. Let M, g be a R-manifold. ∇ is the Levi-Civita connection on M, if it is *compatible with the metric*, i.e. $(dg(X, Y))(Z) = Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z X)$for any three vector fields X, Y, Z, and $\nabla_X Y - \nabla_Y X = [X, Y]$, which is called *torsion-freeness*.

Homework 6.9. *On the one hand we have $d_\nabla \Omega^k(M, T^*M) \to \Omega^{k+1}(M, T^*M)$. These spaces are identified with $C^\infty(M, \Lambda^? T^*M \otimes T^*M)$. We apply $\mathcal{A}$, the antisymmetrization operator to get the next one in the sequence.*

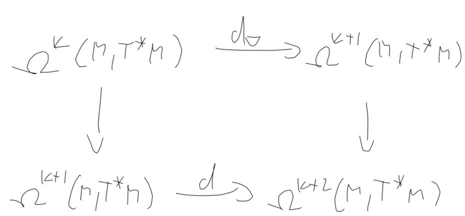

Figure 2: Show that this diagram commutes if ∇ is torsion free.

Remark 6.10. Given a connection, we get many induced connections, for example on S^2T^*M. The metric compatibility condition can be rephrased as $\nabla g = 0$.

Theorem 6.11 (Fundamental theorem of Riemannian geometry)**.** *If (M, g) is a pseudo-Riemannian manifold, then $\exists!$ Levi-Civita connection on M.*

Proof. Write the metric compatibility equation for X, Y, Z in all three cyclic permutations, and add them up with alternating sign. Apply the torsion free property to isolate one covariant derivative to obtain the Koszul formula:

$$2g(\nabla_X Y, Z) = dg(Y, Z)X + dg(Z, X)Y - dg(X, Y)Z + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$

This is true for all vector fields X, Y, Z and one can check that this formula defines a coariant derivative. Uniqueness is obtained simply. □

Remark 6.12. Let X, Y, Z be coordinate frames. Then we know that $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$, and $g(\partial_i, \partial_j) = g_{ij}$ by definition, and that $[\partial_i, \partial_j] = 0$. We can come up with an explicit formula from the Koszul identity for the christoffel Symbols from the components of the metric tensor.

6.2 The Riemannian curvature tensor

$R := F_\nabla$, this is a type $(1,3)$ tensor $R(X,Y)Z \in C^\infty(M,TM)$. Writing it out in a local chart we get the functions $R^l_{ij,k}$. Using the metric we can restate this as a $(0,4)$ tensor as $g(R(X,Y)Z,W)$, in coordinates $R_{ij,kl} = g_{lp}R^p_{ij,k}$.

Theorem 6.13 (Symmetries of the Riemannian curvature)**.** *Let $R^j_{i,kl}$ be the components of the 1,3 curvature tensor, $R_{ij,kl}$ be the components of the 0,4 curvature tensor.*

1. $R^i_{j,kl} = -R^i_{i,lk}$
2. $R_{ij,kl} = -R_{ji,kl}$
3. $R^i_{j,kl} + R^i_{l,jk} + R^i_{k,lj} = 0$
4. $R_{ij,kl} = R_{kl,ij}$

Proof. The first one we already know, every curvature tensor is antisymmetric in its two arguments.
For the second one if $\nabla|_U = d + A^U$ and $A^U \in \Omega^1(U, \mathfrak{gl}(m,\mathbb{R}))$. Then actually $A^U \in \Omega^1(M, \mathfrak{o}(m))$ from the compatibility condition. We know, that $\mathfrak{o}(m) = \Lambda^2\mathbb{R}^m$ as vectorspaces. The identification is as follows $A \mapsto \omega_A$ where $< Av, w > \mapsto \omega_A(v,w)$. This will be antisymmetric because A is so. Now $R \in \Omega^2(M, End_{\mathfrak{o}(m)}TM) = \Omega^2(M,\mathfrak{o}(m)) = \Omega^2(M, \Lambda^2T^*M) = C^\infty(M, \Lambda^2T^*M \otimes \Lambda^2T^*M)$ and we are done.
For the third identity we compute using $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

$$R^i_{j,kl}(\partial_i) = R^i_{p,kl}\delta^p_j(\partial_i) = R_{kl}(\delta^p_j\partial_p) = R_{kl}(\partial_j) = [\nabla_k, \nabla_l]\partial_j$$

Now write the last term out 3 times with cyclically permuted indicies and add them together. Use the torsion freeness at all three terms to get $0+0+0$.
The fourth and last identity is the most mysterious. See Milnor's Morse theory book for a quick proof:) □

Corollary 6.14. *1,2,4 gives us that $R \in C^\infty(M, S^2\Lambda^2T^*M)$. What does the Bianchi identity tell us about the curvature tensor? Define $b(R)(X,Y,Z,W) := \frac{1}{3}(R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W))$, called the Bianchi map in coordinate free form. Clearly $b^2 = b$, and b maps the space of symmetric tensors to itself. From the rank-kernel theorem $S^2\Lambda^2T^*_xM$ decomposes according to b.*

Homework 6.15. *Nontrivial algebra gives us, that $b : S^2\Lambda^2V \to \Lambda^2V \otimes \Lambda^2V$ actually has image $= \Lambda^4V$.*

So R is a smooth section of the bundle $S^2\Lambda^2T^*M \cap ker\ b$.
This helps us compute its dimension: $\binom{\binom{m}{2}+1}{2} - \binom{m}{4} = \frac{1}{6}\binom{m^2}{2}$.

7 Seventh lecture

Correction for last week:

Remark 7.1. Let V be a finite dimensional real vector space endowed with a vector product (i.e. $V = V^*$). We know that there is a canonical decomposition $End(\Lambda^2V^*) = S^2\Lambda^2V^* \oplus \Lambda^2\Lambda^2V^*$. Last time we introdiced a map $b : S^2\Lambda^2V^* \to End(\Lambda^2V^*)$ by $b(R(X,Y,Z,V)) = \frac{1}{3}(R(X,Y,Z,V) + R(Z,X,Y,V) + R(Y,Z,X,V))$. It is clear, that $b^2 = b$, so actually $im\ b \leq S^2\Lambda^2V^*$, consequently by standard linear algebra $S^2\Lambda^2V^* = ker\ b \oplus im\ b$.

Lemma 7.2. $im\ b = \Lambda^4 V^*$

Proof. First if $\omega \in \Lambda^4 V^*$, then $\omega(X, Y, Z, V) = \omega(Z, V, X, Y)$ by antisymmetry. This shows one inclusion, now surjectivity.
If $\alpha, \beta \in \Lambda^2 V^*$ we introduce their symmetric product $\alpha v \beta := \frac{1}{2}(\alpha(X, Y)\beta(Z, V) + \alpha(Z, V)\beta(X, Y))$. We claim, that $b(\alpha v \beta) = \frac{1}{3}\alpha \wedge \beta$. This is a simple check by coordinates:

$$\frac{1}{3}\frac{1}{2}(\alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij} + \alpha_{jk}\beta_{il} + \alpha_{jk}\beta_{il} + \alpha_{ki}\beta_{jl} + \alpha_{jl}\beta_{ki}) = \frac{1}{6}(\alpha_{ij}\beta_{kl} - \alpha_{ik}\beta_{jl} + \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{jl}\beta_{ik} + \alpha_{kl}\beta_{ij})$$

Which is $c \cdot (\alpha \wedge \beta)_{ijkl}$. □ ?constant?

7.1 Representation theory of $SO(4)$

From now on M is an oriented 4-dimensional Riemannian manifold. In particular TM_x is equipped with an $SO(4)$ structure. Our aim is to decompose $S^2\Lambda^2 T^*M \cap ker\ b$ into irreducible representations of $SO(4)$.
The group $SO(4)$ and $Spin(4) = SU(2) \times SU(2)$: Take $(\mathbb{R}^4, <,>)$ with a positive definite scalar product, we identify this with $\mathbb{H}, |.|$.

Proposition 7.3. $SU(3) = S^3 \subset \mathbb{H}$

Proof. $z = wj = q \in S^3$ where $|z|^2 + |w|^2 = 1$ we associate $\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$ and this is a group isomorphism. □

Proposition 7.4. $\pi : S^3 \times S^3 \to SO(4)$ *by* $x \mapsto \xi x \bar{\eta}$ *with the identification above is an index 2 cover.*

Proof. The quaternionic norm is multiplicative, so the maps are in $O(4)$, and actually in $SO(4)$, since quaternionic multiplication preserves orientation.
$\xi_2(\xi_1 x \bar{\eta}_1)\eta_2 = \xi_2\xi_1 x \overline{\eta_1\eta_2}$ so this is well defined homomorphism, since quaternion multiplication is associative.
Finally $ker\pi$ consists of elements of form $\eta = \xi^{-1}$. Then for any quaternion we get that $\xi x = x\xi$, i.e. ξ is in the center of H. Since its norm is 1, $\xi = \pm 1$, and $ker\pi = \mathbb{Z}_2$.
Since the source and the target is equidimensional, and the fiber of this map is discrete, π is a local homeomorphism. Moreover this map is closed, so it is surjective. □

Observations:$\pi_1(SO(\geq 3)) = \mathbb{Z}_2$. To see this consider the fibration $SO(n+1) \xrightarrow{SO(n)} S^n$ (map is projection to the first column) and writing out the first part of the homotopy exact sequence. We define $Spin(n)$ to be the universal cover of $SO(n)$. So the previous computation shows that $S^3 \times S^3 = Spin(4) = SU(2) \times SU(2)$.
We will first compute representations for the universal cover, then pick out those, who descend to $SO(4)$.

7.2 Representation theory of $SU(2)$

7.2.1 Background

Let G be a compact finite dimensional Lie group

Definition 7.5. Let V be a complex vector space, a complex representation of of G is a homomorphism $\rho : G \to Aut(V)$.
If ρ, σ are two representations of G, then they are equivalent/isomorphic, if there exists $U \in Aut(V)$ such that $\sigma = U\rho U^{-1}$ (U is independent of $g \in G$).
ρ is *irreducible* if there is no nontrivial ($\neq 0, V$) invariant subspace of V under the action of G.

Definition 7.6. $\rho : G \to Aut(V)$ is a representation, then $\chi_\rho := tr(\rho)$ is called a *character.*

Lemma 7.7.

1. *χ_ρ is C^∞*
2. *if ρ, σ are equivalent representations of G on V, then $\chi_\rho = \chi_\sigma$*
3. *$\chi_\rho(hgh^{-1}) = \chi_\rho(g)$*
4. *$\chi_{\rho\oplus\sigma} = \chi_\rho + \chi_\sigma$*
5. *$\chi_{\rho\otimes\sigma} = \chi_\rho\chi_\sigma$*
6. *$\chi_{\rho^*} = \overline{\chi}_\rho$ or equivalently $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$*
7. *$\int_G \chi_\rho(g)dg = \dim_{\mathbb{C}} V_G$ where $V_G = \{v \in V | \rho(g)v = v \forall g \in G\}$*

Proof. Analogously to the finite dimensional case. □

Definition 7.8. $< \chi_\rho, \chi_\sigma > := \frac{1}{Vol(G)} \int_G \chi_\rho \overline{\chi_\sigma} dg$. Here dg is the left-right invariant Haar-measure on G, and $Vol(G) = \int_G dg$ is the volume

Lemma 7.9 (Schur orthogonality relation). *If ρ, σ are irreducible complex representations of G on V, W respectively, then $< \chi_\rho, \chi_\sigma >= 1$ iff σ and ρ are equivalent and 0 otherwise.*

Proof. Observe that $\chi_\rho \overline{\chi_\sigma} = \chi_\rho \chi_{\sigma^*} = \chi_{\rho\otimes\sigma^*}$ and $\rho \otimes \sigma^* : G \to Aut(Hom(V, W))$. By (a different) lemma of Schur if $B \in Hom(V, W)$ such that $\rho \otimes \sigma^*(g)B = B$ for all g, then B is an equivariant map from (V, ρ) to (W, σ), and $V = W$ and $B = \lambda I$ follows (otherwise no such B exists).
Stated otherwise, we get that $dim_{\mathbb{C}} Hom(V, W)_G = 1$ if ρ, σ are equivalent representations, and 0 otherwise. Now apply point 7 from the previous lemma $< \chi_\rho, \chi_\sigma >= 1$ or 0, dependent on if ρ and σ are equivalent or not. □

Lemma 7.10 (Schur, lol). *A representation ρ is irreducible iff $||\chi_\rho||^2 = 1$.*

Remark 7.11. Every finite dimensional representation of a compact Lie group decomposes as a sum of irreducible representations. One sees this by constructing a G action invariant scalar product on V by averaging. There are conterexamples if the group is noncompact, since there may not be invariant scalar products, see $GL(2)$ on $\mathbb{R}^2$.

Proof. Take a decomposition of V into irreducible representations $\oplus n_i V_i$, and $\rho = \oplus n_i \rho_i$ where any two ρ_i are inequivalent. $||\chi_\rho||^2 = \sum n_i^2 ||\chi_{\rho_i}||^2$ by the orthogonality relations and $||\chi_{\rho_i}||^2 = 1$ by the previous Schur lemma. The other direction is immediate, there is only one partition of 1 into positive whole numbers. □

Theorem 7.12. *If G, H are compact Lie groups and ρ_G, ρ_G are some irreducible complex representations of them, then $\rho_G \otimes \rho_H$ is an irreducible complex representation of $G \times H$. Conversely any irreducible complex representation of $G \times H$ is of this form.*

Proof.

$$||\rho_G \otimes \rho_H||^2 = \int_G \int_H \chi_{\rho G} \chi_{\rho H} \overline{\chi_{\rho G} \chi_{\rho H}} dh dg = 1$$

Conversly we apply the Schur lemma. Let U be an irreducible $G \times H$ representation, then there exists the following* isomorhism of H-modules

$$\phi : \bigoplus_{W_j \in Irr(H;\mathbb{C})} Hom_H(W_j, U) \otimes W_j \to U$$

This is true by the Schur lemma, $Hom_H(W_j, U)$ is 1 dimensional if W_J isequivalent to U, when we forget about the G-module structure of U.† Moreover viewing $Hom(W_j, U)$ as a G-module, we see, that it decomposes $\oplus n_{ji} V_i$ into irreducible G modules. This means, that $U = \bigoplus n_{ij} V_i \otimes W_j$ □

7.3 Irreducible complex representations of $SU(2)$

Let $V = \mathbb{C}^2$ and let $SU(2)$ act in the standard way by matrix multiplication. On V^* it acts by multiplication by the inverse. These representations are equivalent in the $SU(2)$ case. Consider $S^m V^* = \{p_m(x, y)\}$ homogeneous polynomials of degree m in 2 variables x, y. The dimension is $m + 1$, clearly $S^0 V^* = \mathbb{C}$ with the trivial action. $S^1 V^* = V^*$ is the standard representation.

Definition 7.13. $\rho_m : SU(2) \times S^m V^* \to S^m V^*$, where $g, p_m(\xi) \mapsto p_m(g^{-1}\xi)$.

Lemma 7.14. *ρ_m is an irreducible representation for all $m \geq 0$.*

Proof. By Schur lemma we will see, that if $A : S^m V^* \to S^m V^*$ complex linear map such that $\rho_m(g)A = A\rho_m(g)\ \forall g$, then $g = cI$.
Let $g_a = diag(a, a^{-1})$ with $a \in U(1)$ and introduce a basis $p_k = x^k y^{m-k}$ in $S^m V^*$. $\rho_m(g_a)p_k = a^{2k-m} p_k$ and if $\rho_m(g_a)(Ap_k) = A\rho_m(g_a)p_k$, then its equal to $Aa^{2k-m}p_k$. We can pick a such that its powers are all different, then the a^{2k-m} eigenspace $\rho_m(g_a)$ is spanned by p_k, since we see all of its eigenvectors for different eigenvalues p_*. This means, that $Ap_k = c_k p_k$.
Consider r_t, the real rotation matrix with angle t.

$$Ar_t p_m = a(x \cos t + y \sin t)^m = \sum \binom{m}{k} \cos^k t \sin^{m-k} t (Ap_k) = \sum \binom{m}{k} \cos^k t \sin^{m-k} t (c_k p_k)$$

We also compute

$$r_t A p_m = r_t c_m p_m = c_m r_t p_m = c_m \sum \binom{m}{k} \cos^k t \sin^{m-k} t p_k$$

so $c_k = c_m$, and the representation is irreducible, since $A = c_m I$. □

Remark 7.15. These are all of the irreducible representations of $SU(2)$, but we will not prove it here.

Theorem 7.16 (Clebsch-Gordon formula). $\rho_m \otimes \rho_n = \bigoplus_0^{\min m,n} \rho_{m+n-2j}$

Proof. $\chi_{\rho_m} = \sum_0^m e^{i(m-2k)t}$ since every matrix $SU(2)$ matrix is conjugate to some $diag(e^{it}, e^{-it})$. Some Fourier analysis tells us, that the product of two sums of this form looks like

$$\sum_0^{\min(m,n)} \sum_0^{m+n-2j} e^{i(m+n-2j-2p)t}$$

Homework 7.17. *Check this formula*

* $f \otimes w = f(w)$
† split U into irreducible representations over H

□

Corollary 7.18. *$Spin(4) = SU(2)^+ \times SU(2)^-$, then any irreducible representation is of the form $\rho_m^m \otimes \rho_n^-$, which we denote $\rho_{m,n}$. Moreover we can decompose any representation into irreducibles by the formula. It is clear, that $\dim_{\mathbb{C}} \rho = (m+1)(n+1)$.*

Example 7.19. $\rho = \rho_m^+ \otimes \rho_n^-$, $\sigma = \rho_k^+ \otimes \rho_l^-$. Now applying the commutativity of the tensor product, and the Clebsch-Gordon formula:

$$\rho \otimes \sigma = \rho_m^+ \otimes \rho_n^- \otimes \rho_k^+ \otimes \rho_l^- = (\rho_m^+ \otimes \rho_k^+) \otimes (\rho_n^- \otimes \rho_l^-) = (\oplus \rho_i^+) \otimes (\oplus \rho_j^-)$$

We have two questions:

1. How to get representations of $SO(4)$?

2. How to get *real* irreducible representations of $SO(4)$?

Lemma 7.20. *$\rho_{m,n}$ is a complex representation of $SO(4)$ iff $m \equiv 2 \mod 2$.*

Proof. We have to see, that $\ker \pi$ acts trivially in $\rho_{m,n}$. This is easily seen if m, n are even, orifboth are odd, since $-I$ acts trivially on either both parts of a tensor product basis, or both get multiplied by -1, i.e. $(-1)^2 = 1$. □

Definition 7.21. Representatin σ on V is a real representation off there is a G-invariant decomposition $V = W \oplus iW$, where W is a real G-space. (i.e. there is a real linear map, whose square is the identity, commuting with the image of σ)

Remark 7.22. If ρ is an irreducible (?need this?) complex representation,then $\int_G \chi_\rho(g^2)dg = 1$ if ρ is a real representation, 0 if ρ remains complex and -1 if it admits a $\mathbb{H}$ structure.

Lemma 7.23. *$\rho_{m,n}$ always has a real structure if $m \equiv n \mod 2$.*

Proof. $\tau : S^m V^* \to S^m V^*$ is defined as $p(x, y) \mapsto \overline{p(-\bar{y}, \bar{x})}$. It is clear, that $\tau^2 = id$, we call p real, if $\tau p = p$. Put W to be the$\mathbb{R}$ span of real polynomials, we get an $m+1$ dimensional real subspace of $S^m V^*$. $\forall g \in SU(2)$ if p_m is real, then a simple computation shows that $\tau \rho_m(g) p$ is real as well, if m is even.
The nontrivial computation comes when $m + n$ is even. Then $\tau(PQ) = PQ$ will imply, that $\tau(\rho_m^+ P \rho_n^- Q) = \rho_m^+ P \rho_n^- Q$. □

Remark 7.24. Every odd complex dimensional representation of $SU(2)$ descends to $SO(4)$.

8 Eight lecture

8.1 Representations of SO(4) on p-forms

Let $(W, <,>)$ be an oriented 4 dimensional real inner product space (i.e. $SO(4)$ acts on it with the standard representation). Question: What kind of $SO(4)$ module is $\Lambda^p W$? By assumption, there is a Hodge operator $* : \Lambda^p W \to \Lambda^{4-p} W$.

Remark 8.1. On an $SO(4)$ module V, the operator $*$ is defined for $\alpha, \beta \in \Lambda^p V$ to be the form satisfying $\alpha \wedge *\beta =< \alpha, \beta > vol$. Some basic properties: $*^2 = \pm 1 id_{\Lambda^p V}$, and it commutes with with the induced $SO(n)$ action on $\Lambda^p V$ and $\Lambda^{n-p} V$.

For $p = 2$ we get an automorphism of 2-forms, and $*^2 = 1_{\Lambda^2 W^*}$. So its eigenvalues will be ± 1.

Definition 8.2. $\Lambda^{\pm} W^* := \{\omega \in \Lambda^2 W^* | * \omega = \pm \omega\}$, and call these self dual, and anti self dual forms.

This means $\Lambda^2 W^* = \Lambda^+ W^* \oplus \Lambda^- W^*$, and we get that $\dim \Lambda^{\pm} W^* = 3$. Since $\mathfrak{so}(4) = \mathfrak{su}(2)^+ \oplus \mathfrak{so}(2)^-$ we get that $\mathfrak{so}(4)$ as an $SO(4)$ module is not irreducible.
On the other hand we saw, that $\mathfrak{so}(4) = \Lambda^2 W^*$, where the identification is given by $< x, Ay >= \omega_A(x, y)$ for some antisymmetric matrix A. So if $SO(4)$ acts by the adjoint representation ($A \mapsto gAg^{-1}$) from the previous remark we see that $*$ commutes with this action, and we get that the two factorisations $so(2)^{\pm} = \Lambda^{\pm} W^*$ are isomorphic (equivariantly so).
So $\Lambda^2 W^*$ is reducible, since we just split is apart. We are looking for the irreducible components. Since $\Lambda^p m W^* = \mathfrak{su}(2)^{\pm}$ we get that $\Lambda^{\pm} W^* \otimes \mathbb{C}$ as a complex $Spin(4) = SU^+(2) \times SU^-(2)$ module the action splits on the components (+ only acts on + and vice versa for $-$). Now $\Lambda^{\pm} W^* \otimes \mathbb{C} = S^m V^+ \otimes S^n V^-$ by our classification from before. Since we know the dimensions, $3 = (m+1)(n+1)$, so $m = 2, n = 0$ or the other way around, i.e. $\Lambda^{\pm} W^* = S^2 V^{\pm}$.
For $p = 1$ we get an identification between $\Lambda^1 W^* = \Lambda^3 W^*$, in an SO(4) equivariant manner. Complexify this module, we are looking for $\Lambda^1 W^* \otimes \mathbb{C} = S^1 V^+ \otimes S^2 V^- = V^+ \otimes V^-$, since the other possibility $(4 = 1 \cdot 4)$, we already used for the $p = 2$ case.
The last case $p = 0$ is trivial, we get the trivial module, and its complexification, so $\Lambda^0 W^* \otimes \mathbb{C}\Lambda^4 W^* = S^0 V^+ \otimes S^0 V^-$.

Remark 8.3. This is the classification of the complexified spin(4) modules, and we have to look for the real ones among them. This is clear to see, since $0 + 0 \equiv 2 + 2 \equiv 2 + 0 \equiv 0 \mod 2$, so all of the previous splittings are real representations of SO(4) as well.

8.2 Irreducible splitting of the curvature tensor of an oriented Riemannian 4-manifold

From (M, g) we produced the curvature tensor $R \in C^\infty(M, S^2 \Lambda^2 T^* M \cap kerb)$. Consider the $\mathbb{C}$-linear extension of $R \in C^\infty(M, S^2 \Lambda^2 T^* M$
$otimes \mathbb{C} capkerb^{\mathbb{C}})$. If $x \in M$, then $\Lambda^2_x T^* M = \Lambda^+_x T^* M \oplus \Lambda^-_x T^* M$. In this basis $R = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ as a symmetric map $\Lambda^+_x T^* M \otimes \mathbb{C} \oplus \Lambda^-_x T^* M \otimes \mathbb{C} \to \Lambda^+_x T^* M \otimes \mathbb{C} \oplus \Lambda^-_x T^* M \otimes \mathbb{C}$. Thus we have to split $End(\Lambda^2 T^*_x M \otimes \mathbb{C})$ into irreduibles. Using the fact that $End(V) = V \otimes V^*$, and the fact that the metric canonically identifies V with V^*, substituting $\Lambda^{\pm}_x \otimes \mathbb{C} = S^2 V^{\pm}$ we get the following:

$$End(\Lambda^2_x T^* M) = S^4 V^3 \oplus S^2 V^+ \oplus S^0 V^- \oplus (S^2 V^+ \otimes S^2 V^-) \oplus (S^2 V^- \oplus S^2 V^+) \oplus S^4 V^- \oplus S^2 V^- \oplus S^0 V^-$$

In the above splitting $S^0 V^{\pm} = \mathbb{C} id_{\Lambda^{\pm} \otimes \mathbb{C}}$, $S^2 V^{\pm} = \Lambda^2(\Lambda^{\pm}_x \otimes \mathbb{C})$ and $S^4 V^{\pm} = S^2_0(\Lambda^{\pm}_x \otimes \mathbb{C})$ of traceless symmetric matrices.

Remark 8.4. Given V, acted on by $O(n)$, we get an action on $End(V)$ as well by conjugation. This representation of $O(n)$ splits into three parts, $\mathbb{R} id_v \oplus \Lambda^2 V \oplus S^2_0 V$.

Finally $S^2V^+ \otimes S^2V^- = Hom(\Lambda^\pm_x \otimes \mathbb{C}, \Lambda^\mp_x \otimes \mathbb{C})$. So B and B^* correspond to this component in the splitting, A to S^4V^+ and C to S^4V^-.
We still have not used the Bianchi identity. $S^0V^+ \oplus S^0V^- = \mathbb{C}(id_{\Lambda^+\otimes\mathbb{C}} + id_{\Lambda^-\otimes\mathbb{C}}) \oplus \mathbb{C}(id_{\Lambda^+\otimes\mathbb{C}} - id_{\Lambda^-\otimes\mathbb{C}}) = \mathbb{C}id_{\Lambda^2\otimes\mathbb{C}} \oplus \mathbb{C}*_{\Lambda^2\otimes\mathbb{C}}$ by a base change we get the trivial represenation, and the one given by the Hodge operator. Notice, that $\mathbb{C}* = \Lambda^4 \otimes \mathbb{C}$, since both sides are the trivial one dimensional SO(4) module, or one can say, that $\alpha \wedge *\beta =< \alpha, \beta > *$, and $z \cdot * \mapsto z \cdot vol$ is an equivariant isomorphism.
We saw previously, that $imb^{\mathbb{C}}_x = \Lambda^4_x \otimes \mathbb{C}$, thus the Hodge component of this remaining part is zero (in the $id_{\Lambda^2\otimes\mathbb{C}}$ direction we get the complexified *scalar curvature*).

Definition 8.5. We denote by $s_x \in \mathbb{C}id_{\Lambda^2_x T^*M\otimes\mathbb{C}}$ the complexified scalar curvature. We denote by $W^\pm_x \in S^2_0(\Lambda^\pm_x T^*M \otimes \mathbb{C})$ component, and call them the complexified (anti-) self dual Weyl tensors. We denote by $B_x \in Hom(\Lambda^\pm_x T^*M \otimes \mathbb{C}, \Lambda^\mp_x T^*M \otimes \mathbb{C})$ and call it the complexified traceless Ricci tensor.

By these definitions $R = \begin{bmatrix} W^+_x + s_x/12 & B_x \\ B^*_x & W^-_x + s_x/12 \end{bmatrix}$ is true at every $x \in M$.

8.3 Globalisation, realification

Problem: $V^\pm$ are *spin*(4) modules.

Definition 8.6. If there are complex rank 2 bundles $\Sigma^\pm$ over M, whose fibers are $V^\pm$, then we call $\Sigma^p m$ chiral spinor bundles.

Remark 8.7. If $\Sigma^\pm$ exists, then we can produce the $S^m\Sigma^\pm$ bundles, and all our constructions work globally.

Definition 8.8. We call such manifolds *spin* manifolds.

Remark 8.9. Suppose M to be spin and consider the $S^m\Sigma^+ \otimes S^n \otimes \Sigma^-$ bundle over it. This is a complex spin(4) vector bundle, but if $m+n$ is even, then this reduces to a complex $SO(4)$ bundle, and will inherit a real structure as well. So there exists $W^{(m,n)} \subset S^m\Sigma^+ \otimes S^n\Sigma^-$, a real subbundle, which correspond to the $\rho_{m,n}$ irreducible real representation of $SO(4)$. Moreover, these real bundles exist even if M is non-spin.

Corollary 8.10. *The previous remark is true for every component of the complexified curvature tensor, so the previous decomposition globalises, and with real bundles.*

Theorem 8.11 (Singer-Thorpe '69)**.** *Let (M, g) be an oriented Riemannian 4-manifold, ∇ its Levi-Civita connection and R_∇ its curvature tensor. Then $R_\nabla \in End(\Lambda^2T^*M)$ splits into irreducible $SO(4)$ components as follows:* $R_\nabla = \begin{bmatrix} W^+ + s/12 & B \\ B^* & W^- + s/12 \end{bmatrix}$. □

Remark 8.12. If we forget about the orientation, then $W = W^+ + W^-$ will be invariant, we call it the *Weyl tensor.*
The type $(3, 1)$ Weyl-tensor is a conformal invariant! $W_{f^2g} = W_g$ for any nowhere vanishing function f.
In 4-dimensions R_∇ is a $20 = 1 \oplus 5 \oplus 5 \oplus 9$ dimensional representation, the irreducible components are the scalar curvature, the two Weyl tensors, and the traceless Ricci curvature.

Definition 8.13. $Ric := s+B$ is called the Ricci tensor. A 4-manifold (M, g) is *Ricci-flat*, if $Ric = 0$. We call a 4-manifold *Einstein*, if $B = 0$. We call M *half-conformally flat* if $W^- = 0$, and *half-conformally anti-flat* if $W^+ = 0$.

Homework 8.14. *For an Einstein manifold $Ric = \Lambda g$. Show from the differential Bianchi identity, that the function Λ is a constant. We call this the cosmological constant.*

$$Ric - \frac{1}{2}gs = 8\pi T + \Lambda g$$

where $T \in S^2(T^*M)$ is called the Einstein equation.

9 Ninth lecture

Quick recap and clarification.

Remark 9.1. Let M be a spin 4-manifold, $TM \to M$ has a principal $P_{SO(4)} \to M$ bundle. It is called spin, if we van find a 2 fold cover $P_{spin(4)} \to P_{SO(4)}$, that is a 2-fold cover on each fiber. The obstruction class to this is $w_2(TM)$. If $\rho_{m,n}$ is the complex representation of spin(4) on $S^mV^+ \otimes S^nV^-$, then we can produce the asociated vector bundle.

Definition 9.2. Using this representation we define $S^m\Sigma^+ \otimes S^n\Sigma^- := \tilde{P} \times_{spin(4)} S^mV^+ \otimes S^nV^-$.

Lemma 9.3. *The bundle $S^m\Sigma^+ \otimes S^n\Sigma^-$ over M of complex dimension $(m+1)(n+1)$ and structure group (spin(4)) can have its structure group reduced to SO(4). Moreover there exists an SO(4)-equivariant endomorphism of order two.*
This statement is true iff $m + n$ is even.

Proof. See previously. □

Corollary 9.4. *By last week's calculation we get, that $s, W^\pm, B, B^*$ descends to real operators on the realification of the bundle.*

Homework 9.5. *Do the calculation in the $m \equiv n \equiv 1 \mod 2$ case.*

9.1 Theorems of Atiyah-Hitchin-Singer

Fix (M, g), an oriented riemannian 4-manifold. The curvature tensor of the Levi-Civita connection $R : \nabla : C^\infty(M, \Lambda^2T^*M) \to C^\infty(M, \Lambda^2T^*M)$. Consider the induced connection $\nabla : C^\infty(M, \Lambda^2T^*M) \to C^\infty(M, \Lambda^2T^*M \otimes T^*M)$. We saw already, that this splits as $\nabla^\pm : C^\infty(M, \Lambda^\pm M) \to C^\infty(M, \Lambda^\pm M \otimes T^*M)$ since $\mathfrak{so}(4) = \mathfrak{so}(3)^+ \oplus \mathfrak{so}(3)^-$.

Remark 9.6. Over some open set $\nabla^\pm = d + A^\pm = d + p^pmA$, where $p^\pm : \mathfrak{so}(4) \to \mathfrak{so}(3)^\pm$.

Take their curvatures $F_{\nabla^\pm} \in \Omega^2(M, End(\Lambda^\pm M)) = \Omega^2(M, \mathfrak{so}(3)^\pm) = \Omega^2(M, \Lambda^\pm) = C^\infty(M, \Lambda^2 \otimes \Lambda^\pm) = C^\infty(M, (\Lambda^+ \oplus \Lambda^-) \otimes \Lambda^\pm)$. We take the self-dual and anti self-dual parts of these forms, as sections of the bundles $C^\infty(M, \Lambda^+ \otimes \Lambda^\pm)$ and $C^\infty(M, \Lambda^- \otimes \Lambda^\pm)$. Denote the self-dual and anti self-dual parts by a superscript $+$ and $-$ respectively.
It is easy to read off, which component of the Singer-Thorpe theorem these new splittings correspond to, i.e. the self-dual part of F_{∇^+} will be $s/12 + W^+$.

Remark 9.7. 1. if $B = 0$, then F_{∇^+} is self-dual and F_{∇^-} is anti self-dual

2. if $W^- = s = 0$, then F_{∇^-} is self-dual

3. if $W^+ = s = 0$, then F_{∇^+} is anti self-dual

Theorem 9.8 (Atiyah-Hitchin-Singer, '78)**.** *Let (M, g) be an oriented Riemannian 4-manifold*

1. *if M, g is Einstein ($B = 0$), then the induced $SO(3)^+$ connection on Λ^+T^*M (or the $SU(2)^+$ connection on Σ^+, if M is spin) is self-dual; moreover the $SO(3)^-$ connection on Λ^-T^*M (or the $SU(2)^-$ connection on Σ^-, if M is spin) is anti self-dual*

2. *if M, g is half-conformally flat ($W^- = 0$) and $s = 0$, then the induced $SO(3)^-$ connection on Λ^-T^*M (or the $SU(2)^-$ connection on Σ^- if M is spin) is self-dual*

3. *if M, g is half-conformally anti-flat ($W^+ = 0$) and $s = 0$, then the induced $SO(3)^+$ connection on Λ^+T^*M (or the $SU(2)^+$ connection on Σ^+ if M is spin) is anti self-dual*

Proof. The previous discussion. □

Remark 9.9. ∇ is an SO(4) connection on TM. We complexify it to an SO(4) connection on $TM \otimes \mathbb{C} = \Sigma^+ \otimes \Sigma^-$. These latter two bundles only exist locally, the spin condition guarantees them to exist as global complex vector bundles, this is where we can split the connection in the parenthesis part of the theorem.

Definition 9.10. Let M, g be an oriented Riemannian 4-manifold, and let E be a rank 2 SU(2) bundle, and ∇ an SU(2) connection. We call ∇ (anti) self-dual, if the corresponding curvature F_∇ is (anti) self-dual.

Remark 9.11. What kind of equation is this for the connection? It is enough to understand this locally. $\nabla|_U = d + A^U$, and $F_\nabla|_U = dA^U + A^U \wedge A^U$ in local coordinates. One has to write $A^U = A_i dx^i$, where $A_i : U \to \mathfrak{su}(2)$ for $i = 1..4$. $dA^U = (\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}) dx^j \wedge dx^i$.

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j] = \pm\sqrt{\det g}\left(\frac{\partial A_{4-i}}{\partial x^{4-j}} - \frac{\partial A_{4-j}}{\partial x^{4-i}}\right)$$

Example 9.12. Over flat $\mathbb{R}^4$ the metric determinant goes away, and the Hodge operator acts in the way we expect.

Now we give a method to solve these equations: conformal scaling.

Lemma 9.13. *M, g as before. If $f : M \to \mathbb{R}$ is a nonwhere zero function, then $*_{f^2g} = *_g$.*

Proof. $\omega = \omega_{ij} dx^i \wedge dx^j$, then $(*\omega)^{ij} = \frac{1}{2}\sqrt{\det g} g^{ik} g^{jl} \epsilon_{klmn} \omega_{mn}$, where $\epsilon_{ijkl} = 1$ if $ijkl$ are in even permutation, -1 if its odd, and 0 if there are repeating indicies.
Now after the scaling $\det(f^2 g) = f^8 g$, and $(f^2 g)^i k = f^{-2} g^{ik}$, and we get the claim. □

Now consider the last two cases of the AHS theorem, take a conform scaling of it, the equation stays the same, since the Hodge operator is conform invariant. Since the Weyl tensor is also conform invariant, only the scalar curature can change, but that we understand.

Lemma 9.14. *$\tilde{g} = f^2 g$, then $f^3 s_{\tilde{g}} = f s_g - 6\Delta_g f$*

Corollary 9.15. *If we do the scaling with a harmonic function, we stay in the case 2 or 3 from which we started.*

This gives a method to produce self-dual solutions over M, g from harmonic functions $\nabla_g f = 0$.

10 Tenth lecture

10.1 The 1-instanton moduli space over $\mathbb{R}^4$

Think of $\mathbb{R}^4$ as the quaternions $\mathbb{H}$, and $su(2)$ will be thought of as the imaginary quaternions, where the bracket is $[x, y] \mapsto im(xy)$. The identifiaction is given by $\begin{bmatrix} ix & z \\ -\bar{z} & -iz \end{bmatrix} \mapsto xi + zj$.

Homework 10.1. *$tr(xy^*) \mapsto -2Re(x\bar{y})$ under this isomorphism between $su(2)$ and $\mathbb{H}$.*

Let $E = \mathbb{R}^4 \times \mathbb{C}^2$ the trivial $SU(2)$ bundle.

Lemma 10.2. *Define the connection $\nabla := d + A : \Omega^0(\mathbb{R}^4, E) \to \Omega^1(\mathbb{R}^4, E)$, where $A = im(\frac{xd\bar{x}}{1+|x|^2})$. We claim, that ∇ is self-dual, and $||F_\nabla||^2 := -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} tr(F_\nabla \wedge *F_\nabla) = 1$. Similarly, for $b \in \mathbb{H}$ and $\lambda \in \mathbb{R}_+$, then $\nabla_{b,\lambda} := d + A_{b,\lambda}$, where $A_{b,\lambda} := im(\frac{(x-b)d\bar{x}}{\frac{1}{\lambda^2}+|x-b|^2})$ is a 5-parameter sequence of self-dual connections, where the energy (pairwise gauge-inequivalent).*

Proof. $A(x) = A_j(x)dx^j = \frac{x^1dx^0 - x^0dx^1 + x^3dx^2 - x^2dx^3}{1+|x|^2} i + \frac{x^2dx^0 - x^3dx^1 - x^0dx^2 + x^1dx^3}{1+|x|^2} j + \frac{x^3dx^0 + x^2dx^1 - x^1dx^2 - x^0dx^3}{1+|x|^2} j$
Denote the coefficients $\frac{1}{1+|x|^2}(\theta_i i + \theta_j j + \theta_k k) = A(x)$. Compute curvature:

$$dA + A \wedge A = im(d(\frac{x}{1+|x|^2})) \wedge d\bar{x} + \frac{x}{1+|x|^2} d\bar{x} \wedge \frac{x}{1+|x|^2} d\bar{x} =$$

$$= im(\frac{(1+|x|^2)dx - x(d\bar{x}x + \bar{x}dx)}{(1+|x|^2)^2}) \wedge d\bar{x} + \frac{xd\bar{x} \wedge xd\bar{x}}{(1+|x|^2)^2} = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}.$$

Note, that $* : \Omega^0 \to \Omega^2$ acts in a simple way. $dx^0 \wedge dx^1 \mapsto dx^2 \wedge dx^3$, the sign can be computed simply from the definition $\alpha \wedge *\beta = <\alpha, \beta> vol$, in the other cases $dx^0 \wedge dx^2 \mapsto -dx^1 \wedge dx^3$, $dx^0 \wedge dx^3 \mapsto dx^1 \wedge dx^2$. This gives us a basis for Ω^+, namely $dx^0 \wedge dx^1 + dx^2 \wedge dx^3, dx^0 \wedge dx^3 - dx^1 \wedge dx^3, dx^0 \wedge dx^3 + dx^1 + dx^2$. We can compute $dx \wedge d\bar{x} = -2(dx^0 \wedge dx^1 + dx^2 \wedge dx^3)i - 2(dx^0 \wedge dx^1 - dx^2 \wedge dx^3)j - 2(dx^0 \wedge dx^3 + dx^1 \wedge dx^2)k$.
Now we can compute $tr(F_\nabla \wedge *F_\nabla) = tr(F_\nabla \wedge F_\nabla) = \frac{1}{(1+|x|^2)^4} 2Re(\dots)vol$. A simple check shows, that a constant of -48 appears after taking the trace. Finally the "instanton charge" $\frac{48}{8\pi^2} \int \frac{dvol}{(1+|x|^2)^4} = \frac{48}{8\pi^2} 2\pi^2 \int_0^\infty \frac{r^3}{(1+r^2)^4} dr = \frac{1}{4} 48 \frac{1}{12} = 1$ as stated.
The 5-parameter family comes from the fact, that $\mathbb{R}^4$ is inariant under coordinate change of the form $x \mapsto x - b$, and we noted, that that the self-duality equation is invariant under conformal coordinate change, so we produce $x \mapsto \lambda(x - b)$, this produces $A_{b,\lambda}$ as $im \frac{\lambda(x-b)d(\overline{\lambda(x-b)})}{1+|\lambda(x-b)|^2} = \frac{\lambda^2(x-b)^2 d\bar{x}}{1+\lambda^2|x-b|^2} = im \frac{(x-b)d\bar{x}}{\frac{1}{\lambda^2}+|x-b|^2}$ will also be self-dual, energy 1 connections. The only thing in need of checking, is that the integral stays equal to 1 after scaling by λ. The curvature can be calculated similarly to be $\frac{\frac{1}{\lambda^2} dx \wedge d\bar{x}}{(1/\lambda^2 + |x-b|^2)^2}$.
Inequivalence is seen from the curvature. It transforms by simple conjugation, and we see, that two matrices of the form calculated previously cannot be conjugated into each other by $SU(2)$ elements. □

Remark 10.3. As $\lambda \to \infty$, the metric $|F_{b,\lambda}|^2 dvol$ converges weakly to $\frac{d\bar{x}}{x-b}$, which is δ_b.
If $\lambda \to 0$, then $|F_{b,\lambda}|^2 dvol$ approaches a metric called the "centerless instanton". Substitute x with q/λ. The connection form approaches $Im \frac{qd\bar{q}}{1+|q|^2}$.

Remark 10.4. The $\nabla_{b,\lambda}$ connections can be obtained by conform scaling (Jackiw-Rebbi method) with the choice $f_{b,\lambda} := 1 + \frac{1/\lambda}{|x-b|^2}$ or $\lambda + \frac{1}{|x-b|^2}$, so the centerless instanton is obtained from the scaling $f_b(x) = \frac{1}{|x-b|^2}$.

Homework 10.5. *Prove, that* $\exists g : \mathbb{R}^4 \to SU(2)$ *such that* $gA_{b,0}g^{-1} + gdg^1 = A_{0,0}$. *(This is the reason it is called "centerless").*

Lemma 10.6. *Let* $p : S \setminus \{\infty\} \to \mathbb{H}$ *be the stereographic projection,* $\nabla_{b,\lambda}$ *as before. Consider* $p^*\nabla_{b,\lambda} : \Omega^0(S^4 \setminus \{\infty\}, p^*E) \to \Omega^1(\dots)$*, the pullback connection. Then ther exists a gauge transformation from a neighborhood of infinity to* $SU(2)$ *such that* $\nabla'_{b,\lambda} = gp^*\nabla_{n,\lambda}g^{-1}$ *extends smoothly to a connection on an* $SU(2)$ *bundle* $E' \to S^4$*, whose second Chern class* $c_2(E') = 1$*. Moreover* $\nabla'_{b,\lambda}$ *will be a self-dual connection according to the e.g.* $R = 1$ *standard metric of* S^4*.*

Proof. Uhlenbeck singularity removal theorem, in this special casewe should be able to compute it explicitly. □

Homework 10.7. $\frac{d}{dt}A^{-1} = -A^{-1}A'A^{-1}$.

Homework 10.8. $f : \mathbb{C}^* \to \mathbb{C}$ *a holomorphic function, and its local* L^2 *norm is finite, then* f *extends to* $\mathbb{C}$.

Consider the spherical S^4, and the rank 2 $SU(2)$ vector bundle over it with second Chern class 1. We found $\nabla_{b,\lambda}$, a 5-parameter family of connections, where $||F_{\nabla_{b,\lambda}}||^2 = C_2(E) = 1$. Accepting the fact, that there are no other "instantons", we just found a parametrisation $\mathcal{M}_{S^4}(1) = B^5$ as the compactification of $S^4 \times \mathbb{R}_+$ by gluing together the (b, ∞) connections.
Another example is $\overline{\mathbb{C}P^2}$ with the Fubini-Study metric.

Lemma 10.9. *Let E be the SU(2) vector bundle over* $\overline{\mathbb{C}P^2}$ *with* $c_2(E) = 1$*. Then there exists a 1-parameter family of energy 1 self-dual connections* $\nabla_{b,t}$*, where* $A_t = \frac{1}{1+|x|^2-t^2}(\theta_i i + t\theta_j j + t\theta_k k)$ *with* $t \in [0, 1]$*.*

Proof. We don't prove this. □

Remark 10.10. As $t \to 0$, we get $\frac{1}{1+|x|^2}\theta_i i$, which is an $U(1) \subset SU(2)$ self-dual connection, which will turn out to be reducible. The moduli space will

11 Eleventh lecture

Lemma 11.1. *Consider* $\mathbb{R}^4$ *with the standard Euclidean metric, and* $E = \mathbb{R}^4 \times \mathbb{C}^2$ *the trivial* $SU(2)$ *bundle and* $\nabla_{b,\lambda}$ *a self-dual connection on it. Then there exists a smooth extension* $\tilde{\nabla}_{b,\lambda}$ *on the* $SU(2)$ *bundle over* S^4 *with* $C_2(\tilde{E}) = 1$*, where* $\tilde{\nabla}_{n,\lambda}$ *is self-dual with respect to the spherical radius* R *metric of* S^4*.*

Proof. We saw last week, that $1 = ||F_{b,\lambda}||$ with the L^2 norm on $\mathbb{R}^4$, we calculate the integral $\frac{1}{8\pi^2}\int_{\mathbb{R}^4} |F_{b_\lambda}|dx = -\frac{1}{8\pi^2}\int tr(F_{b,\lambda} \wedge *F_{b,\lambda})$. We know that $*$ is conform invariant, identifying $\mathbb{R}^4$ with $S^4 \setminus \{\infty\}$ we see $-\frac{1}{8\pi^2}\int tr(F_{b,\lambda} \wedge *F_{b,\lambda})$. By the Lebesgue theorem this is the same,as if we would integrate over the whole S^4. Self duality stays intact, we can omit the $*$, and we see the Chern-Simons formula for $C_2(\tilde{E})$. So $\nabla_{b,\lambda}$ can only extend over the $C_2 = 1$ SU(2) bundle, if anywhere.
Now we do the extension. $S^4 \setminus \{\infty\} = \mathbb{H}$, and we introduce a new coordinate, $1/y = x - b$. Moreover $S^4 = H_+ \cup H_-$ and $H_+ \cap H_- = S^3 \subset H$, and we identify this S^3 with $SU(2)$. We take the gauge transformation $g(y) = y$ to glue together this bundle and compute how $A_{n,\lambda}$ transforms. $g(y)^{-1} = \bar{y}$, and computing $yIm\frac{1/yd(1\bar{/}y)}{\lambda^2+1/|y|^2}$ we see, that it is smooth as $y \to 0$, thus we have achieved the extension. □

Denote by $\mathcal{M}_{S^4,g_r}(1)$ the moduli space of finite energy self dual connections over the radius R sphere and the bundle with secondChern number 1. We know this to be an open B^5, which we can compactify by "ideal" instantons, which will become a closed 5-ball. Notice that its boundary is the original S^4.

11.1 Moduli space over CP2 with the Fubini-Study metric

11.2 Yang-Mills theory and YM-type classical field theories

Definition 11.2. Let M, g be a closed Riemannian 4-manifold, called "spacetime". Let G be a compact Lie group (SU(2)) (called the 'Gauge' group, note, that this is not $\mathcal{G}_E$).

12 Twelfth lecture

From now on M will denote a simply connected closed oriented manifold.

Definition 12.1. Let (M, g)be an oriented riemanian 4-manifold, E an $SU(2)$ bundle, where $c_2(E) = k \in \mathbb{Z}$ as arbitrary. ∇ is an $SU(2)$ connection on E, $S(\nabla) = \frac{-1}{8\pi^2} \int_M tr(F_\nabla \wedge *F_\nabla)$ the Yang-Mills functional.

Remark 12.2. So S is an $\mathcal{A}_E \to \mathbb{R}_+$ non-linear map, it is also clear that it is gauge-invariant. This implies, that it descends to $\mathcal{B}_E$.

12.1 Variation and YM equations

Theorem 12.3. *On a given E bundle the Euler-Lagrange equation associated to the YM function is of the form $d^*_\nabla F_\nabla = 0$.*

Definition 12.4. The $d_\nabla F_\nabla = 0, d^*_\nabla F_\nabla = 0$ system of equations is called the vacuum YM equations. A connection $\nabla \in \mathcal{A}_E$, which solves these equations is called a YM-field. Note that the first equation is just the differential Bianchi identity.

Proof. If $\nabla, \nabla' \in \mathcal{A}_E$ then we know that $\exists a \in \Omega^1(M, End\ E)$ such that $\nabla' = \nabla + a$. This identifies $T_\nabla \mathcal{A}_E$ with $\Omega^1(M, End\ E)$, $\nabla + ta$ is a one paramter subgroup, representing a tangent vector. We also saw that $F_{\nabla+ta} = F_\nabla + td_\nabla a + t^2 a \wedge a$.
The YM functional gives an L^2 inner product, we denote $S(\nabla) = (F_\nabla, F_\nabla)$. Now we compute the variation $S(\nabla+ta) = (F_{\nabla+ta} = F_\nabla + td_\nabla a + t^2 a\wedge a, F_{\nabla+ta} = F_\nabla + td_\nabla a + t^2 a \wedge a) = (F_\nabla, F_\nabla) + 2t(F_\nabla, d_\nabla a) + O(t^2)$. This implies, that $\frac{d}{dt} S(\nabla + ta) = 2(F_\nabla, d_\nabla a)$. This has to vanish for all a, using the formal adjoint d^*_∇ we get $2(d^*_\nabla F_\nabla, a) = 0$, this happens if and only if $d^*_\nabla F_\nabla = 0$. □

Definition 12.5. $d^*_\nabla : \Omega^2(M, End\ E) \to \Omega^1(M, End\ E)$ is the formal L_2 adjoint of d_∇.

Lemma 12.6. *If $d^*_\nabla : \Omega^p(M^m, End\ E) \to \Omega^{p-1}(M^m, End\ E)$, then $d^*_\nabla = (-1)^{1+p(1-m)+p^2} * d_\nabla *$.*

Proof. Let $\omega, \eta \in \Omega^p(M^m, End\ E)$. By the closedness of M we get

$$0 = \int_M dtr(\omega \wedge *\eta) = \int_M tr(d_\nabla \omega \wedge *\eta) + (-1)^p \int_M tr(\omega \wedge d_\nabla * \eta) =$$

$$\int_M tr(d_\nabla \wedge *\eta) + (-1)^{p-(m-p)p} \int_M tr(\omega \wedge * * d_\nabla * \eta)$$

And thus $(d_\nabla \omega, \eta) = (-1)^{1+p-(m-p)p}(\omega, *d_\nabla * \eta)$, this means that $d^*_\nabla = (\pm 1) * d_\nabla *$. □

Corollary 12.7. *If ∇ is (anti-) self dual, i.e. $*F_\nabla = \pm F_\nabla$, then it solves the YM equations.*

Proof. $d_\nabla F_\nabla$ is always true, and by the lemma $d^*_\nabla F_\nabla = \pm * d_\nabla * F_\nabla = \pm 1 * d_\nabla F_\nabla = 0$. □

Lemma 12.8. *Let E be an $SU(2)$ bundle with second Chern-class $C_2(E) = k$. Then $S(\nabla) = ||F_\nabla||^2 \geq |C_2(E)|$, and an (anti-) self dual connection is a global minimum of the YM functional. Thus such a connection attains the minimum.*

Proof. We claim, that if $\alpha \in \Omega^+(M)$, and $\beta \in \Omega^-(M)$, then we can compute their pointwise scalar product induced by the metric

$$(\alpha, \beta) = -tr(\alpha \wedge *\beta) = -tr(*\beta \wedge \alpha) = -tr(*\beta \wedge **\alpha) = (*\beta, *\alpha)$$

So $*$ is an orthogonal operator at every $x \in M$. Now $(\alpha, \beta) = (*\alpha, *\beta) = -(\alpha, \beta) = 0$, thus $\Omega^+ \perp \Omega^-$.

$$||F_\nabla||^2 = \frac{-1}{8\pi^2} \int_M |F_\nabla|^2 * 1 = \frac{-1}{8\pi^2} \int_M (|F_\nabla^+|^2 + |F_\nabla^-|^2) * 1 \geq \frac{-1}{8\pi^2} \int |F_\nabla^\pm|^2 * 1$$

If ∇ is (anti) self dual, then this is equal to

$$= \frac{\mp 1}{8\pi^2} \int_M tr(F_\nabla \wedge F_\nabla) = |C_2(E)|$$

□

Remark 12.9. The Levi-Civita connection on $SO(4)$ solves the YM equations, but is not a minimum.

Definition 12.10. ∇ is called an (anti) instanton, if F_∇ is (anti) self dual, and $S(\nabla) < \infty$.

Remark 12.11. Over a compact manifold being self dual is the same as being an instanton. A change of orientation exchanges instantons and anti-instantons.

12.2 The structure of the instanton moduli-space and reducible connections

From now on we fix an orientation, and consider only instantons.

Lemma 12.12. *Let E be an SU(2) bundle over M, and $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ a non-flat connection, and $\nabla^{End\ E} : \Omega^0(M, End\ E) \to \Omega^1(M, End\ E)$ the induced connection on the associated lie-algebra bundle. TTFAE:*

1. *The factor of the stabiliser by the center $G_\nabla/\mathbb{Z}_2 = U(1) \subset SU(2)$*
2. *$\nabla^{End\ E}$ has nontrivial kernel*
3. *There is an U(1) bundle L with $E = L \oplus L^{-1}$ and $\nabla = \nabla_L \oplus \nabla_{L^{-1}}$*
4. *$G_\nabla/\mathbb{Z}_2 \neq 1$*

Proof. $1 \to 2$:There is $g_t \in \mathcal{G}_E$ such that $g_t \nabla g_t^{-1} = \nabla$, where $g_t(x) = diag(e^{it\phi(x)}, e^{-it\phi(x)})$, denote $g_0'(x) = u(x) = diag(i\phi(x), -i\phi(x))$. Take the derivative w.r.t. t to see that $[u, \nabla] = 0$. This means, that $\nabla^{End\ E} u = 0$ and $u \neq 0$, since by definition $End\ E \subset E \otimes E^*$, in the E component we take derivative with ∇, and in the E^* component we get a sign flip by pullback.
$2 \to 3$ Choose $u \neq 0 \in ker \nabla^{End\ E}$. Since $u(x) \in \mathfrak{su}(2) \forall x \in M$, we get that if $u(x) \neq 0$, then its nonzero in an open neighborhood of $x \in U$ where u is nonzero, then there is a $\lambda : U \to \mathbb{R}$ whith $u(x) = diag(i\lambda(x), -i\lambda(x))$. There also exists a local section $e \in \Gamma(U, E)$ with $ue = i\lambda e$, moreover it can be chosen so $(e, e) = 1$. Take the

covariant derivative $u\nabla e = id\lambda e + i\lambda\nabla e$ and $2Re(\nabla e, e) = 0$. Imaginary part of the pointwise scalar product of the first equation with e gives $d\lambda = im(u\nabla_e, e) = -im(\nabla e, ue) = \lambda Re(\nabla e, e) = 0$ from the previous relation. This means that λ is constant, this e extends to a global section, and splits the bundle. We want to show, that $\nabla^E e_\pm|_U = 0$. The section $u|_U = i\lambda e_+ \otimes e_+^* \oplus (-\lambda i)e_- \otimes e_-^*$ by the spectral theorem. Applying $\nabla^{End\ E}u = 0$ gives us the previous equation. If $\nabla^E e = 0$, then $Ae \sim de \sim e$, and the connection matrix splits.
$3 \to 4$: by the splitting of the connection $G_\nabla/\mathbb{Z}_2 \supseteq U(1)$ by fixing the two components, so its nontrivial.
$4 \to 1$: requires the holonomy group. This is where we use the non-flatness condition. Note, that $G_\nabla = C_{SU(2)}(Hol(\nabla))$ the centralizer. If this is bigger than $U(1)$, then it can only be a discrete subgroup, which is ruled out by the flatness assumption. □

Definition 12.13. We call a connection satisfying any of the 4 properties a non-flat reducible connection.

Corollary 12.14. *In the $\mathcal{B}_E$ orbit space the (equivalence classes of) reducible connections are singular points, understanding them is paramount.*

Notice, that if ∇ is flat on E, then $E = M \times \mathbb{C}^2$ and $[\nabla]$ is unique.

Proof. $F_\nabla = 0$, thus $C_2(E) = 0$, thus its the trivial bundle. Moreover by holonomy theory once again, flat connections are in bijection with conjugacy classes of representations of $\pi_1(M)$ to $SU(2)$, so by the simple connectivity assumption every flat connection is gauge-equivalent with the $\nabla_0 = d$ trivial connection. □

∇ being flat is equivalent with $G_\nabla = SU(2)$.

Lemma 12.15. *Let M be a simply connected closed 4-manifold with $b^- = 0$, L an arbitrary $U(1)$ line bundle over M. Then for every Riemannian metric g $\exists!\nabla_L$ self-dual connetion on M.*

Proof. Since $b^- = 0$, we know that $H^2 = H^+$, which means that $H^2(M, \mathbb{Z}) \subset \mathcal{H}^+$. By the Hodge decomposition, and Chern Weyl theory we get that if L is a U(1) bundle over M, then there is a unique cohomology class $[\omega] = c_1(L)$ classifying it, and ω can be taken to be self-dual. Since ω is closed, by the Poincaré lemma $\omega|_U = dA^U$ shows us, that there is a ∇_L U(1) connection such that $F_\nabla = \omega$, since $F_\nabla = dA^U$ in the abelian case (which U(1) is). By the choice of ω this connection is self-dual, there is an instanton on L w.r.t. M, g. For uniqueness let ∇'_L another self-dual connection with curvature F'_∇. Since $c_1(L) = [F_{\nabla_L}] = [F_{\nabla'_L}]$, there is $a \in \Omega^1(M)$ with $F_{\nabla'_L} = F_{\nabla_L} + da$. a is not unique, $a' = a + df$ also suffices for our purposes. We use this freedom to achieve $d^*a = 0$. Thus we want to solve the equation $d^*a' = d^*a + d^*df = d^*a + (d^*d + dd^*)f = d^*a + \triangle_0 f$. Is there an f with $-d^*a = \triangle_0 f$? This is a second order linear elliptic PDE for f over a closed M. This is posible if and only if $-d^*a$ is L^2 orthogonal to $ker\triangle_0$. By the maximum principle $ker\triangle_0$ consists of the constant functions, so we need to show, that $-c\int_M d^*a = 0$, which is clear, so we can suppose $d^*a = 0$.
Applying d^* to the equation $0 = d^*F_{\nabla'_L} = d^*F_{\nabla_L} + d^*da$, so $0 = d^*da + dd^*a = \triangle_1 a$, so $a \in ker\triangle_1 = H^1(M) = 0$ by the Hurewicz theorem, and Hodge decomposition, so $a = 0$. □

Lemma 12.16. *If $b^- > 0$, then for a generic metric a line bundle with nonzero c_1 has no self-dual connection.*

Proof. Trivial, since if $b^- > 0$,then $H^+ \subset H^2(M, \mathbb{Z})$ has nonzero codimension. For a generic choice of metric the subspace H^+ avoids the lattice $H^2(M, \mathbb{Z})$. □

Corollary 12.17. *On an E bundle the gauge equivalence classes of reducible connections are in bijection with $\alpha \in H^2(M) : -\alpha^2 = c_2(E)$.*

Remark 12.18. $1 + c_2(E) = c(E) = c(L \oplus L^{-1}) = 1 + c_1(L) + c_1(L^{-1}) - c_1(L)^2$

13 Thirteenth lecture

Beginning remarks. $\alpha \in [\alpha] \in H^2_{dR}(M)$, we wish to compute $q_M(\alpha, \alpha) = \int_M \alpha \wedge \alpha = (\alpha, *\alpha)_{L^2}$. For a self-dual form this is $\pm(\alpha, **\alpha) = \pm||\alpha||^2_{L^2}$. This means, that the definite part of the intersection form coincides with the self-dual and anti-self dual forms.

Remark 13.1. M closed simply connected 4-manifold with $b_- = 0$, $L \to M$ a line bundle, then $\exists!$ anti self-dual $U(1)$ connection on it. We give a different proof without relying on PDE theory.

Proof. Existence is the same, let ∇, ∇' be self dual connections on L, $[F_\nabla] = [F_{\nabla'}] \in H^2$ by Chern-Weyl theory. This implies the existence of $a \in \Omega^1(M, L)$ such that $F_{\nabla'} = F_\nabla + da$. We need to show, that there is a map $f : M \to U(1)$ such that $A'^U = fA^U f^{-1} + fdf^{-1} = A^u - dlog(f)$ since $U(1)$ is abelian. This exists by simple connectivity. This shows gauge equivalence of the two connections. □

13.1 The structure theorem

Let M be a closed simply connected 4-manifold, g a Riemannian metric and $E \to M$ an SU(2) bundle with $C_2 = k$. $\mathcal{A}_E/\mathcal{G}_E = \mathcal{B}_E$ is the space of connections modulo gauge equivalence. $\nabla \in \mathcal{A}_E$ we denote its class by $[\nabla]$.

Definition 13.2. $\mathcal{M}_k(g) \subset \mathcal{B}_E$ denotes the (equivalence classes of) (M, g) self-dual SU(2) connections over E. We call this *the moduli space.*

Theorem 13.3 (Atiyah-Hitchin-Singer '78)**.** *Let M be a closed simply connected oriented 4-manifold with indefinite intersection form. Then for generic metrics $\mathcal{M}_k(g)$*

- *only consits of a single point if $k = 0$*
- *is an $8k - 3(1 + b_-)$ dimensional* smooth manifold for $k \neq 0$*

Theorem 13.4. *If M is as in the previous theorem, but positive definite, then for generic metrics $\mathcal{M}_k(g)$ is*

- *a single point if $k = 0$*
- *a smooth manifold of dimension $8k - 3$ at irreducible points, and is modelled by a cone over $\mathbb{C}P^{4k-2}$ at reducible points for $k \neq 0$*

Moreover $\forall k \in \mathbb{Z}$ there are finitely many reducible (i.e. singular) points. Their number is equal to the number of 2-cohomology classes α which satisfy $-\alpha \wedge \alpha = c_2(E)$.

In the $k = 1$ case the moduli space consits of reducible points, and by the grafting theorem of Taubes we have concentrated connections at the open end of the moduli space, our goal is to make this happen.

*in particular its empty if the dimension is negative

13.2 Quick PDE summary: Sobolyev spaces of sections.

Let M be a closed manifold with a riemannian metric. Let E be a G-bundle with G compact endowed with a G-invariant positive definite inner product $<,>$. Let ∇ a fixed arbitrary G-connection on E.

Definition 13.5. Let $s \in \Gamma(M, E)$ a section, $||s||_{C^k(M)} := \sum_0^k ||\nabla^{(n)}(s)||_{C^0(M)}$, the sum of the sup-norms of the first k ovariant derivatives of s. We also denie $||s||^p_{L^p_k(M)} := \sum_0^k ||\nabla^{(n)}(s)||^p_{L^p(M)}$ with the L^p norm defined by $(\int_M |\cdot|^p)^{1/p}$.

Homework 13.6. $L^\infty_k = C^k$.

Definition 13.7. $C^k(M, E)$ is defined to be the completion of $C^\infty(M, E)$ w.r.t. the C^k metric, and similarly we define $L^p_k(M, E)$ to be the complection w.r.t. the L^p_k norm.

Remark 13.8. These spaces are independent of the choice of g and ∇.

Theorem 13.9 (Sobolyev embedding). *For compact M, there exists a continous embedding $L^2_k(M, E) \hookrightarrow C^l(M, E)$ if $k > \dim M/2 + l$. I.e. there is a constant c such that $||s||_{C^l} \leq c||s||_{L^2_k}$ for each section s.*

Corollary 13.10. $\cap_k L^2_k = C^\infty$

Let $P : C^\infty(M, E) \to C^\infty(M, F)$ *a kth order elliptic partial differential operator* (e.g. the Laplace operator,which is a second order PDO). For example, locally it should look like $P|_U = \sum a_{ij} D_i D_j + \sum b_i D_i + c_i$, ellipticity means, that the eigenvalues of the symmetric matrix $(a_{ij}) : M \to \mathbb{R}$ has only positive eigenvalues, order means to which order are we taking derivatives.

Theorem 13.11 (Existence, unicity and regularity). *Let M, g be a closed riemannian manifold, E,F two G-bundles over M and P a linear kth order elliptic PDO (inparticular rk E = rk F). We extend this operator to some generalised weak function class. In this case $Pu = f$ has a solution iff $f \perp$ coker P. Moreover the solution is unique if ker $P = 0$. Finally if $s \in L^p_l(M, E)$, then $\forall l\ \exists c_l$ so that $||s||_{L^p_{l+k}} \leq c_l||Ps||_{L^p_l} + c||s||_{L^p_0}$, and $c = 0$ can be assumed if ker $P = 0$.*

Corollary 13.12. *If $Pu = 0$, then $u \in C^\infty(M, E)$.*

Theorem 13.13 (Sobolyev multiplication theorem). *Let M be a closed n-manifold, then there exists a continous embedding $L^{p_1}_{k_1} \otimes L^{p_2}_{k_2} \hookrightarrow L^p_k$ if $k - \frac{n}{p} \leq (k_1 - \frac{n}{p}_1)(k_2 - \frac{n}{p}_2)$.*
In other words $||fg||_{L^p_k} \leq c||f||_{L^{p_1}_{k_1}}||g||_{L^{p_2}_{k_2}}$ for some c dependent only on M, g.

Remark 13.14. If $pk > n$, then $L^p_k \subset C^0$ and the theorem is trivial, the statement is interesting if $pk < n, p_i, k_i < n$.

13.3 Uhlenbeck theorems

Theorem 13.15 (Coulomb gauge fixing theorem '82). *Let S^4, g_R be the round sphere and E_0 the trivial SU(2) bundle and ∇ an arbitrary SU(2) connection on E_0. In this case there exists $\epsilon > 0, N < \infty$ constants such that if $||F_\nabla||_{L^2} < \epsilon$, then*

1. *$\exists g \in \mathcal{G}_{E_0}$ with $\nabla' = g\nabla g^{-1}$ there is a trivialisation of E_0,where $\nabla' = d + A'$ there $d^*A' = 0$, called the Coulomb gauge*

2. $||A'||_{L_1^2} < N||F_{\nabla'}||_{L^2}$

Proof. We are unable to prove this now. □

Remark 13.16. $||A'||_{L_1^2} := ||A||_{L^2} + ||dA||_{L^2}$ on E_0 by choosing the trivial connection.

Theorem 13.17 (Singularity removal global form)**.** *Let M be closed 4-manifold, $E^\times, \nabla$ as before over $M \setminus \{*\} = M^\times$ with ∇ self-dual and $||F_\nabla||_{L^2} < \infty$. We claim that there exists $g \in \mathcal{G}_{E^\times}$ such that the transformed ∇' extends smoothly to some $E \to M$ SU(2) bundle as an SU(2) connection.*

Proof. We again lack the resources for a proof, we have to believe. □

Remark 13.18. $||F_{\nabla'}||_{L^2}$ gives the type of E.

13.4 Ideal connection, weak and strong convergence

Definition 13.19. M as before and let $E \to M$ an SU(2) bundle with second Chern class k. A $(\nabla_B, \{x_1, \ldots, x_l\})$ tuple is an *ideal connection* if

- $0 < l \leq k$
- $[\nabla_B] \in \mathcal{M}_{k-l}(g)$
- $x_1, \ldots, x_l \in M$ are not necessarily different points

Definition 13.20. The curvature density of a $[\nabla_B], \{x_1, \ldots, x_l\}$ ideal connection is $\frac{1}{8\pi^2}|F_B|^2 + \sum_1^l \delta_{x_j}$ where $F_B := F_{\nabla_B}$ and δ_y is the $y \in M$ Dirac measure.

Remark 13.21. The energy of an ideal connection is defined as $\frac{1}{8\pi^2}\int_M |F_B|^2 + \int_M \sum_1^l \delta_{x_j} = k$.

Definition 13.22. Let $\nabla_{A_n} \in \mathcal{M}_k(g)$ instantons for all n on E with second Chern class k, as before. We call this sequence weakly convergent to a $(\nabla_B, \{x_i\}_1^l)$ ideal connection if

- for every integrable function $f : M \to \mathbb{R}$ we have $\int_M f|F_{A_n}|^2 \xrightarrow{n\to\infty} \int f|F_B|^2 + \sum f(x_j)$
- for every $K \subset M \setminus \cup\{x_j\}$ compact subset $[\nabla_{A_n}|_K] \to [\nabla_B|_K]$, i.e. for every n, there is a $g_n \in \mathcal{G}_{E|_{M\setminus\cup\{x_j\}}}$ such that $B|_K - A'_n|_K \in \Omega^1(K, End\ E)$ satisfies $||B - A'_n||_{C^k(K)} \to 0$ $(n \to \infty)$ for every k.

Remark 13.23. For $l = 0$ the ideal connection is just a normal connection, and the convergence notion is the strong convergence.

14 Fourteenth lecture

Corrections.

Proposition 14.1. *M simply connected closed 4-manifold, $L \to M$ complex line bundle, then there exists a unique self-dual instanton over it.*

Proof. We saw, that such a ∇ exists. If ∇' is another such connection, then their curvatures are equal. We also need, that there is a $g : M \to U(1)$ such that $\nabla' = g\nabla g^{-1}$. Since $F_\nabla = F_{\nabla'}$, over some U subset they have the form dA^U and $dA^{U'}$. This means, that $A^U - A^{U'}$ is closed. Also $\nabla' = \nabla + a$ for some $a \in \Omega^1(M)$. So $a|_U = A^{U'} - A^U$ and a is closed as well, and by simple connectivity its cohomology clas has to be zero, thus $a = df$ for some $f : M \to \mathfrak{u}(1)$. Now we can define $g := e^{if}$. □

14.1 Uhlenbeck weak convergence theorem

Proposition 14.2. *Over the trivial $\mathbb{C}^2$ bundle over S^4 let ∇_i be a weakly self-dual sequence of connections, where weakly self-dual means, that for every L^1 function f we have $\int_{S^4} |F_i^-|^2 f = 0$, and where $||F_i|| < \epsilon$, where ϵ comes from the Uhlenbeck gauge fixing constant, then there is a subsequence ∇_{i_k} and g_i gauge transformations such that the transformed conections ∇'_{i_k} converge in the C^∞ topology to a ∇ self-dual connection.*

Proof. For $k = 1$ the Coulomb gauge theorem gives us that for $0 < \epsilon$ we have ∇' with $d^*A' = 0$ and a constant N such that $||F_{A'}|| \leq N\epsilon$.
For $k = 2$ if ∇ is self dual, take $(dA + A \wedge A)^- = F_A^- = 0$ at least weakly.
Note, that $\sqrt{\Delta} = \pm(d + d^*)$,easy to see by squaring. Secondly, since M is closed then $kerd + d^* = ker\Delta = H^k(M)$ $0 = (\Delta\phi, \phi) = ||d\phi||^2 + ||d^*\phi||^2$, implying the statement. So if the manifold has no homology in some degree, then there the Dirac operator has no kernel.
Thirdly if $dimM = 4$, then $d + d^* : \Omega^1 \to \Omega^2 \oplus \Omega^0$ is elliptic.
Apply elliptic regularity to $||A'||_{L^2_2} \leq c_1||(d+d^*)^- A'||_{L^2_1} + c_2||A'||_{L^2}$. Since $kerd+d^* = ker\Delta_1 = H^1(S^4) = 0$, we can choose $c_2 = 0$. From the Coulomb gauge theorem we can omit the d^* operator, and $dA^- \sim (A \wedge A)^-$ so in the end we get the upper bound $c_2||A'||^2_{L^4_1}$ by the multiplication theorem since $1 - 4/2 \leq 0$. Finally we can return to the L^2_1 norm by the Hölder-Minkowski inequality: if M is a compact manifold, we write $||f||^p_{L^p} = \int_m |f|^p = \int_M (|f|^q)^{p/q} \cdot 1^{1-q/p} \leq (\int_M |f|^q)^? (\int_M 1)^?$.
For $k \geq 3$ we play the same game, $||A'||_{L^2_k}$ will be bounded above by $c||A' \wedge A'||_{L^2_k} \leq c||A'||^2_{L^2_k}$.
Finally we apply the Arzela-Ascoli theorem to get a convergent subsequence.

□

Remark 14.3. The Uhlenbeck Coulomb gauge theorem and the above weak compactness theorem was stated for small curvature self-dual connections over S^4, but using bump functions they can be extended to similar theorems over $B^4 \subset S^4$. This can be interpreted to mean that these are 'local' forms of the theorems, now we state the global forms, valid for manifolds.

Theorem 14.4. *Let M be a closed simply connected manifold, $E \to M$ an SU(2) bundle, ∇_i a sequence of self dual connections over E. Suppose, that there are points $x_1, \ldots, x_l \in M$ such that for each $x \neq x_i$ in M has a neighborhood U_x disjoint from the x_i such that $||F_{\nabla_i}||_{L^2} < \epsilon$, where ϵ is from the Coulomb gauge theorem.*
Then there is a subsequence i_k where ∇_{i_k} converges over the compact subsets of $M \setminus (\cup\{x_i\})$ in the C^∞ topology to a self-dual connection ∇ over $M \setminus (\cup\{x_i\})$.

For the proof, we need that M decomposes as a union of B^4's, and we patch together the previous local theorems.

Theorem 14.5 (Uhlenbeck weak compactness '82)**.** *Let M, E as before, $k = c_2(E)[M]$ and $[\nabla_i] \in \mathcal{M}_k(g)$ an instanton sequence. Then there is a $[\nabla_j]$ subsequence which weakly converges to a $\nabla, x_1, ..., x_l$ ideal connection.*

Remark 14.6. Here ∇ is representing an element of $\mathcal{M}_{k-l}(g)$. In particular it is defined on the whole of M.

Proof. Consider the measure induced by the curvature of ∇_i, $\mu_i := \frac{1}{8\pi^2}|F_i|^2$. By boundedness of this measure, there is a subsequence ∇_j where the measures μ_i converge weakly to some μ measure on M.

Suppose, that there is a point with positive measure $x : \mu(x) > 0$. If* $\mu(x) \leq \epsilon$, then there is some U_x neighborhood of it whose measure is also less than ϵ. Then by the local convergence theorem $\nabla_i|_{U_x}$ converges strongly to a connection on U_x, and $\mu(x) = 0$.
Thus we assume, that $\mu(x) \geq \epsilon$. Since $\mu(M) = k$, we know that there can be at most $l < k/\epsilon$ many such points, x_i. By removing these points, we can apply the previous theorem, there is a subsequence which converges in a strong sense (over compacts) to some self-dual connection ∇ over $M \setminus (\cup\{x_i\})$.
We compute its energy, since $tr(F_A \wedge F_A) = dCS(A)$, we can compute the integal by removing a ball around one of the x_i points. There the sequence of connections converges, and we get that we lose some whole numbers from the energy. From this observation it follows that $l \leq k$ and $[\nabla] \in \mathcal{M}_{k-l}^{M\setminus\{x_1,\ldots,x_l\}}(g)$. Thus on the complement of the x_i's ∇ has finite energy, and we can apply the Uhlenbeck singularity removal theorem to produce ∇ over the entire M, which will b e automatically self-dual on M, g, and smooth. □

14.2 Donaldson's theorem and a fake $\mathbb{R}^4$

Let M be a closed simply connected 4-manifold, whose intersection form is positive definite. Furthermore let $E \to M$ be an SU(2) bundle with $c_2(E)[M] = 1$. From the structure theorem we get, that for generic metrics $\mathcal{M}_1(g)$ is an $8 \cdot 1 - 3 = 5$ dimensional manifold outside the reducible points.
Note, that the number of (gauge classes of) reducible connections is the number of topological reductions of E, i.e. $(\alpha, -\alpha) \in H^2 \times H^2 : \alpha^2 = 1$

Lemma 14.7. *If $c_2(E)[M] = 1$ and $\{(\alpha_1, -\alpha_1)..., (\alpha_t, -\alpha_t)\}$ is the set of topological reductions of E, then $\{\alpha_i\}$ is an independent set in $H^2(M, \mathbb{Z})$.*
Moreover if M has a diagonalisable intersection form over $\mathbb{Z}$, then $t = b_2$.

Proof. $(\alpha_i + \alpha_j)^2 = 1 + 2\alpha_i\alpha_j + 1$ gives us that $\alpha_i\alpha_j = 0$, so they are independent.
For the other statement, one can take the elements of a diagonal basis to represent the topological reductions. □

Notice also, that because $k = 1$ we know that $l = 1$ by asumption, from the weak compactness theorem $\mathcal{M}_1(g)$ may be compactified as $\mathcal{M}_1(g) \cup M$.

Theorem 14.8 (Donaldson collar theorem '82)**.** *The compactified moduli space $\overline{\mathcal{M}_1(g)}$'s end is diffeomorphic to $M \times [0, \epsilon)$.*

Theorem 14.9 (Donaldson's nonexistence theorem '82)**.** *Let M be a closed simply connected positive definite smooth 4-manifold, then q_M can be diagonalised over $\mathbb{Z}$, i.e. $q_M \simeq n < 1 >$.*

Proof. By the previous theorems we know that the space $\overline{\mathcal{M}_1(g)}$ has a boundary component diffeomorphic to M, and singular points, which are locally diffeomorphic to cones over $\mathbb{C}P^2$. By cutting off neighborhoods of the singular points, we get an oriented† cobordism between M and $k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}$. We know that the signature is an oriented cobordism invariant, so $\sigma(M) = k - l$, but since q_M is positive definite, $\sigma(M) = b^+ = b_2$, since $k + l = b_2$ as well by the cobordism, we get that $l = 0, k = b_2 = b^+$. □

To see exotic $\mathbb{R}^4$'s we state the following theorem as well.

*the ϵ is from the Coulomb gauge theorem, we will not state this further, it is always assumed
†this, we didn't state

Theorem 14.10 (Freedman '81)**.** *There is an X simply connected closed 4-manifold with intersection form* $-E_8 \oplus -E_8 \oplus < 1 >$.

Corollary 14.11. *X is non-smoothable.*

Proof. By Donaldson. ☐

Theorem 14.12 (Freedman, Quinn, Gompf)**.** *$X \setminus \{pt\}$ is smoothable.*

Theorem 14.13. *There is an exotic $\mathbb{R}^4$.*

Proof. X has no smooth structure, while $X^\times := X \setminus \{x_0\}$ does admit one. Let $x_0 \in X$, consider a neighborhood of it $U_{x_0} \setminus \{x_0\}$, which is homeomorphic to $S^3 \times [0,1]$. We state that by identifying S^3 with the hopf fibration, that $S^3 \times I$ gets identified with $\nu(\mathbb{C}P^2 \subset \mathbb{C}P^2) \setminus \mathbb{C}P^1$, diffeomorphic. We glue to the "outer" S^3 boundary part a B^4 to get back $\mathbb{C}P^2$ with a sphere removed. This space is homeomorphic to $\mathbb{R}^4$, but we claim, that it inherits a non-standard differentiable structure from $\mathbb{C}P^2$.
In the standard $\mathbb{R}^4$ any compact subset can be covered with a smooth ball. We claim, that in this space there are compact subsets

☐