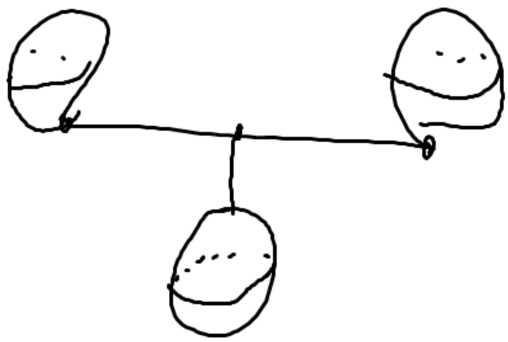
1.

Grading is the same as last year.

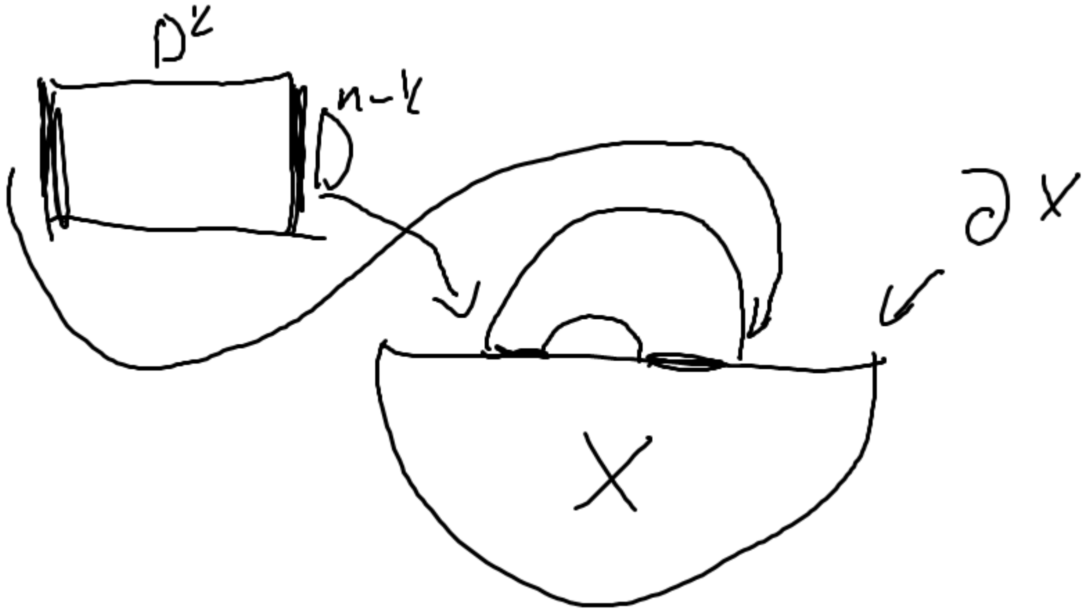
Heegaard-Fleur homology, but before 4-manifolds and 3-manifolds, Lagrangian-Fleur homology.

Today:Handle-decomposition, Kirby diagrams and 4-manifolds

Def.: CW-complexes. X is a top. space. A CW decomposition is a chain $X^0 \subset X^1 \dots$ satisfying the following: X^0 is discrete, X^n is X^{n-1} with some D^n glued to it on the boundary. Also, $\cup X^n = X$. C stands for closure-finite, the closure of an open cell meets only finitely many cells, W is for weak topology.



Def.: An n-dimensional k-handle attached to the n-dimensional manifold X with boundary ∂X is $D^k \times D^{n-k}$ glued to X through the embedding $\phi : S^{k-1} \times D^{n-k} \hookrightarrow \partial X$



By repeating this attachment we get a handlebody. If $k = 0$ then $\partial D^0 \times D^n = \emptyset$, which can be attached to the empty set (which is itself az n-manifold) so the buildup process starts.

Theorem: Suppose that X is a smooth compact manifold. Then X can be presented as a handlebody.

> 💡 If X is not compact we need to modify the definition a little, maybe infinitely many handles. Also some non-smooth manifolds admit andle decompositions, but not all.

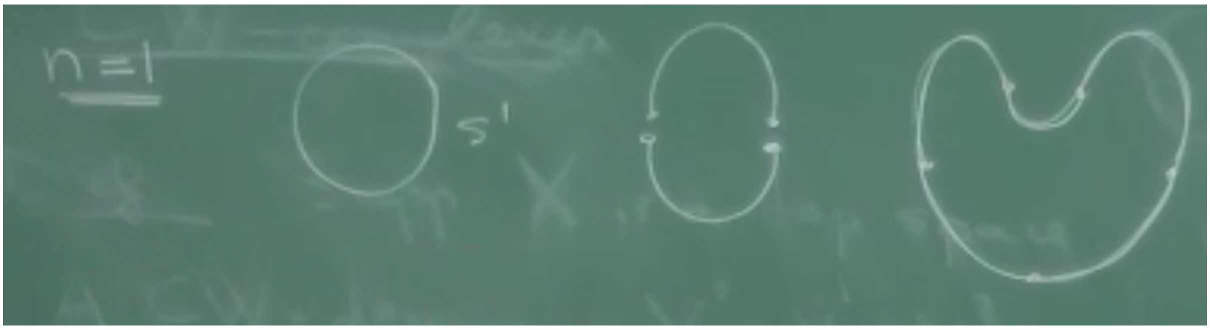
How to specify the data for attaching a handle?

$\phi : S^{k-1} \times D^{n-k} \hookrightarrow \partial X$. $\phi|_{S\times 0}$ is a (parametrized) embedding of a k-1 sphere into the boundary of X . This embedding comes with a trivial normal bundle, indeed this bundle is trivialized.

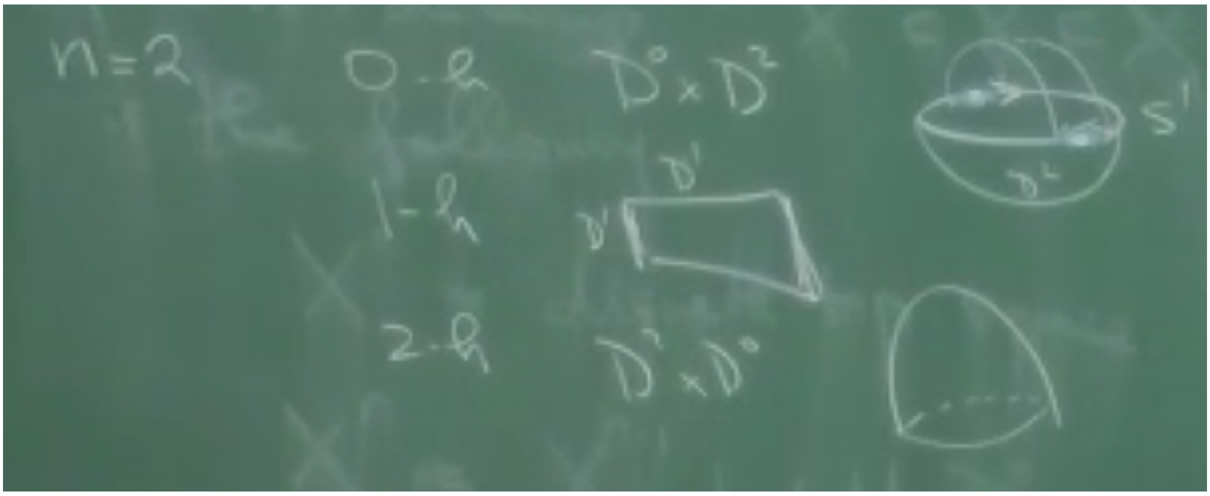
We relate two trivializations by a map $S^{k-1} \to GL(n-k, \mathbb{R})$. Ergo an element $\pi_{k-1}(GL(n-k, \mathbb{R})) = \pi_{k-1}(O(n-k)$ by Gram-Schmidt orthogonalization.

> 💡 Beware, the space of trivializations doesn't have a distinguished element! This is like having an affine space over the homotopy group pictured above.

Example $n = 1$. Only S^1 is compact. We have an abvious decomposition, but also many others.

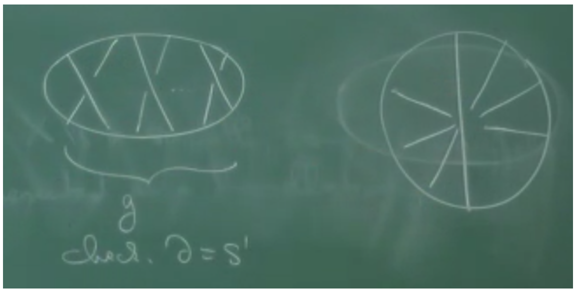


For $n = 2$ a 0-handle is a half-sphere, a 1-handle is an honest band, a 2-handle is again a half-sphere (but the upper half 😀). Here we see, that we have two ways of attaching a one-handle, because $\pi_0(O(1)) = \mathbb{Z}_2$. From now on we assume manifolds orientable to counteract this.



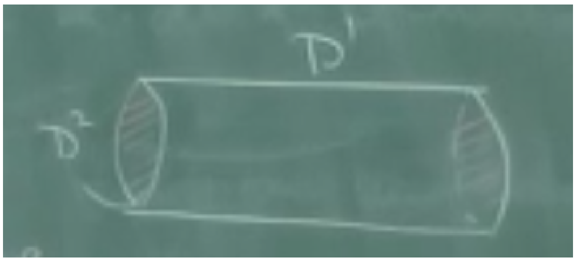
We attach another band to the 0-handle to get a space with boundary a circle. An $S^1 \to S^1$ embedding is essentially unique, so we can close it up by a 2-handle. We could compute the euler characteristic, or homotope the gluings a little to see that this space is the torus.

Observation: with enough patience we can create every surface. The below two constructions give the genus g surface.

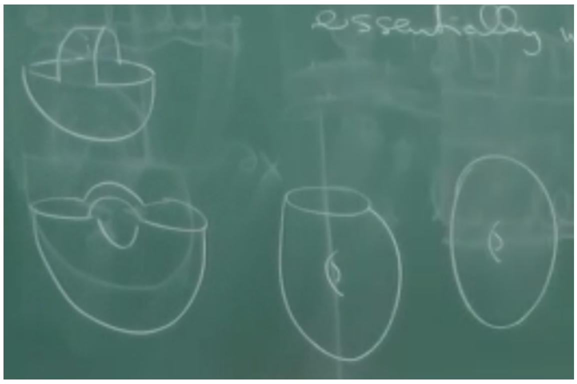


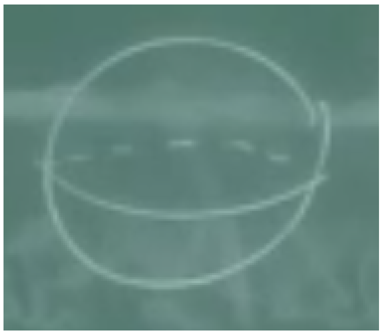
$n = 3$ case. The 0-handle is a D^3.

The 1-handle is $D^1 \times D^2$, the gluing part is two disks of this henger.



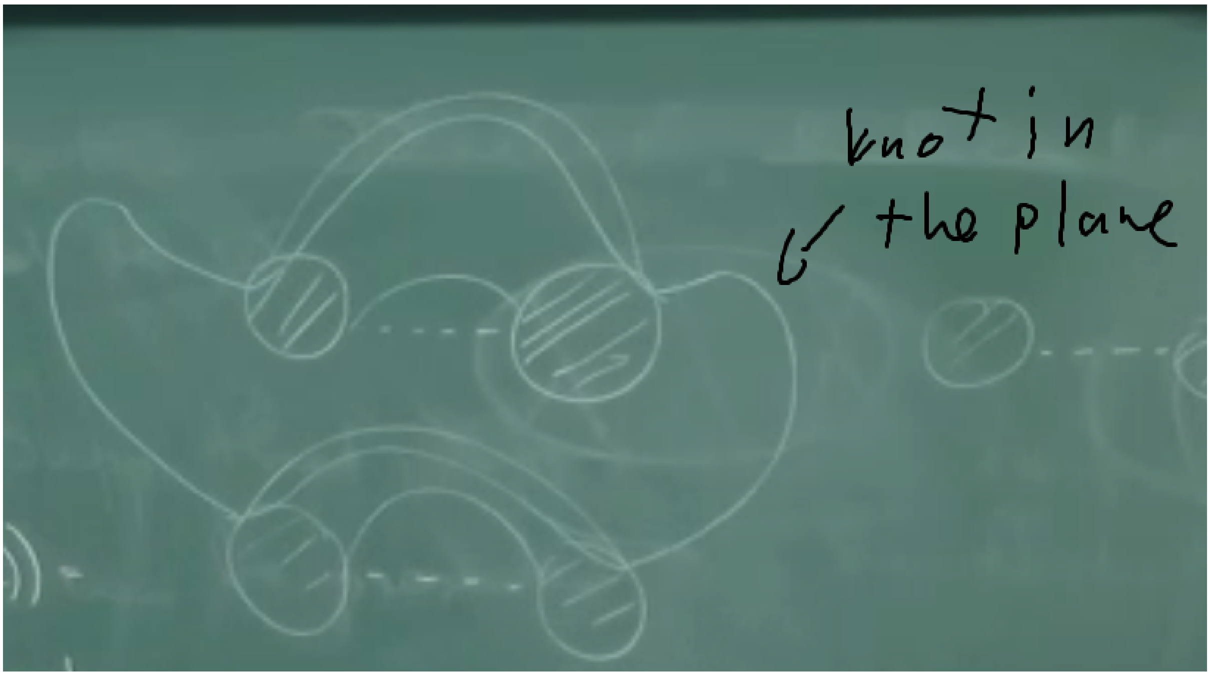
3-handle is again a D^3.





2-handles are the same, viewed the different way, attaching along $S^1 \times D^1$

The 1-handles are circles on the plane, we imagine them being connected outside the plane. The orthogonal group has 2 components in any dimension, since we assumed orientability, we have no choice at the 1-handles. the 2-handles are knots in this space, the framings live in $\pi_1(O(1)) = 1$, there is no framing issue.



We always assume that the “neck” of the 1-handles are very short, so the knot doesn’t have time to wrap around them.

After attachment of the 1-handles we have a genus g surface, the attaching circles give the “holes”. We have a dual picture:

Plane with g circle pairs, and some circles	A genus g surface with red circles indicating the position of 1-handles and blue circles indicating the 2 handles

A 3-handle can only be attached, if the boundary of the 2-skeleton is S^2 exactly.

Definition: A Heegaard diagram is a 3-tuple $H = (\Sigma, \alpha, \beta)$ satisfying the following:

- Σ is a genus g oriented surface
- α is a set of circles $\alpha_1 \dots \alpha_g$ where each α_i is an embedded simple closed curve in Σ, pairwise disjoint (these are the red circles) such that $\Sigma \setminus \cup\alpha_i$ is connected.
- β is again a collection of g circles, with the same properties as α.

- [x] ~~Homework#1: Show that $\Sigma \setminus \cup\alpha$ is connected iff $< [\alpha_i] > \leq H_1(\Sigma, Z_2)$ is of dimension g.~~

> By a diffeomorphism we can make the α -s look like the red circles, but then the β -s will be horrible, or vice versa. We can't make both nice, 3-manifolds are complicated!

Theorem: (Σ, α, β) HD gives a closed oriented 3-manifold, and every such manifold arises in this way.

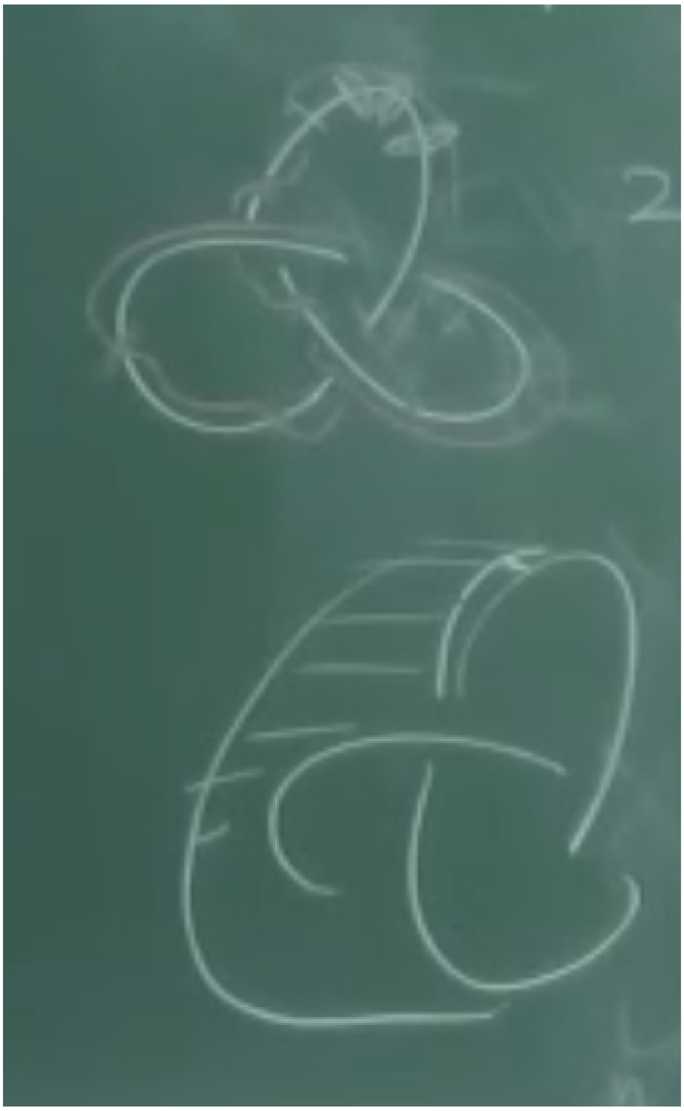
$n = 4$ 0-handles are $D^0 \times D^4$, the attaching region is S^3.

$D^1 \times D^3$ is the 1-handle, ergo a parametric family of 3-balls. Attaching region is two S^2-s.

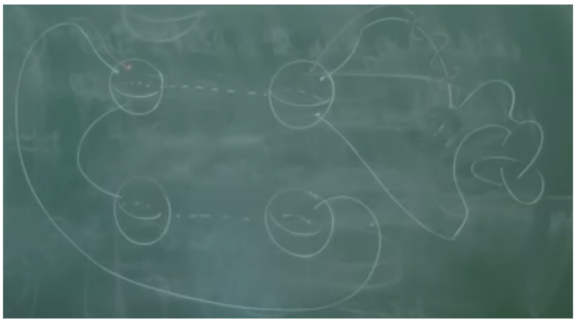
2-handle is $D^2 \times D^2$ attached along $S^1 \times D^2$.

If we don't have 1-handles, then we want $\phi : S^1 \times D^2 \to S^3$ maps to attach to the 0-handle. $\phi|_{S^1 \times 0}$ is a knot. We have $\pi_1(O(2))$ choice of framings, this is $\mathbb{Z}$. To get a trivialisation of the normal bundle we need only one section, since the other one is unique from the orientation, ergo we need a nearby translated knot (by the normal section). We identify this number with the linking number of these knots. We can declare a 0-linking number knot by flowing down on a Seiffert surface of the knot. This information, the knot and the number gives us enough information to attach the 2-handle.

What happens if we do have 1-handles? We have S^3 (R^3), and we have to find 3-ball pairs. 2-handles are attached along knots (which may teleport around the attaching parts of the 1-handles in 3-space. To specify the trivialisation, we draw another knot close to the first one, and assume in the "wormholes" of the 1-handles it doesn't do anything crazy. We crush the 1-handles, and put a circle around where they were, this way our knots will suddenly be in S^3 completely, and we can apply the previous construction, and we can attach the linking number as specification of the normal section.



$D^4 = D^2 \times D^2$, consider the disc $D^2 \times 0$, take its complement, ergo $D^2 \times (D^2 \setminus 0) = D^2 \times D^1 \times S^1$. Also $\partial D^2 \times 0 = S^1 \subset \partial D^4$. Attaching a 1-handles does the same thing. The drawn dotted circle on the diagram indicates that we need to push this circle into D^4 and delete it there. The point is that attaching a one handle to a 0-handle also gives an $S^1 \times D^3$, this justifies that we can indeed replace the attaching parts of 1-handles by the dotted circles without losing information.



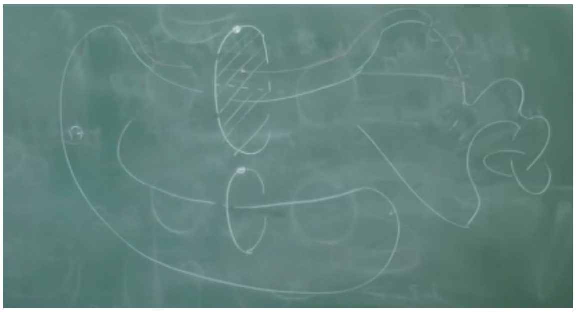


Diagram of a 4-manifold consists of dotted circles represetning the collapsed 1-handles (they should form an unlink), and other circles with a number representing 2-handles. This is called a Kirby diagram of a 4-manifold.

Fact: If X is closed, and $Y \subset X$ is the union of 0,1,2-handles, then $\partial Y = \#(S^1 \times S^2)$. The 3 and 4-handles can be attached uniquely.

A 4-dimensional 3-handle is $D^2 \times D^1$ attached along $S^2 \times D^1$, and we can't draw this :(.

<u>Theorem</u> (Laudenbach-Poenau): Any diffeomorphism of $\#S^1 \times S^2$ extends to a diffeomorphism of $\#S^1 \times D^3$ connected along the boundary ($:= \natural S^1 \times D^3$).

$\natural S^1 \times D^3$ connected along the boundary is exactly the 3 and 4-handles. This theorem tells us, that it doesn't matter how we attach the higher dimensional handles.

Fact: If X is closed, and $Y \subset X$ is the union of 0,1,2-handles, then $\partial Y = \#(S^1 \times S^2)$. The 3 and 4-handles can be attached uniquely.

Conclusion: Take a link with each component either dotted or numbered, the dotted ones representing an unlink, if the result of attaching the 0,1,2-handles has boundary $\#S^1 \times S^2$

Theorem: Every closed orientable 4-manifold arises this way.

We will give an indication on how one can prove the above theorems.

Suppose M a smooth manifold. $f : M \to R$ smooth. $p \in M$ is *critical* if $df : T_pM \to R$ vanishes, ergo in a local chart each partial derivative is zero (which is independent of the coordinate charts chosen). A critiacl point p is *nondegenerate* if the matrix of the second derivatives in a local chart is nondegenerate as a quadratic form.

Lemma (Morse): If p is a nondegenerate critical point of $f : M \to R$, then there is a coordinate chart U near p such that $f|_U = f(p) - \sum_1^\lambda x_i^2 + \sum_{\lambda+1}^n x_i^2$. This λ is called the *index* of p.

f is a Morse function, if all its critical points are nondegenerate.

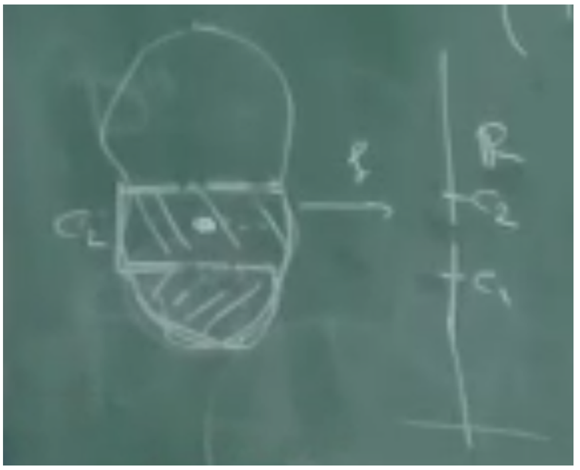
💡 A C^2 open dense subset of smooth functions are Morse.

The main idea of Morse theory is to look at $\{f \leq c\}$ and $\{f = c\}$.

Theorem: Suppose f Morse and $c_1, c_2 \in \mathbb{R}$ satisfy:

1. No critical values in $[c_1, c_2]$, then $f \leq c_1$ and $f \leq c_2$ are diffeomorphic (also the lower one is a deformation retract of the upper one.
2. If there is a unique critical point p with index λ, and $f(p) \in [c_1, c_2]$ then $\{f \leq c_1\} = \{f \leq c_2\} \cup$n-dimansional λ-handle.

> Milnor writes good books, because he's stupid lol.



Def.: $\phi : \mathbb{R} \times M \to M$ is a one parameter family of diffeomorphisms if its smooth, and $\phi(t, \cdot) : M \to M$ is a diffeomorphism of M, and it forms a group under composition.

▼ Fact: Every ϕ determines, and each is determined by a smooth vector field (assuming M compact).

Consider $\phi_t(p)$, which is a smooth curve, and take its tangent vector, this gives the desired smooth field.

Conversely we can take the integral curves of a vectorfield, which will be globally defined by compactness.

Pick a metric $g : TM \times TM \to \mathbb{R}$. The gradient vector field of f w.r.t. g is defined by the formula$< \nabla f, v >= v(f)$.

Theorem: If M closed smooth manifold, then

- M admits a Morse function f
- We can assume that f admits a unique local max and min (assuming M connected)

- f can be assumed to be self-indexing (ergo for a critical point $p : f(p) = ind(p)$

> 💡 Only the 1-handle has its attaching region disconnected (compare with CW complexes).

How to chart 4-manifolds?

For every finitely presentable group G there is a closed X^4 such that $\pi_1(X^4) = G$.

> 💡 There is no Y^3 with fundamental group $\mathbb{Z}_4$, also no 3-manifold with fundamental group $\mathbb{Z}_2$ (this is harder) (but $\mathbb{Z}_3$ exists of course).

> 💡 There are a lot of simply connected 4-manifolds .

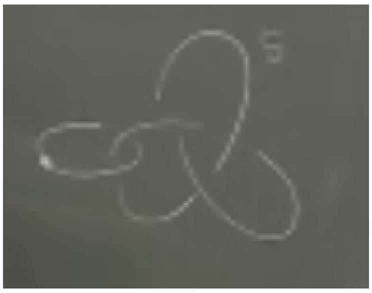
Suppose X^4 simply connected. Then $H_0 = Z, H_1 = 0, H_2 = Z^{b_2}, H_3 = 0, H_4 = Z$, so from homology we get a single number, also by duality we get this for the cohomologies as well. But we have a symmetric bilinear form $Q : H^2 \times H^2 \to H^4;\ \alpha, \beta \mapsto \alpha \cup \beta$, which is called the intersection form of X.

From simple connectivity and the universal coefficient theorem we get that H^2 doens't have torsion.

2.

What is low dimension? 0-4, we go in descending order. Today: overview of 4-manifolds.

A Kirby diagram is a link with two types of knots, either a number is written on a component, or it has a dot. The dotted circles have to form an unlink.



The dotted circle corresponds to a 4-dimensional 1 handle, ergo $S^1 \times D^3$. Take a disk on the surface of the 4-ball, with boundary the component, we push in the disc bounded by it, and delete it, this is the same as attaching a 1-handle. The numbered knot is attaching a 2-handle. The knot itself is the $S^1 \times 0$, and the number determines the framing, which is a trivialization of the normal bundle. We take a nearby translate of the knot, and their linking number is written on the knot, this gives a trivialization, since the other normal vector is determined by the first.

If X_L has boundary $\# S^1 \times S^2$, then $S^1 \times D^3$ can be attached to it along the boundary in only one way. Thus L represents a closed oriented 4-manifold.

We want to understand how can we deform a diagram without changing their manifold.

> 💡 4-Manifolds are complicated! Every finitely presented group can be the fundamental group of a closed oriented smooth 4-manifold.

We restrict to simply connected 4-manifolds. This would be a really trivial class for 0-3, but simply connected 4-manifolds are a very rich class.

What do we see on the homologies? $H_4 = Z$, since we have closed oriented manifolds. $H_0 = Z$ by connectivity, $H_1 = 0 = H^3$ by simple connectedness, we claim $H^2 = Z^{b_2}$, so a free group. The only interesting cohomological product is $\cup : H^2 \times H^2 \to H^4 = Z$, where $(\alpha, \beta) \mapsto < \alpha \cup \beta, [X] >$, so we take the cup product, and evaluate it on the fundamental class of X, this map, denoted by Q_X is a symmetric bilinear function on Z^{b_2}. By Poincaré duality this map is *unimodular*. We can represent this bilinear form by a matrix. Pick a basis $\alpha_1 \dots \alpha_{b_2}$ of the second cohomology, the matrix will be $(Q_X(\alpha_i, \alpha_j))$.

A bilinear form Q_X is unimodular, if the determinant of its matrix is ± 1.

$< n >$ is unimodular if $n = \pm 1$, $(0, 1; 1, 0)$, and $(0, 1; 1, \ell)$ are unimodular.

- [ ] HW1 $(0,1;1,\ell) \sim (0,1;1,k)$ (as bilinear forms) iff $\ell \equiv k \mod 2$ (we are over Z!).

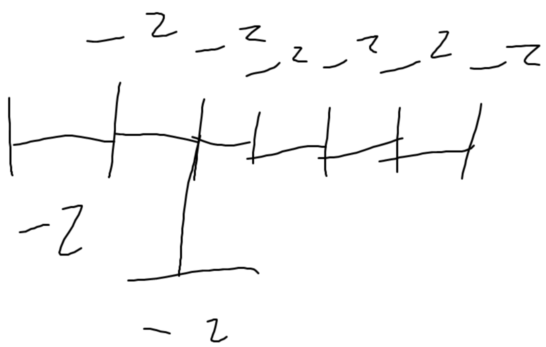
Take Q symmetric unimodular bilinear over Z. It has a rank, as the dimension of the space on which it is defined. It has a signature $\sigma(Q)$ defined as follows. Extend Q to $Z^b \otimes R$ to get Q_R on R^b, now we have Sylvester's theorem guarantees, that Q_R can be diagonalized, and we can take the trace of a matrix of Q_R (ergo the difference of the dimension of the positive definite subspace of Q_R, and the negative definite). The third information is the *type* of Q, it can be even, or odd.

Definition: Q is *even*, if $Q(\alpha,\alpha)$ is even for all integers α. It is called *odd* otherwise. Q is even iff its matrix has only even numbers on the diagonal. (This solves half of the homework lol).

Theorem: Suppose that Q_1, Q_2 are symmetric bilinear indefinite unimodular forms. Q_1 and Q_2 are equivalent iff the above 3 numerical quantities (rank, signature, type) coincide.

Example: -I is a fantastic negative definite bilinear form.

This matrix defines another one. The diagonal is -2, and the off-diagonal elements are -1 iff the vertices are connected on the graph. This is E_8, it will be negative definite, but even (notice that $-I$ is odd).



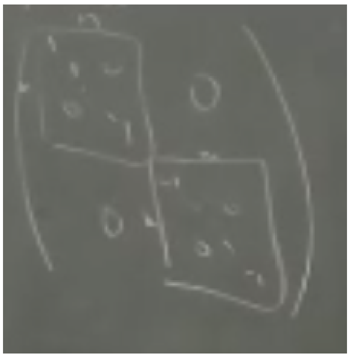
- [ ] HW2 Show $9 < -1 >$ and $E_8 \oplus < -1 >$ are different. (They are both odd).

There are at least 10^{50} rank 40 definite symmetric bilinear unimodular forms.

In fact if Q is indefinite, then its equivalent to $n < 1 > \oplus m < -1 >$, or $kE_8 \oplus l(0,1;1,0)$.

The former is the odd case, the latter is the even case. Notice that it has even rank, and signature divisible by 8.

In the future Donaldson will say, that we only have indefinite forms, and one (of the very many) definite



forms in 4-topology.

Theorem: Suppose that Q_1, Q_2 are symmetric bilinear indefinite unimodular forms. Q_1 and Q_2 are equivalent iff the above 3 numerical quantities (rank, signature, type) coincide.

The proof rests on two other theorems.

1. Hasse-Minkowski: Let Q be as before, and suppose $\sigma(Q) = 0$. Then there exists $x : Q(x, x) = 0$. (Also, x can be chosen *primitive* [$x = dy$ implies $d = \pm 1$]).
2. Wall: Suppose that $Q = n < 1 > \oplus m < -1 >$ with $n, m > 0$, (it can be diagonalized over Z) and $x, y \in Z^b$ are two primitive elements. Then there is an automorphism ϕ of (Z^b, Q) with $\phi(x) = y$, provided that both of x, y are *characteristic*, or neither of them are.

Definition: $x \in (Z^b, Q)$ is characteristic if $Q(x, \alpha) \equiv Q(\alpha, \alpha) \mod 2$ for all α.

Example: In $(Z, < 1 >)$ 1 is a characteristic element. In $(Z^2, < 1 > \oplus < 1 >)$ the vector $(1, 1)$ is characteristic. It is clear that this property is preserved by automorphisms.

Proof of the theorem: First consider Q s.t. $\sigma(Q) = 0$, so we get a vector s.t. $Q(x, x) = 0$. Unimodality gives $y : Q(x, y) = 1$. Take $< x, y > \oplus Z^{b-2}$ decomposition, so we can cut out from Q a $(0, 1; 1, Q(y, y))$ block, this has signature 0, so the remaining block has signature 0, and we repeat. If all the $k_i = Q(y, y)$ are even, than the whole form is even. If k_j is odd, then the matrix $(0, 1; 1, k_j) =< 1 > \oplus < -1 >$, easily seen, by a change of basis $e_1 \mapsto -e_1 - e_2, e_2 \mapsto e_2$.

☐ HW3: $(0, 1; 1, k) \oplus < -1 >=< 1 > \oplus 2 < -1 >$ for even and odd k.

This finishes the proof for the $\sigma(Q) = 0$ case. If the signature is positive, we add to it a -1 block to decrease the signature. $Q \oplus < -1 >= b_1 < 1 > \oplus b_2 < -1 >$ since we made it odd. If Q is even, then the generator of the extra subspace is characteristic, and then we can apply the Wall theorem, so the extra vectors can switch places, so the orthogonal complements go to each other as well, so the suspended Q-s are as well.

Examples of intersection forms: CP^2 is always oriented, the intersection is $< 1 >$, since $H^2 = Z[g]/g^2$. If we take the other orientation, the intersection form will be $< -1 >$. By taking connected sums we can make any $n < 1 > \oplus m < -1 >$.

$S^2 \times S^2$ ha cohomology $Z \oplus Z$, the intersection form will be $(0, 1; 1, 0)$. Again, we can take connected sums, also this torus is orientation-reversing diffeomorphis to itself, so the other orientation gives the same form.

$K3 = \{[z_1 : \ldots : z_4] \subset CP^3 : \sum z_i^4 = 0\}$ is a complex codimension 1 manifold, it is smooth by the implicit function theorem, since it has Jacobian$= 0$ only at the origin. We claim its intersection form is $2E_8 \oplus 3(0, 1; 1, 0)$.

Theorem (Rokhlin): If X is smooth, Q_X is even, then $16|\sigma(Q_X)$ (differently, its form is $kE_8 \oplus lH$, where k is even).

$11/8$ conjecture: X smooth oriented closed. Q_X indefinite and even, then$Q_X = 2mE_8 \oplus lH$ and $l \geq 3m$ has to be satisfied. $(H = (0, 1; 1, 0))$

For this, we want to decompose the m-fold connected sum of $K3$-s into some manifold, and an $S^2 \times S^2$.

$10/8$ theorem: If X is smooth, $Q_x = 2mE_8 \oplus lH$, then $l \geq 2m$.

Theorem (Donaldson): X smooth closed oriented, and Q_X definite, then $Q_X = n < 1 >$.

This is the *realization problem*, can we construct a 4-manifold with a given intersection form. Stated differently, can we realize a given ring as the cohomology ring of a 4-manifold?

Why is it called an intersection form?

Claim: If $\sigma \in H^2(X, Z)$, then there is a $\Sigma^2 \hookrightarrow X$ such that $i_*([\Sigma]) = \sigma$.

A cohomology class $c \in H^2$, then there is a complex line bundle associated to it. Take a section, and let F be the zero section. This is of codimension 2 in X, so $\dim F = n - 2$, and $[F] = PD(c) \in H^{n-2}$ is the Poincaré dual of c. Take $\alpha, \beta \in H^2$, and surfaces for them by the previous discussion. We can presume they intersect transversely, and we claim $Q_X(\alpha, \beta) = \#(\Sigma_\alpha \cap \Sigma_\beta)$ counted with sign. Let $p \in \Sigma_\alpha \cap \Sigma_\beta$, and take an oriented basis at p of the tangent spaces of both surfaces. Concatenate these to get a basis, if its oriented we add $+1$, if not, we put a

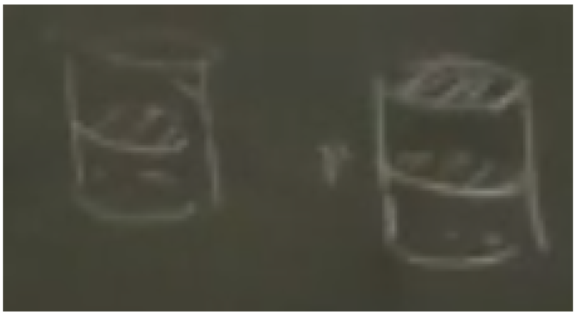
-1. It doesn't even depend on which order we concatenate in, since the dimension is 4.

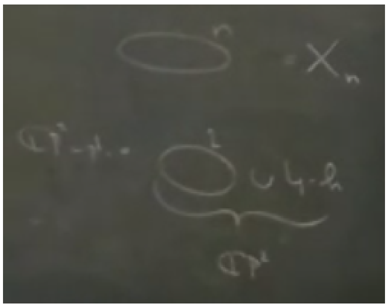
Examples: The Kirby diagram of S^4 is the 0 component empty link, there is no second cohomology, no intersection form.

Take a D^2 bundle over S^2, ergo the unit ball bundles of complex line bundles. There are Z many line bundles over CP^1, we have to specify the class.

> Delete a point from CP^2, we claim this is a bundle over CP^1 with this number equal to 1. Take a projective line in CP^2, and delete a point outside of it. The line bundle structure is given by connecting the deleted point with a point of our chosen projective line, this will be the projection. A section will be another line, it will intersect our 0-section line in 1 point, and complex lines always intersect positively.

We would like to draw the Kirby diagram of every ball bundle. Decompose S^2 as two D^2-s. We have to glue $D^2 \times D^2$ to another copy of this, the first part will be the fiber, the second part is the upper/lower hemisphere. The Kirby diagram for X_n starts with an unknot, which will glue to give the S^2. From the section property we need to write n on this circle, since it should meet a nearby circle exactly n times.





$Q_{X\#X'} = Q_X \oplus Q_{X'}$ is clear from the surface description of the intersection form.

For $S^2 \times S^2$ the diagram will be the Hopf link, with 0 written on both components, next week.

Suppose L a Kirby diagram of X with no 1-handles.

Claim: The intersection form is just the linking matrix of $L(=lk(K_i, K_j)$, where the K_i are the components of L, if $i = j$ write the number written on the knot).

As a CW-complex these manifolds look like a point, with 2-handles attached. Every knot bounds a Seifert surface, this closes off the attachment of the 2-handles, and we get representatives of the homology classes. We can push one Seifert surface into the 4-ball, and look at the intersection of these, which is exactly the definition of the linking number, and also the intersection form.

3.

Theorem (Freedman '82): Suppose X_1, X_2 are two simply connected oriented closed smooth 4-manifolds, then X_1 is homeomorphic to X_2 iff $Q_{X_1} = Q_{X_2}$.

There is also a version of this for non-smooth manifolds, the theorem either holds, or there is a 2-1 correspondance with manifolds, and their intersection forms.

> 💡 Σ_1 homeomorphic/diffeomorphis to Σ_2 iff their homologies are isomorphic, which is again equivalent to their genera being equal.

$Y_1^3 \sim Y_2^3$ diffeomorphic/homeomorphic iff their fundamental group are "equal"? This theorem is, strictly speaking, false, we have to restrict ourselves to (oriented) manifolds, which don't decompose as direct sums.

The fundamental group was invented exactly for this purpose, Poincaré believed that homology will tell you everything, but that turned out to be false by Dehn, then he searched for another invariant.

For smooth simply connected 4-manifolds the above theorem of Freedman says, that the cohomology ring is the only thing that matters.

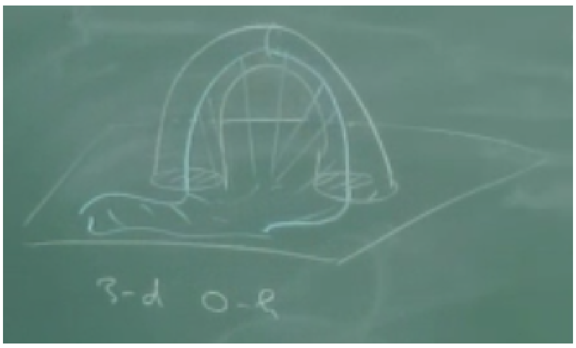
Counterexample $S^2 \times S^2$ and $CP^2 \# CP^2$, their homologies are $Z, 0, Z \oplus Z, 0, Z$, and for the other the same, but the multiplication is either $(0, 1; 1, 0)$ for the torus, for the other $(1, 0; 0, -1)$, so it is not enough to consider the graded additive group structure of the cohomology ring. But $S^2 \times S^2 \# CP^2 = CP^2 \# CP^2 \# CP^2$ up to diffeomorphism, so we can't even expect unique decomposition of 4-manifolds.

Last time we looked at Kirby diagrams, a link determines a 4-manifold. $L = (K_1, \ldots, K_n)$ where the first few components are framed, the other ones are dotted and form an unlink, this determines a manifold with boundary, if this boundary is $\# S^2 \times S^2$ a connected som of tori, then it determines a unique closed 4-manifold.

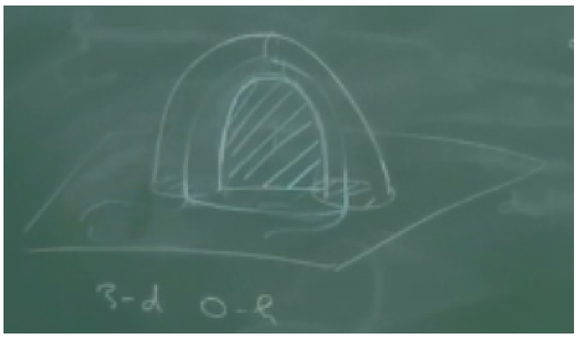
We want to discuss the moves, which leave the resulting manifold invariant. These are the Kirby moves.

1. isotopies of course leave everything unchanged
2. handle slides
3. handle cancellation/creation

We start with 3., sometimes we can cancel a 1-h/2-h pair, or a 2-h/3-h pair. The rule is the following: a 1-h/2-h pair linking geometrically once can be deleted, or introduced. K can do anything outside, but pierces the 1-handle exactly once, and there are no other lines piercing the spanning disc of the doted circle.



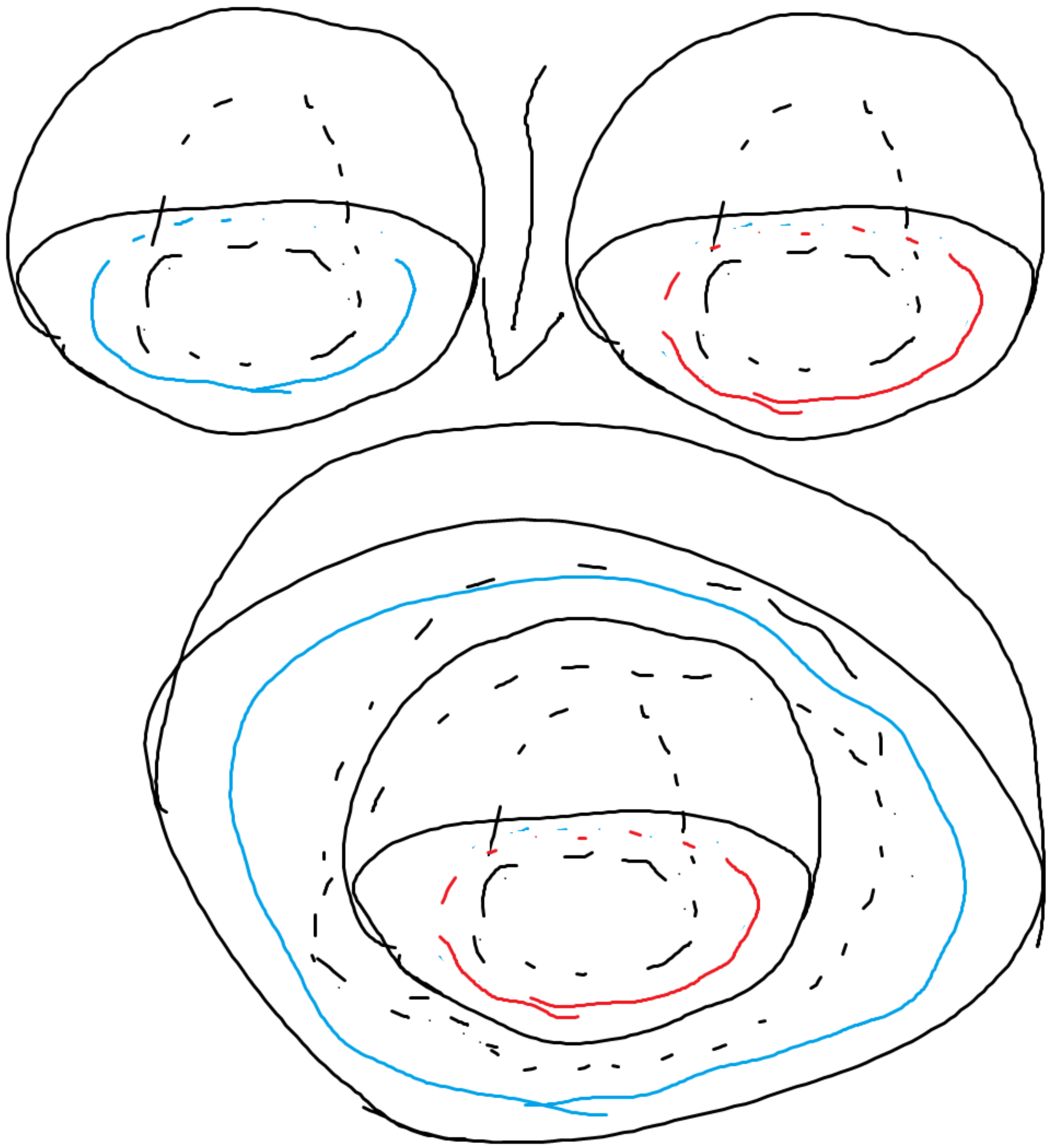
The 1 and 2 handle before the homotopy



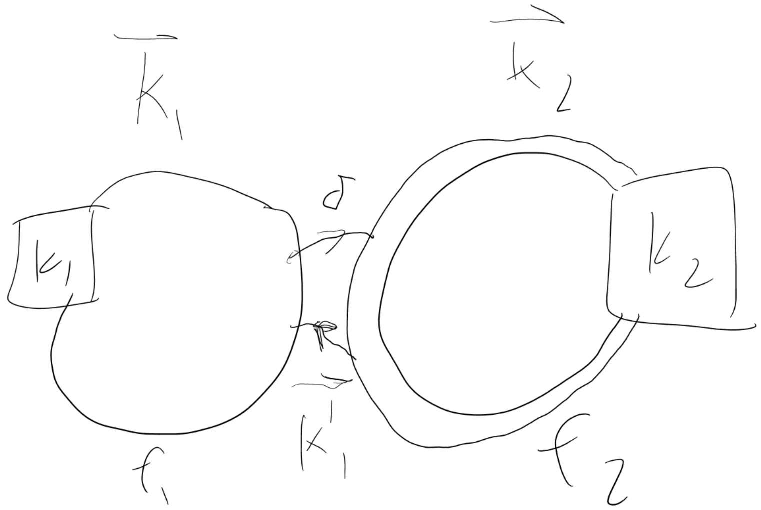
and after.

In 3-d we attach a 1-handle, and a 2-handle intersecting the midline of the 1-handle exactly once. We homotope the 2-handle which is glued onto it, and we see, that it closed off the 1-handle exactly, we can push it back into the 0-handle. The situation is similar in 4-d, also it is enough to think about one case, since we can take a dual decomposition (-f morse function).

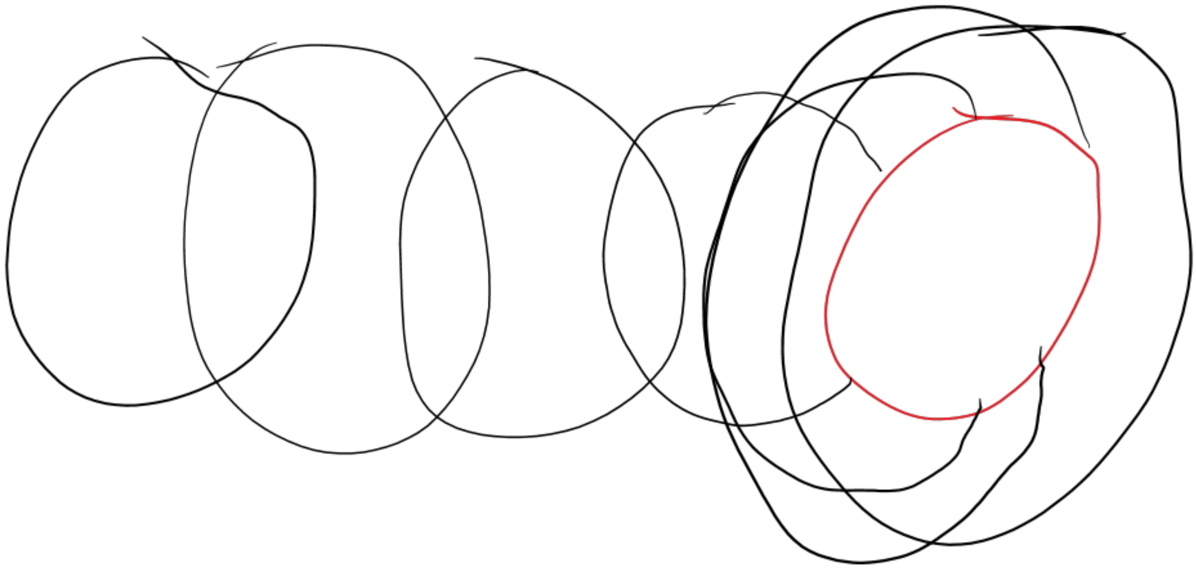
Now handle slides. We can slide 2-handles in 3d.



We take a nearby knot to the second one, and take the connected sum. What number do we write on the new knot? $f_1' = f_1 + f_2 + 2lk(K_1, K_2)$. This is the 4d handle slide. The new framing is the square by the intersecion form of $K_1 + K_2$.



We actually slide the attaching part along the other handle, but only draw the part in S^2.



Theorem (Fundamental Theorem of Kirby Calculus): Two diagrams $L_1 \& L_2$ represent diffeomorphic 4-manifolds iff L_1 can be transformed to L_2 by a finite sequence of Kirby moves.

One direction is kinda easy, we saw that the moves don't change the diffeomorphism type, they are essentially isotopies, or gluing a nullhomotopic part.

What about the other direction? We get handle decompositions from Morse functions. $X \xrightarrow{f_1, f_2} R$ two Morse-functions giving L_1, L_2 Kirby diagrams. Take $F : [0,1] \times X \to R$ a one parameter family between them. Beware, $f_t = F|_{t \times X}$ is not always Morse, but we can take the homotopy itself generic. It turns out that there will be only finitely many t-s which are non-Morse, and locally one should see that these points correspond to slides and handle cancellation/creation.

Now we move on to classifying 4-manifolds. How does one find suh a sequence of moves? How do we see that the diagrams cannot be moved into each other?

Brute force doesn't work, there are infinitely many ways to pick the path for handle slides, the cancellations and creations and so on.

Definition: Suppose X a smooth oriented closed 4-manifold. $g_X : H_2(X, Z) \to N$ is called the *genus function* of X, defined by

$$g_X : \alpha \mapsto min\{g(\Sigma^2) : \exists \Sigma^2 \overset{\iota}{\hookrightarrow} X,\ \iota_* \Sigma^2 = \alpha,\ \iota \in C^\infty(\Sigma^2, X)\}.$$

Suppose X can be given by a Kirby diagram L with no 1-handles. We look at the diagram, and the Seiffert surfaces give a surface for this function, but the slice genus gives an even better upper bound, since the surfaces can be pushed into the 4d 0-handle.

Definition: The *shake genus* $g_{shake}(K)$ is defined as $g(\alpha_k)$ in $X = D^4 \cup D^2 \times D^2$ attached along K with framing 0 (where α_K is the generator of the 2-homology corresponding to K).

▼ Lemma (Trace embedding lemma): $K \subset S^3$ is slice iff $X_0(K)$ smoothly embeds into S^4 (same construction as before, $D^4 \cup D^2 \times D^2$ along K with framing 0), this is called the *knot trace*, or *0-trace*.

$S^4 = D^4 \cup D^4$. Look at K in the intersection of the two hemispheres, which is an S^3. First of all, if $K$is slice, put the disc on top of it in the bottom hemisphere. Take a tubular neighbourhood of it, this set and the upper hemisphere gives exactly the $X_0(K)$ knot trace.

In the other direction, embed $X_0(K)$ into the 4-sphere. We see a D^4 and a 2-handle embedded. We have to think about that the complement of a smooth image is another D^4. This is true, ince we can locally approximate the function by a linear one, shrink the 0-handle and ...?

Observation: Suppose $K_1, K_2 \subset S^3$ two knots, which satisfy:

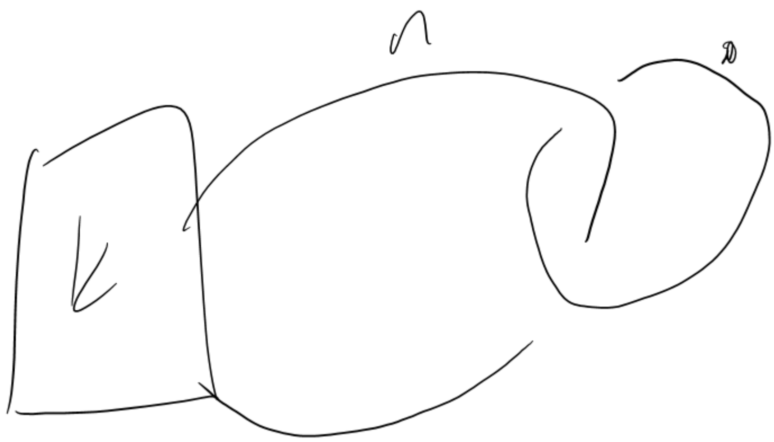
- K_1 is slice
- K_2 is not slice
- $\partial X_0(K_1) = \partial X_0(K_2)$ by diffeomorphism

Then there is a smooth 4-manifold X which is homeomorphic to S^4, but not diffeomorphic to it. b r u h, 4 dimensional Poincaré conjecture.

K_1 is slice, so $X_0(K_1)$ smoothly embeds into the 4-sphere. Let $X = S^4 \setminus int(X_0(K_1)) \cup_{\partial X_0(K_1)} X_0(K_2)$. This is the construction. This is a smooth manifold first of all. By Freedman, all we have to do to recognize this manifold is to compute the cohomology ring. We get $H^2 = 0$, and $\pi_1(X) = 1$, so this space is homeomorphic to S^4, but this space cannot be diffeomorphic to S^4 by the trace embedding lemma.

Homework: $X(K, n) = X(K', n')$ iff $n \equiv n' \mod 2$.

Determine the boundary if possible.



one circle has n on it, the other has a 0 on it, not a dot!

Three-manifolds

Assume Y closed oriented smooth.

Examples: S^3, $S^1 \times \Sigma_g$, or $S^1 \times S^1 \times S^1$.

The second example can be generalised to the following. $Y \xrightarrow{S^1} \Sigma_g$ a principal S^1 bundle. $S^1 = SO(2) = U(1)$ so this can be regarded as arising from a real plane bundle, or a complex line bundle, this is characterised by one number, the Euler class/first Chern class. The total space of a complex line bundle over a 2-surface is a 4-space, a generic section has some number of zeros. This will be $Y(g, n)$ (g the genus, n the class). ???

We can also create another bundle, this over S^1. Let $\phi : \Sigma_g \to \Sigma_g$ be an orientation preserving diffeomorphism. Cut the circle up, and glue it back $[0, 1] \times \Sigma_g/(0, x) \sim (1, \phi(x))$. It is clear that every Σ_g bundle over S^1 arises this way. Isotopic maps of course glue together into the same bundle, so we need to look at $\Gamma_g = Diff^+(\Sigma_g)/Diff^\circ(\Sigma_g)$, where we quotient out by the maps isotopic to the identity. This is called the *mapping class group*. This is a discrete infinite noncommutative group in general. $\Gamma_0 = 1,\ \Gamma_1 = SL_2(Z)$, but the other ones avoid understanding. Γ_g is finitely presented, at the very least, but this is also a theorem of Thurston from 1980.

Two more classes are hyperbolic 3-manifolds (ergo manifolds admitting a metric of constant -1 curvature). For example H^3 is the upper halfspace, with metric $1/x_3^2(dx_1^2 + dx_2^2 + dx_3^2)$. Consider $G = Isom(H^3)$ and take $\Gamma \leq G$ such that H^3/Γ is a manifold. This gives examples of closed hyperbolic manifolds. In fact this will give all of them.

The other extra class is Seifert fibered spaces. These are 3-manifolds which admit a decomposition into the union of circles such that near each circle the union of nearby circles give $S^1 \times D^2$. An example is a circle bundle over a surface, ergo $Y_{g,n}$, and from these by surgery on finitely many fibres we get all possible closed SFPs.

We claim now, that *in some sense* these are all 3-manifolds!

> Definition: Y is prime, if $Y = Y_1 \# Y_2$ implies $Y_1 = S^3$ or $Y_2 = S^3$.

Theorem (Kervaire-Milnor): Any 3-manifold decomposes as a connected sum of prime 3-manifolds, and this decomposition is unique up to order and S^3.

The proof is similar, we need to find something similar to a norm, that is less on the components of the direct sum, than the sum itself.

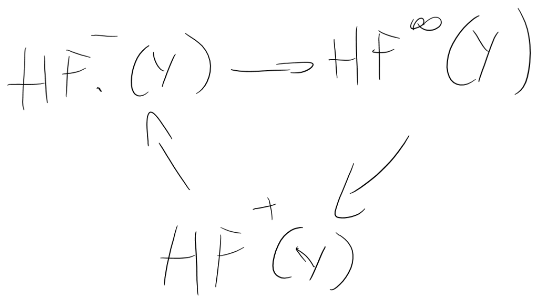
Let Y be a prime 3-manifold.

Theorem (Geometrization theorem, Perelman '04): Y can be decomposed along finitely many T^2-s so that the components are either hyperbolic, or Seiffert fibered.

The problem is, there are many self-diffeomorphisms of the torus.

The rest of the semester will be spent developing a homology theory answering questions about 3-manifolds. This will be Heegaard-Fleur homology.

A functor $Y \mapsto \hat{HF}(Y)$, and a long exact sequence



Also if there is a smooth cobordism from Y_1 to Y_2, get a map $F_X : \hat{HF}(Y_1) \to \hat{HF}(Y_2)$. This isn't exactly true, we will give some extra struture to our manifolds in the future. Now we focus on Heegaard.

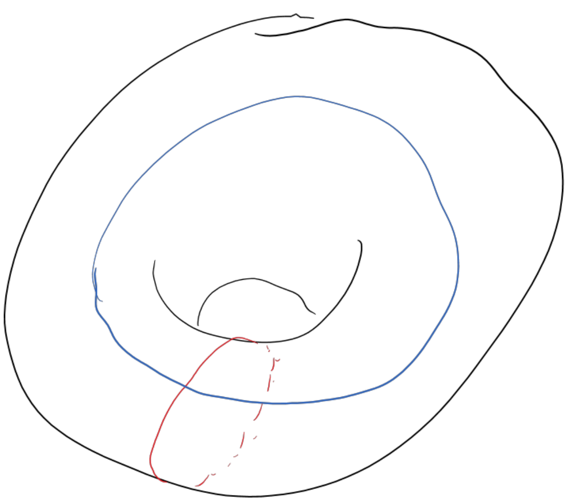
Definition: A Heegaard diagram is a 3-tuple $H = (\Sigma, \alpha, \beta)$ satisfying the following:

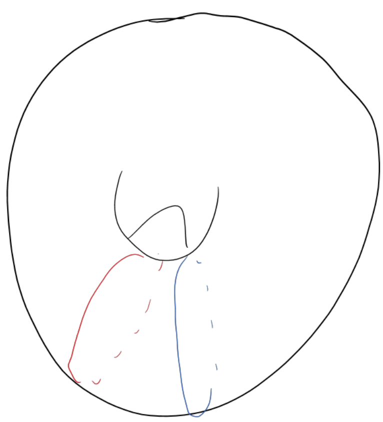
- Σ is a genus g oriented surface
- α is a set of circles $\alpha_1 \dots \alpha_g$ where each α_i is an embedded simple closed curve in Σ, pairwise disjoint (these are the red circles) such that $\Sigma \setminus \cup\alpha_i$ is connected.
- β is again a collection of g circles, with the same properties as α.

Every such diagram determines a closed Y_H manifold. Take $\Sigma \times I$, draw the α curves on the top, the others on the bottom, and attach 1 handles to the bottom circles, a 2-handles to the top. The boundary will be an S^2 on top, and another on the bottom, close it up by a 0 and a 3-handle.

Heegaard moves can move around the diagram without changing the end result.

- isotopy. The two sets of curves can be isotoped independently of each other.
- handle slide, as we already saw.
- stabilization/destabilization





Heegaard diagram of $S^1 \times S^2$

Heegaard diagram of S^3

Stabilization is connected union with the left diagram, which is an S^3.

Theorem: Suppose $H_1(\Sigma_1, \alpha_1 \beta_1) \& H_2(\Sigma_2, \alpha_2, \beta_2)$ are two Heegaard diagrams of diffeomorphic manifolds. Then H_1 can be transformed into H_2 by a finite sequence of Heegaard moves.

> Heegaard fled from Germany during/before WW2, to the US through Siberia. Kirby is still alive.

4.

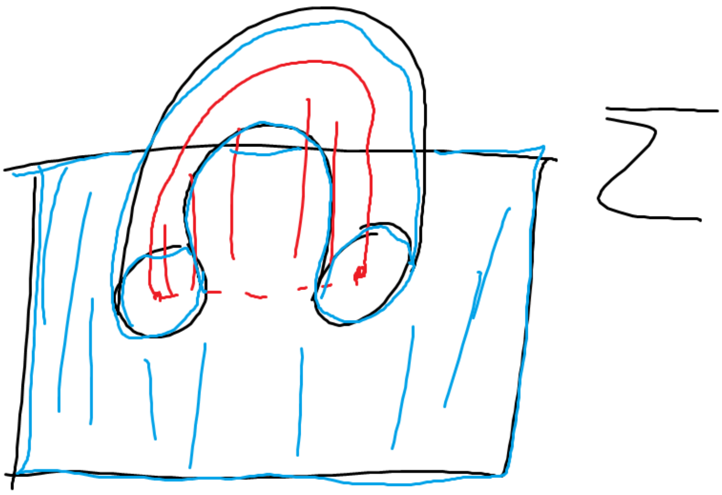
Last time: 3-manifolds. They are presented by Heegaard diagrams, $H = (\Sigma_g, \alpha, \beta)$ s.t. the α curves span a g dimensional subspace in the first homology of Σ_g, pairwise disjoint. Our 3-manifolds are smooth closed oriented. All diagrams represent a unique 3-manifold, it can be capped off. This is an upgrade of the Heegaard splitting, which we learned before, $Y_H = U_\alpha \cup_\Sigma U_\beta$ glued by a diffeomorphism of Σ.

This mapping class group is complicated.

Fact: Every smooth closed oriented 3-manifold can be represented as Y_H. This H is unique up to Heegaard moves (isotopy, handle slide, stabilization/destabilization).

Theorem 1: Suppose that Σ_1 and Σ_2 are two Heegaard surfaces in Y (ergo $Y \setminus \Sigma_i$ is the union of two handlebodies). Then there is a sequence of stabilizations such that $\Sigma_1' \& \Sigma_2'$, the stabilized surfaces are isotopic.

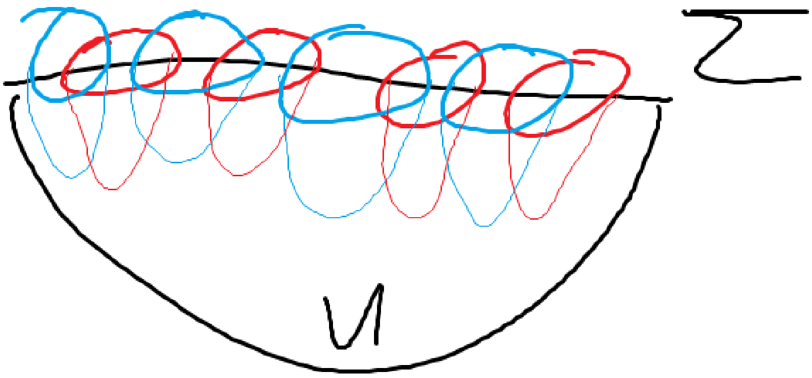
Stabilizing Σ means that we introduce a cancelling pair, we attach an “unknotted” tube ot the surface s.t. it could be pushed into Σ.



A surface after a stabilization along the red curve.

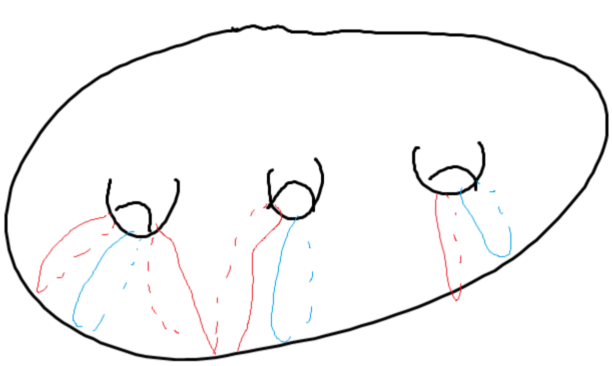
People spent much time trying to understand how many stabilizations one needs, before two surfaces become isotopic for a given manifold.

Theorem 2: Suppose $\alpha \& \alpha'$ two collections of g simple closed curves on Σ_g such that U_α is diffeomorphic to $U_{\alpha'}$ such that the diffeomorphism is isotopic to the identity (said differently, there is a handlebody U with boundary Σ s.t. both $\alpha \& \alpha'$ bound discs in U). Then α can be turned into α' by a sequence of handle slides.

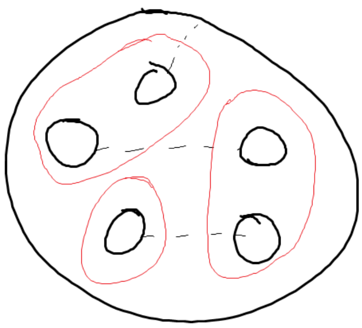


The two families of curves and the discs bounded by them.

Proof: Assume $\alpha \cap \alpha' = \emptyset$. Then consider the α' curves in $\Sigma \setminus \cup\alpha$. The α'-s are complete circles in $\Sigma \setminus \cup\alpha$. This space stays connected by the assumption on the α curves for a diagram.

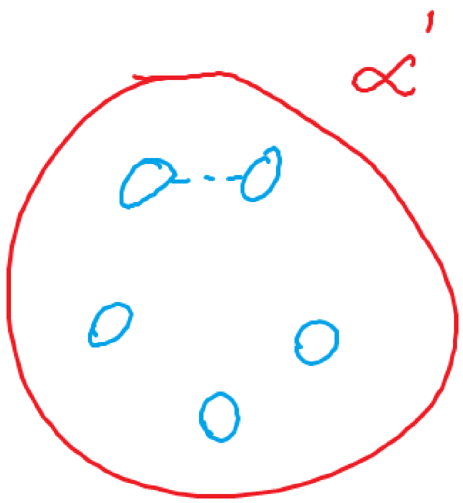


The circles before the cutting

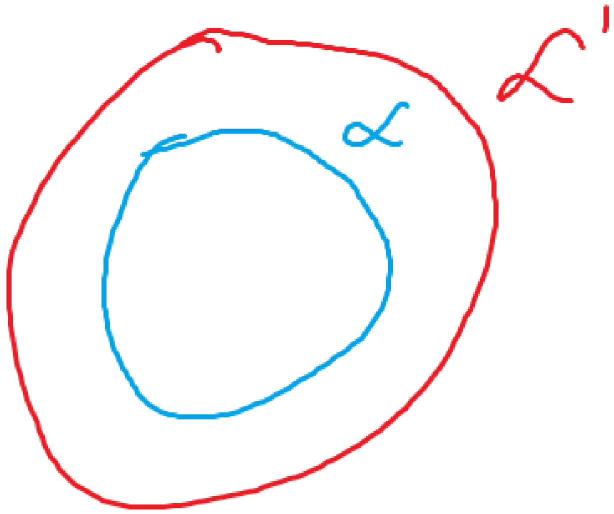


and after.

Now α' cuts $\Sigma \setminus \cup\alpha$ into $g-1$ components, so there exists a component, which has only one α' boundary component, take that.



The “innermost” circle before the sliding



and after the sliding.

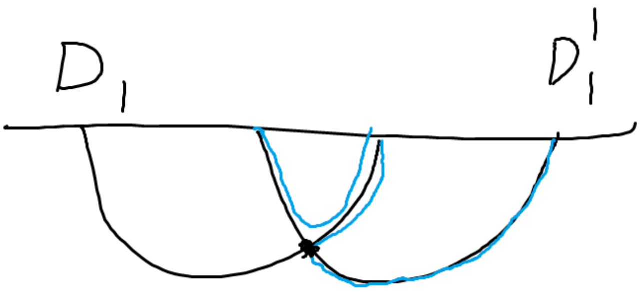
We state, that we cannot have only pairs inside. If we can pair up every other boundary type inside, we found a subsurface bounding this circle from α', so its 0 in homology, but we picked the linearly independently. So there is on component at least, which has no friend inside. Now take every other circle inside, and slide it along the one, which has no pair, this will remove them from the inside of this circle.

In the end only one boundary component will remain, so they are isotopic, and by induction, we are done.

Now in the general case, the $\alpha \& \alpha'$ are not necessarily disjoint. Induction by the number of components of the intersection of the discs bounded by the two curve families. Generically the intersection will be a union of circles and intervals. Firstly we can get rid of circles. Consider an "innermost" circle of the intersection on D^1. They bound a circle on D^1 and D^2, together with the part of the discs they bound they form a sphere. The handlebody is a prime manifold, so it has to bound a ball, and we can get rid of them.

Now the union can be supposed to be a union of intervals. Now take an "innermost" interval, ergo one of the complements of the interval is disjoint from D'.

The intersection of two discs with interval components, the innermost intervals denoted blue.



Switching D_1' to another disc.

Instead of D_1', we take two discs bound by the two components after cutting out an interval. $D'' = D'$ except D_1', and instead of it take one of the discs constructed before. The bounding circles sum to a curve, which satisfies linear independence, so at least one of the new bounding curves satisfy it also, and we pick that one. The property of the circles bounding discs stays, so this is the same handlebody up to diffeomorphism isotopic to the identity. The $\#$ of intersection points lowers by at least one, so by induction we are done.

These two theorems show, that really $Y_H \sim Y_{H'}$ iff $H \sim H'$ by Heegaard moves.

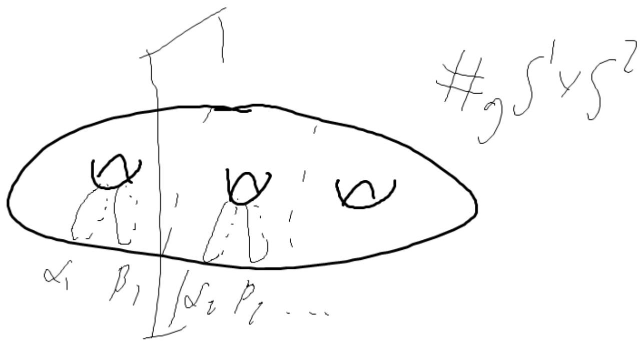
What can we read off the diagram? The fundamental group for example. As we saw in kindergarten, 1-cells are the generators, and 2-cells are the relations.

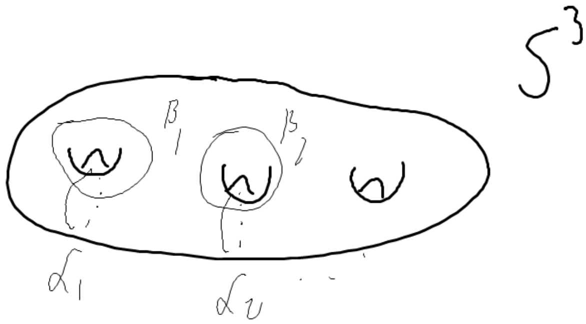
> 💡 Anatomy of a handle. $D^k \times D^{n-k}$ is a handle attached along $\partial D^k \times D^{n-k}$. This portion is called the attaching region. $\partial D^k \times 0$ is the attaching sphere, and $D^k \times 0$ is called the core of the handle. $0 \times \partial D^{n-k}$ is called the belt circle/sphere, and $0 \times D^{n-k}$ is called the cocore of the handle.

The α-s are the belt circles of the 1-handles. After attachment, the β circles tell us how many times the 2-handle goes through the 1-handles.

$\pi_1(Y) =< \alpha | r_{\beta_j} = 1 >$. For the relations orient all circles, consider β_j and a point on it. Go around β_j, and write down if there is an intersection with an α_i, and add it to the word r_{β_j} with exponent given by the orientation, if $[\alpha_i] \smile [\beta_j]$ (in this order) gives $[\Sigma]$ or not.

- [ ] Homework #1: $H_1(Y_H, Z) = H_1(\Sigma; Z)/ < [\alpha_i], [\beta_j] >$ (we have to orient the circles once again!)

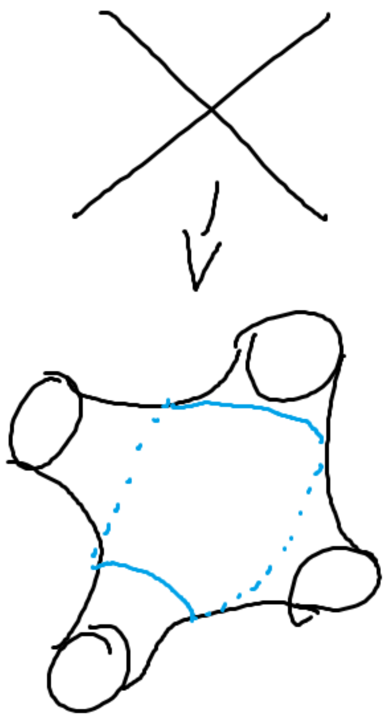


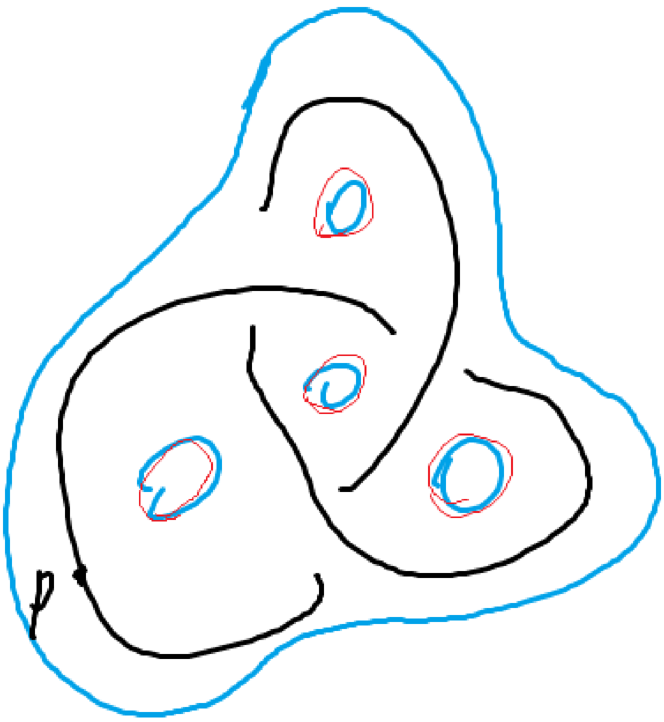


<u>Claim</u>: Consider a diagram H with two points in $w, z \in \Sigma \setminus \{\cup\alpha \cup \cup\beta\}$. Then H gives a 3-manifold, and the two points give a knot in Y_H. The knot K is constructed as follows: connect w to z in $\Sigma \setminus \cup\alpha$, then connect z to w in $\Sigma \setminus \cup\beta$. Push the first arc into the handlebody of α and the second to the handlebody of β. This is unique up to isotopy, by avoiding the α curves, we are drawing on the surface of D^3, ergo

the handlebody with the handles removed, this is unique up to isotopy, so both "sides" of the knot is well defined.

We claim every knot and manifold pair can arise in this way. Example, any $K \subset S^3$ arises this way. Take a projection and a tubular neighborhood of it in R^3. Let $\Sigma = \partial(\nu(\text{projection}))$, for the trefoil, this is a genus 4 surface. Mark the projected knot somewhere. The α circles are defined as follows. Take $\Sigma \cap R^2$, these are the circles of the projection. These will be the α-s, sans one circle, which is next to the marked point p (there will be $g+1$ circles, so we need to leave one out).





For the betas we will do what the left diagram indicates, and take a meridian around the tube close to p.

This really recovers the knot, we can connect the two points in the complement of the alfas easily, the complement of the betas tells us, if a crossing is over or under, which specifies the knot up to isotopy.

Consider $T_\alpha = \alpha_1 \times \cdots \times \alpha_g \subset \Sigma \times \cdots \times \Sigma$, modulo the S_g action, the symmetric power, and the products of the betas in the same space.

- [ ] Homework #2: $T_\alpha \cap T_\beta$ are in 1-1 correspondence with the Kauffmann states. (P, p) marked projection of K in S^3. consider the Heegaard diagram constructed above. Then the previous statement is true.

$Symm^g(\Sigma) = \times_1^g \Sigma / S_g$, but the action is not free! Thus in general this quotient is a *very* singular space, except in our case, when the dimension is 2.

Example $Symm^g(C)$ is in 1-1 correspondence with monic $\deg g$ polynomials by the map $(z_1, \ldots, z_g) \mapsto \prod_1^g (z - z_i)$. A monic polynomial can also be identified with C^g by the coefficients, so these are equivalent. For example $Symm^2(S^1)$ will not be very singular, but will be only a manifold with boundary, namely the Möbius band. If we extend even further, $Symm^g(CP^1) = CP^g$, and indeed $\forall\, \Sigma : Symm^g(\Sigma)$ is a smooth manifold of dimension $2g$.

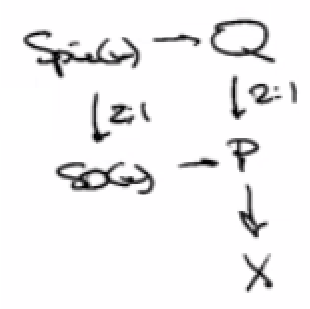
$Spin^c$ and $Spin$

$GL_n(R)$ a nice Lie group, homotopy equivalent to $O(n) := \{A | AA^t = id\}$, this falls into two components, $SO(n)$, and the other one. $\pi_1(SO(n)) = Z_2$ for $n \geq 3$, $SO(1) = 1$, $SO(2) = S^1$, $SO(3) = RP^3$.

Suppose we have on oriented Riemannian manifold, how do we turn this into a simply connected one? $SO(n)$ has a universal cover, which we call the $Spin(n)$ group, this cover is $2:1$. For example $Spin(3) = SU(2)$ which is the unit quaternions also. We know how to construct the universal cover, take the paths from the basepoint modulo homotopy relative to the endpoints. Paths can be multiplied (pointwise), and we have the endpoint map, mapping back down to G.

$q \in SU(2)$ defines a map on the quaternions by conjugation, call it ϕ_q, this gives an action on the imaginary quaternions, and defines an element in $SO(3)$. On the next level $SU(2) \times SU(2) \to SO(4)$ is again isomorphic, the map is $h \mapsto q_+^{-1} h q_-$ the twisted conjugation by a quaternion pair. The kernel is $\{(1,1), (-1,-1)\}$.

A spin structure on a manifold X means, that the lift of the principal orthonormal frame bundle to a $Spin(n)$ principal bundle. $TX \to X$ the tangent bundle, from this we get the oriented orthonormal frame bundle, which is an $SO(n)$ bundle, and we take the fiberwise twofold cover. Or, if we can lift the cocycle structure defining the frame bundle to a spin group.



Theorem: X admits a spin structure iff $w_2(X) = 0$.

$0 \to Z_2 \to Spin(n) \to SO(n) \to 0$, we can view it as a principal Z_2 bundle, since $Z_2 = 0(1)$ this defines also a real line bundle. $H^1(SO(n), Z_2) = Z_2 \; n \geq 3$, so there are two cases, the spin group is the nontrivial one.

Complex line bundles are parametrized by $H^2(SO(n), Z) = Z_2 \; n \geq 3$, so there are also two complex line bundles, and we take $Spin^c(n)$ to be the nontrivial $U(1)$ bundle over $SO(n)$.

This is equivalent to having $Spin(n) \times S^1/Z_2$, where $Z_2 =< (1,1), (-1,-1) >$

We claim $Spin^c(3) = U(2)$, and $Spin^c(4) = \{(A, B) \in U(2) \times U(2) | \det A = \det B\}$

Theorem: X admits a spin structure iff $w_2(X) = 0$.

Also, the number of different Spin structures are parametrized by $H^1(X, Z_2)$.

Theorem: X admits a $Spin^c$ structure iff $\exists c \in H^2(X, Z) : c = w_2(X)$

missing stuff

For every (oriented) Y^3 its TY is trivial, so $w_2(TY) = 0$, hence every 3-manifold admits a spin structure. X a simply connected 4-manifold admits a spin structure iff Q_X is even (e.g. CP^2's intersection form is odd, so it doesn't have).

Theorem: Every closed oriented 4-manifold admits a $Spin^c$ structure.

Suppose Y^3 a 3-manifold (smooth closed oriented). It has tangent bundle trivial, so the principal bundle is also trivial. A $Spin^c$ structure is a $U(2)$ bundle over Y, or equivalently a complex plane bundle, there are two Chern classes, and it is determined by $c_1 \in H^2(Y, Z)$, since Y is 3dim, and $H^4 = 0$, so c_2 is also 0. $TY = L \oplus R$ as a complex line bundle, and a real trivial bundle, i.e. a nowhere zero section of TY, the orthogonal complement will be the complex line bundle.

Definition: A $Spin^c$ structure on Y is an equivalence class of nowhere zero vector fields where $v_1 \& v_2$ are considered equivalent if they are homotopic away from a point.

$TY \to Y$ is trivial, fix a trivialization, $TY = Y \times R^3$, a nowhere zero vector field is a map $Y \to S^2$. $[Y, S^2]$ can be understood by the Pontrjagin-Thom construction, explicitly this space is the same, as the framed 1-manifolds in Y up to framed cobordism.

> Take a map, smooth it locally, an inverse image of a regular point will be a 1-manifold in Y, and if we pull back the tangent space, we get a framing too.

The $Spin^c$ structure will be similar, but we forget about the framing.

Heegaard-Fleur theory will associate to $Spin^c$ manifolds all kinds of invariants. Next time we will do the $T_\alpha \cap T_\beta$ construction more generally, this will be called Heegaard states, every intersection gives a different $Spin^c$ structure.

5.

Y is a 3-manifold, a $spin^c$ structure is a nowhere vanishing vectorfield up to homotopy away from a point.

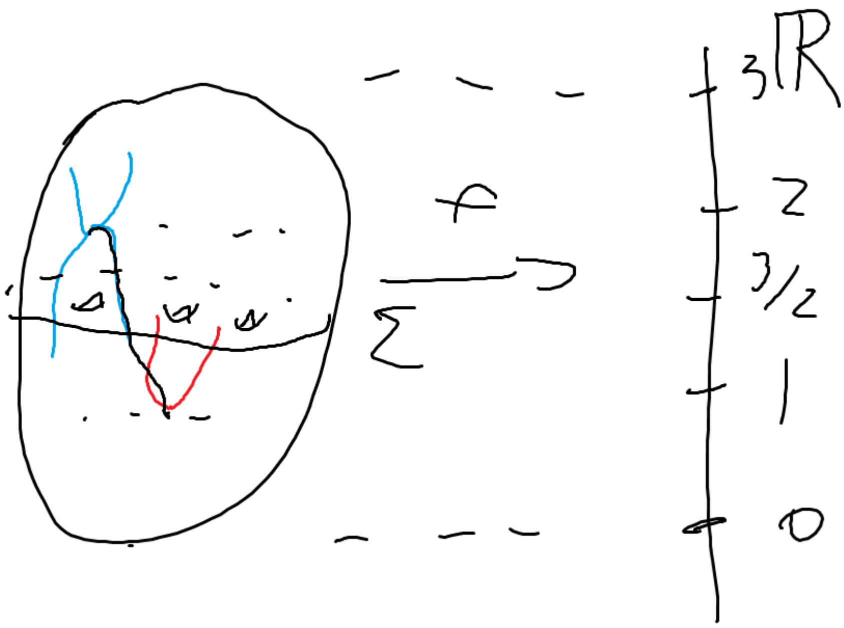
Σ, α, β Heegaard diagram, we considered the intersection points in $T_\alpha \cap T_\beta \subset symm^g(\Sigma)$.

Example, where the curves intersect in multiple points.

We claim, that such ai intersection point in the symmetric product gives ries to a $spin^c$ structure. Take a self-indexing Morse function, which has a unique local minimum and maximum.

If we fix a metric g on a manifold M, we get a vectorfield from f, namely $\nabla_g f$, defined by $< \nabla_g f, v >= v(f)$. We can consider $\gamma : \mathbb{R} \to M$, the gradient flow defined by $d\gamma/dt = (\nabla_g f)_{\gamma(t)}$. Its clear, that $f^{-1}(3/2) = \Sigma$, the surface of the Heegaard diagram. Consider the flowlines, which flow into the index 2 critical points, and which flow out of them, these will be the ascending, and the descending submanifolds of the critical points. We can consider the intersection of the ascending manifold with Σ, these will give us the blue β curves, and similarly with the α curves and the index 1 critical points.

Now we claim, that an intersection point of the curves in the symmetric product corresponds to a collection of flowlines connecting an index 1 and an index 2 critical point. Also every such intersection point determines a flowline (or a collection of them). This gives rise to a natural equivalence classes, namely two intersections x, y will be equivalent, if $\gamma_x - \gamma_y \in H_1(Y, Z)$, which we get by concatenating them, and orienting one of the “down”, the other one “up”, is zero in homology.



The self indexing Morse-function with the ascending and descending manifolds and Σ.

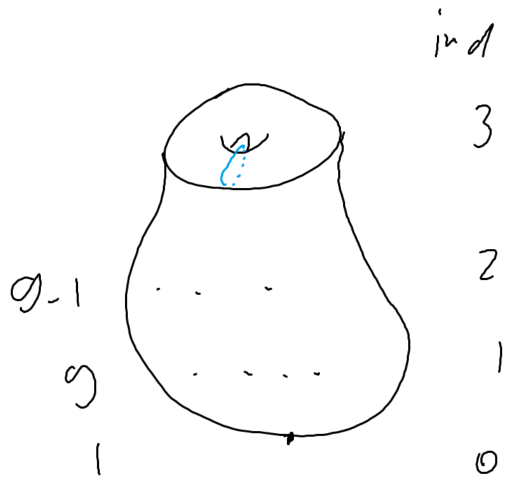
We want to associate to an intersection $\underline{\mathrm{x}}$ a $spin^c$ structure. Take a neighborhood of each trajectory. By the Poincaré-Hopf theorem, $\nabla_g f$ on the boundary of this neighborhood extends inside as a nonzero vector field, and thus we can remove the problems with the gradient along the 1-2 index flowlines. It is zero also in the minimum and maximum, to fix this, we pick a point away from α, β, the flowline of this connects the minimum and maximum, and we can fix is also to get a nonzero vector field. So from now on we consider Heegaard quadruplets Σ, α, β, w. If we change the metric, everything will change by a homotopy, and this $s_w(\underline{\mathrm{x}})$ $spin^c$ structure is defined really up to homotopy away from w.

Fact: $\underline{\mathrm{x}} \sim \underline{\mathrm{y}}$ in the sense as before, iff $s_w(x) = s_w(y)$

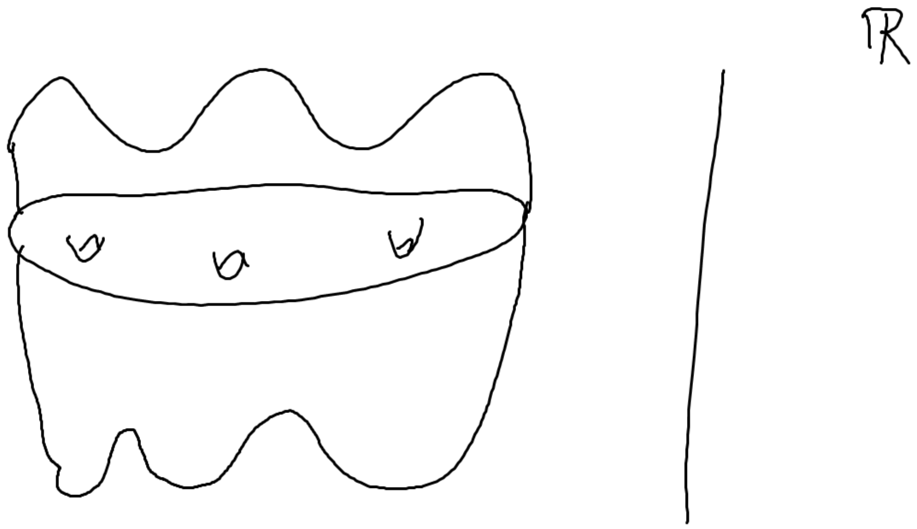
> 💡 $s_w(x)$ might depend on the choice of w.

Last time we saw, that 2 points in the complement of the curves gives a knot in Y. We see this more easily now, the upward flow through w and the downward flow through another point z concatenates to give an oriented knot in Y.

If Y, K are given, there is a $\Sigma, \alpha, \beta, w, z$ presenting it. Take $Y \setminus \nu(K)$, and pick a Morse function on it. We can suppose it has a constant maximum on the boundary, a unique minimum, and a couple of index 1 and 2 critical points. It will have g index 1 and g-1 index 2 to be exact. We pick another Σ between them, we get g red curves, and g-1 blue curves. The boundary will be a torus, we take a meridian on it, and flow it down along the gradient flow to Σ, to get the last blue curve, this tells us how we need to glue back the $D^2 \times S^1$ to close up the manifold, this gives back Y, since we glue the handle back on the meridian.



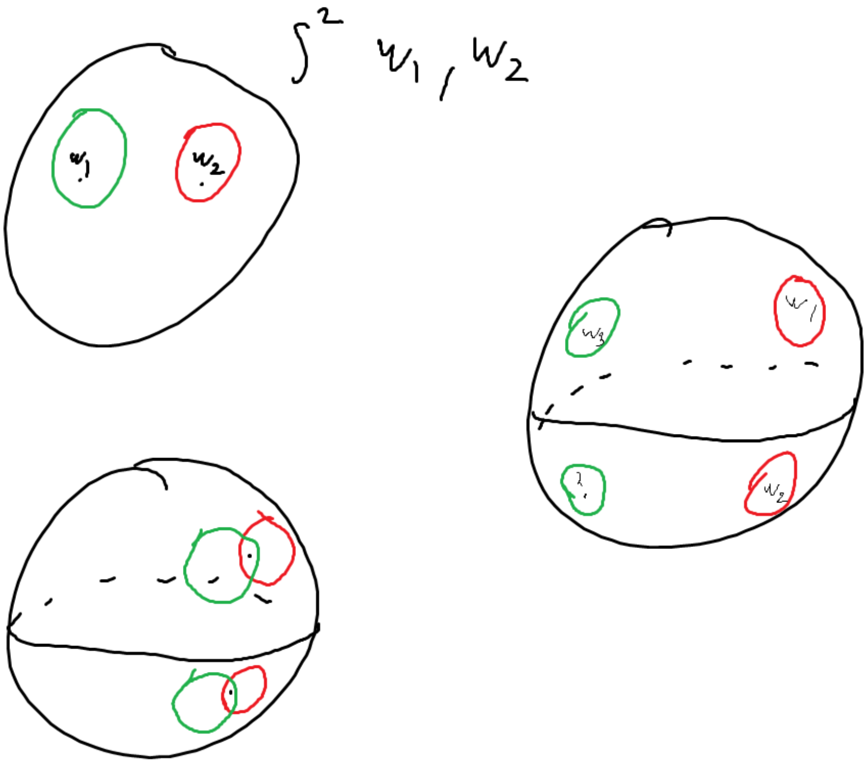
For links, assume we have another Morse function with k local min, and max. We get $g+k-1$ α curves, and the same amount of β curves, pairwise disjoint again. $<[\alpha_i]> \leqq H_1(\Sigma, Z_2)$, and we still request the this subspace is g dimensional, similarly with the β-s. If we require this property, it gives rise to a 3-manifold again, we get a disc with holes, the other curves cut it into components, and each component will correspond to a different minimum.



Multiple minima and maxima

Now we pick k pairs of points z_i, w_i such that any component of $\Sigma \setminus \alpha$ contains a unique z_i, w_i pair, and of course also for the β-s.

Example $Y = S^3, \Sigma = S^2$, $\alpha = \beta = \emptyset$, and any w will do, but for other diagrams there may be no way to satisfy these restraints.



Examples and counterexamples. For 2-2 disjoint circles the restraints cannot be satisfied, but if we move them close, it can easily.

> 💡 Consider S^3 with $n-1$ local min and max split along a torus, we exactly recover the grid diagrams from last semester.

Any dimensional manifold detour

A Morse function gives every smooth manifold a CW structure, which can be used to define CW homology on them. Consider the critical points of f, this is a finite set, graded by index. This gives rise to $CM(f) = \oplus_{x \in Crit(f)} F < x >= \oplus CM_i(f) = \oplus_{x \in crit(f)} \oplus_{ind(x)=i} F < x >$.

Pick a metric on M, consider $\nabla_g f$, consider intersections of descending and ascending manifolds. The coefficient of a critical point in the boundary of another in an index 1 higher is just the number of flowlines between them.

Differently, $M_{x,y} = \{\gamma : R \to M : \lim_{-\infty} \gamma = x, \lim_{\infty} \gamma = y, \text{flowline for } -\nabla_g f\}$, this is the space of flowlines between the two points.

Fact/Theorem: For generic $f \& g$, $M_{x,y}$ is a smooth manifold of dimension $\lambda(x) - \lambda(y)$.

$M_{x,y}$ admits an R action, which is free, if $x \neq y$ by translation. We can consider $\hat{M}_{x,y}/R$, also a smooth manifold of dimension$\lambda(x) - \lambda(y) - 1$.

Definition: The Morse-Smale cell complex $(CM(f,g), \partial)$, where $\partial x = \sum_{y \in Crit(f), \lambda(x)-\lambda(y)} \#\hat{M}_{x,y} \cdot y$.

$\hat{M}_{x,y}$ will be compact, since in general it can be compactified by broken trajectories. The space of broken flow lines will be given the local compact open topology.

We want now, that $\partial^2 = 0$, and that H_* is independent of the choice of f and g, and is an invariant of M.

$\partial^2 x = \sum \hat{\#} M_{x,y} \partial y = \sum \#\hat{M}_{x,y} \sum \hat{\#} M_{y,z} z$, so we need the number of points of $\hat{M}_{x,y} \times \hat{M}_{y,z}$. Consider $\hat{M}_{x,z}$, where the difference of indices is 2 now, and this space can be compactified, and a compact 1-manifold has an even amount of boundary points.

Now independence from f and g. We focus on g, the critical points, and thus the modules stay the same, but the boundary map changes. We want a $\phi_{01} : CM(f, g_0) \to CM(f, g_1)$ and a similar ϕ_{10} such that these are chain maps, and chain homotopy equivalences, so we need also $H_0 : CM(f, g_0) \to CM(f, g_0)$.

Consider a g_t path of Riemannian metrics between g_0 and g_1. We ca also extend this to R by keeping constant after 1 and before 0.

Consider $M^{g_t}_{x,y} = \{\gamma : R \to M : x \xrightarrow{\gamma} y, d\gamma/dt = (-\nabla_{g_t} f)_{\gamma(t)}\}$, this will be still a smooth manifold of dimension $\lambda(x) - \lambda(y)$. $\phi_{01} x = \sum \# M^{g_t}_{x,y} \cdot y$ where the sum is over ys of equal indices as x, all critical points.

Motivation for Floer homology

Suppose $f : X \to X$, we want to see, if it has a fixpoint. Consider $\Lambda(f)$, the Lefschetz number, defined by $\sum (-1)^i Tr f_*|_{H_i(X,R)}$, if $\Lambda(f) \neq 0$, then f has a fixed point. Consequently, if f is homotopic to the identity, and $\chi(X) \neq 0$, then f has a fixed point.

Suppose M a smooth manifold, $\omega \in \Omega^2(M)$. This 2-form ω is symplectic, if on each tangent plane it is a symplectic form, ergo a nondegenerate skew-symmetric bilinear form (this can only happen on even dimensional vector spaces). This is also equivalent to saying, that $\omega \wedge \cdots \wedge \omega$ the n-fold product is nonzero, if M is oriented, then we assume this volume form is positive. The other condition is that $d\omega = 0$, that it is closed. From the second assumption, we get that $[\omega] \in H^2_{dR}(M, R)$, and the first says, that $[\omega]^n \neq 0$ in $H^n_{dR}(X, R)$, so $[\omega] \neq 0$ in the second cohomology, thus it is nonzero. For example, only S^2 is symplectic, with the area form.

For a submanifold $N \subset (M, \omega)$, it can happen, that $(N, \omega|_N)$ is symplectic. A restriction of a closed form is still closed, the first condition can fail however. These are called *symplectic submanifolds*. If $\omega|_N \equiv 0$, then we call the submanifold *isotropic*. In this case $[\omega] \in H^2(M, N, R)$. An isotropic submanifold satisfies $\dim N \leq \dim M/2$. If a submanifolds achieves equality here, we call it *Lagrangian*.

Arnold's conjecture: Some maps $f : (M, \omega) \to (M, \omega)$ have way more fixed points, than expected.

Definition: $\phi : (M, \omega) \to (M, \omega)$ is a *Hamiltonian diffeomorphism*, if there exists $H : R \times M \to R$ such that for the vector field X_t satisfying $i_{x_t}\omega = dH_t$ such that $\phi(x) = \Phi_1(x)$ where Φ_1 is the time 1 map of the flow generated by X_t, and $\phi^*\omega = \omega$. Here $i_{X_t}\omega = \omega(X_t, \cdot)$, and $H_t(x) = H(t, x)$.

Now more precisely the conjecture is $|Fix(\phi)| \geq \sum b_i(M)$. Notice, from Lefschetz, we would only get $\geq \sum (-1)^i b_i(M)$.

Floer's idea

Consider the graph $Gr(\phi) \subset M \times M$, the fixpoints are the diagonal $\Delta \cap Gr(\phi)$. $Gr(\phi)$ and Δ are Hamiltonian isotopic if ϕ is a Hamiltonian diffeomorphism, and both are Lagrangian.

Floer proposed to find a homology theory for pairs of Lagrangian submanifolds $L_0, L_1 \subset (X, \omega)$ such that $HF(L_0, L_1) = H_*(CF(L_\S, L_1))$ generated by the intersection points. This might require some choices, but the homology should be independent of these. This should be invariant under Hamiltonian isotopy, and lastly $HF(L, L) = H_*(L, R)$. This would be enough to solve Arnold's conjecture.

We are given $((M,\omega),L_0,L_1)$ a symplectic manifold and two Lagrangian submanifolds. Consider $\Gamma=\{\gamma:I\to M|\gamma\in C^\infty M,\gamma(0)\in L_0,\gamma(1)\in L_1\}$. This is an "infinite dimensional manifold", and there is a function $A:\Gamma\to R$, which will play the role of a Morse function, such that the critical points of A will be the intersections $L_0\cap L_1$, a critical point is nondegenerate iff $L_0 \pitchfork L_1$ there. We have to figure out the gradient flow equation.

Later we will run this machine for $M=Symm^g(\Sigma),\ L_0=T_\alpha,\ L_1=T_\beta$ to get an invariant of a 3-manifold.

6.

(M,ω) a symplectic manifold, and $L_0,L_1\subset M$ Lagrangian submanifolds ($\omega|_{L_i}=0\&\dim L_i=\dim M/2$) , and $L_0\pitchfork L_1$.

Idea: Set up a chain complex $CF(L_0,L_1)$ generated by $L_0\cap L_1$ and appropriate ∂, so $HF(L_0,L_1)=H(CF(L_0,L_1),\partial)$ is an invariant (under Hamiltonian diffeomorphism).

Take $V=\{\gamma:I\to M|\gamma(0)\in L_0,\ \gamma(1)\in L_1\}$ the space of smooth paths between the two submanifolds. The Morse function will be $A:V\to\mathbb{R}$, but we will not define it lol.

Suppose M,ω is an exact symplectic manifold, ergo ω is an exact 2-form, i.e. $\omega=d\alpha$.

This is very restrictive, this means for example that M is not closed.

> This is true for example if M is a *Stein* manifold, this means that it has a complex structure, and a **plurisubharmonic** function ϕ, and we can choose $dd^c\phi=\omega$.

Assume also, that L_i is also exact, ergo $\alpha|_{L_i}=df_i$, where $f_i:L_i\to\mathbb{R}$. In this case we define $A(\gamma)=f_0(\gamma(0))-f_1(\gamma(1))+\int_0^1\gamma^*\alpha$.

Observation: If $\gamma_0 \sim \gamma_1$ are homotopic paths (with the boundary conditions), let u be the homotopy between them. If we compute $A(\gamma_0) - A(\gamma_1) = \int_{I\times I} u^*\omega$ by Stokes' theorem. We are interested in the derivative $d/dtA(u|_{t\times I}) = \partial_t|_{t=0} \int u^*\omega$. We will demand this in the general case, there may or may not be a function inducing it.

Suppose that $u : [-\epsilon, \epsilon] \times I \to M$ is variation of γ. The derivative is

$$\partial_t|_{t=0} \int_{(s,t)\in[-\epsilon,\epsilon]\times I} u^*\omega = \partial_t|_{t=0} \int \omega(\partial_t u, \partial_s u) dtds.$$

For the constant path, this is 0. Also if $\forall v\ \partial_t|_{t=0} \int \omega(v, \partial_s u)dtds = 0$, then $\partial_s u \equiv 0$.

Suppose (M, ω) symplectic. A metric on TM is the same as reducing $GL_n(R)$ to $O(n)$. Picking an orientation is equivalent to reducing $GL_n(R)$ to $GL_n^+(R)$, if we do both, we reduce the structuregroup to $SO(n)$. Similarly $Sp(2n) = \{A \in M_n | A \text{ respects } \omega\}$, and giving a syimplectic form reduces the structure group to this.

<u>Definition</u>: An *almost complex structure* on a smooth manifold M is a map $J : TM \to TM$ covering the identity such that $J \circ J = -id_{TM}$. This gives a complex structure to every tangent space.

Similarly M is a *complex manifold*, if we have a covering U_α, and maps $\phi_\alpha : U_\alpha \to C^n$ such that $\phi_\beta \circ \phi_\alpha^{-1}$ is a holomoprhic map. Observe that every complex manifold has an almost complex structure, but not in reverse.

This also means, that the structure group of TM reduces to $GL_n(C)$.

<u>Fact</u>: $Sp(2n) \cap O(2n) = SP(2n) \cap GL_n(C) = GL_n(C) \cap O(2n) = U(n)$, so specifying 2 of the above 3 gadgets (g, J, ω) implies the third one.

> 💡 This doesn't mean, that every almost complex manifold has a symplectic structure, since the magic equation $d\omega = 0$ might not be true.
> For example because $[\omega] \in H^2(M, L_0 \cup L_1; R)$ is a nonzero class.

In $\dim = 2$ every oriented manifold admits a complex structure, not just an almost complex one.

For 3-manifolds its easy again, for 4-manifolds:

<u>Theorem</u> (Hirzebruch-Hopf): X^4 closed oriented 4-manifold iff $\exists\, c \in H^2(X, Z)$ such that $c \equiv w_2(TX) \mod 2$, and $c^2 = 3\sigma(X) + 2\chi(X)$.

> The tangent bundle can be reduced to a $TX \xrightarrow{U(2)} X$ bundle, $c_2(X) = e(x)$, and we will choose $c_1(X) = c$. In the other direction we will need that $p_1(TX)([X])/3 = \sigma(X) \in Z$, which is a theorem of Rokhlin/Hirzebruch.

<u>HW#1</u>: Show that $X = CP^2 \# CP^2 \# CP^2$ admits an almost complex structure, using the Hirzebruch-Hopf theorem.

<u>Fact</u>: X admits no symplectic structure.

<u>HW#2*</u>: Show that $CP^2 \# 2C\bar{P}^2 = S2 \times S^2 \# C\bar{P}^2$ by using the Kirby diagrams below.

Kirby diagrams representing the two manifolds.

Pick a compatible (meaning $\omega(v, w) = -g(v, Jw)$) Riemannian metric on M, call it g. This gives us an almost complex structure on M also. This induces a Riemanian metric (formally) on V. A tangent vector in V is a map $v : I \to TM$ over the point $\gamma \in V$ (complying with the boundary conditions also, so $v(i) \in TL_i$). The induced inner product will be $< v, w >= \int g(v, w)dt$.

Now we write the gradient flow equation.

$$\partial_t|_{t=0} A(s \mapsto u(t, s)) = \int \omega(v, \partial_s u) ds$$

We are looking for a 1-parameter family of paths in M, i.e. a map $u : I \times R \to M$ such that $\{i\} \times R \mapsto L_i$ and when $t \to \pm\infty$ $u(s, t) \to x, y \in L_0 \cap L_1$.Using the relation between the metric and the symplectic form the equation will be $\partial_t u + J\partial_s u = 0$, which is a more general form of the Cauchy-Riemann equations.

<u>Definition</u>: Suppose M, ω compact symplectic, L_0, L_1 transverse Lagrangian submanifolds. We define $CF(L_0, L_1) = \oplus_{x \in L_0 \cap L_1} F < x >$, and $\partial : CF \to CF$ given by

$$x \mapsto \sum_{y \in L_0 \cap L_1} \sum_{\phi \in \pi_2(x,y),\, \mu(\phi)=1} \#\hat{M}(\phi) \cdot y$$

Fix an almost complex structure J on M, ω, consider

$$\{u : I \times R \to M | \partial_t u + J\partial_s u = 0,\ \{i\} \times R \subset L_i,\ u \xrightarrow{\infty} y,\ u \xrightarrow{-\infty} x\},$$

which will be denoted by $M(x, y)$. Observe that if there is a solution, we can shift it in time to get another solution. Now look at $\hat{M} = M/\mathbb{R}$ the factor by this action.

> 🤯 $\Lambda^0\mathbb{R}^0 = \mathbb{R} < \emptyset >$

u is a Whitney strip if satisfies everything in the definition of M except the CR equations. If $u \in M$ it is a (pseudo-)holomorphic Whitney strip. These are biholomorphic to the complex unit disc with $\pm i$ removed. So equivalently $\{\mathbb{D} \to M | u(i) = y,\ u(-i) = x,\ u(\partial\mathbb{D} : re < 0) \subset L_0\ u(\partial\mathbb{D} : re > 0) \subset L_1\}$ are the Whitney discs, we can assume the CR equations also to get the holomorphic ones. This will be similar to the Whitney trick but with holomorphic discs, the term comes from here at least.

Problems:

- Why is M a manifold?
- What is the dimension? This will be μ, the change in the number of positive to negative eigenvalues of the Hessian.
- Is $\hat{M}$ compact?

Consider m, the space of u-s that solve the CR for some a.c.s. J. We have a mapping $m \to j$ to the space of a.c.s.-s. If m, j are finite dimensional we have to look for regular values. We have to upgrade Saard's lemma to the Saard-Smale theorem to cover the infinite dimensional case.

Theorem: If j^p denotes the paths of a.c.s.-s on M. We can modify the CR equations so the solutions satisfy the equation with complex structure at time t $j(t)$, where $j \in j^p$ is a path. These will give the m^p space. Now applying SS to the j^s pseudoholomoprhic solutions the space M_{j^s} is a smooth manifold.

$A : V_1 \to V_2$ a linear map, what is $\dim ker A =$? The index of A, defined as $\dim Ker - \dim Coker$ can be followed much easily. In finite dmension this is

$\dim V_1 - \dim V_2$.

Definition: $F : V_1 \to V_2$ is *Fredholm*, if imF is closed, and $\dim ker\ F, \dim coker\ F < \infty$.

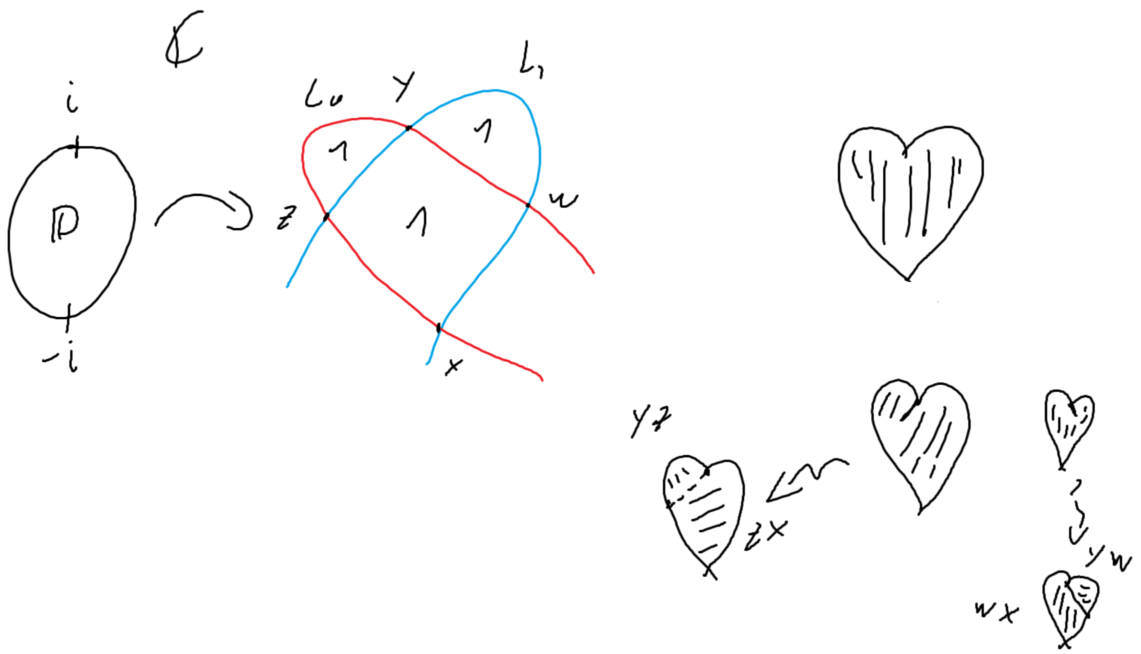
Fact: The space of Whitney discs might not be connected. $\pi_2(x, y)$ is defined as the space of homotopy classes of Whitney discs. So now we have to refine the previous definition of the boundary operator.

$$x \mapsto \sum_{y \in L_0 \cap L_1} \sum_{\phi \in \pi_2(x,y),\ \mu(\phi)=1} \#\hat{M}(\phi) \cdot y$$

This number is called the Maslow index of ϕ.

How can things degenerate?

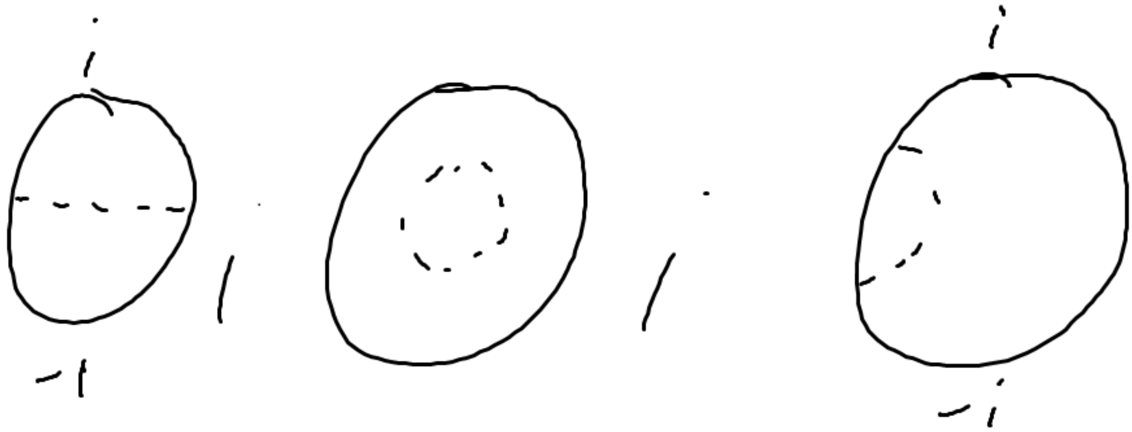
$\mathbb{C}$ equipped with the standard structures to be a symplectic manifold. There are 2 1-parameter families of discs onto the broken heart.

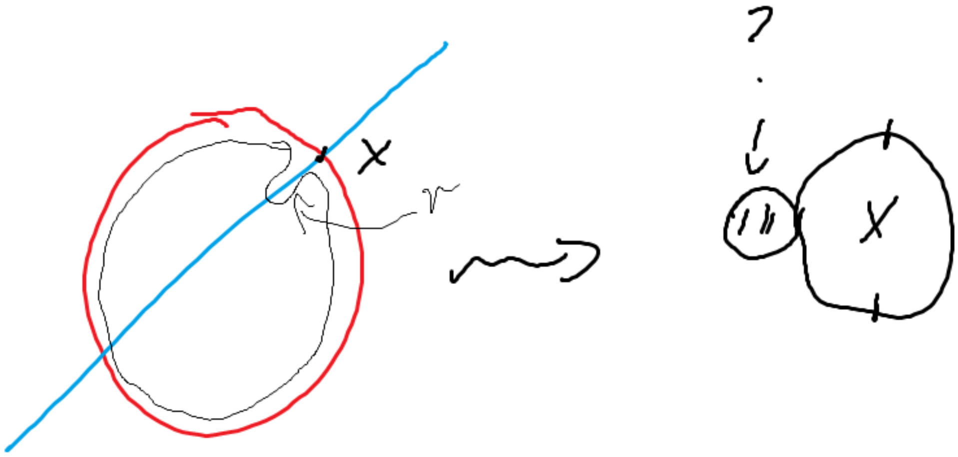


The limit of the 'flowlines' of Whitney discs breaks the heart in two.

The disc can break in two inside, or by a line going from one component of the boundary to the other, or to the same. This last possibility is bad, since the flowlines doesn't converge to a broken flowline.

The second one is called a sphere bubble, the other two are the boundary bubbles.





Gromov's theorem: Adding these degenerations we get a compact space.

Theorem (Floer): M, L_0, L_1 as before, also $\pi_2(M) = 0, \pi_2(M, L_0) = 0, \pi_2(M, L_1) = 0$. Then for a sufficiently generic path $\{j^s\}$ of a.c.s.-s and $\phi \in \pi_2(x, y)$ the moduli space $M(\phi)$ is a smooth manifold of dimension $\mu(\phi)$, and $\hat{M}$ is compact if $\mu(\phi) = 1$.

> 💡 The first excludes sphere bubbles, since there would be $[\omega]([S^2]) > 0$, but every sphere is nullhomotopic. Here we used that $d\omega = 0$! The other two excludes the boundary bubbles on the same component, here we also need, that $[\omega]$ is also a relative homology class.

Therefore ∂ "exists", and $\partial^2 = 0$.

$$x \mapsto \sum_{y \in L_0 \cap L_1} \sum_{\phi \in \pi_2(x,y),\, \mu(\phi)=1} \# \hat{M}(\phi) \cdot y$$

We have another proble, now we know that the coeficients exists, but we dont know whether the $\pi_2(x, y)$ are finite!

7.

Claim: $\#_n CP^2$ admits an a.c.s. iff n is odd, and a symplectic iff n=1

We only address the first claim. By the theorem from previous class we need a $c \in H^2(X, Z)$ such that $c \equiv w_2 \mod 2$, and $c^2 = 3\sigma(X) + 2\chi(X)$.

$H^2(\#_n CP^2) = Z^n$ is clear, since each CP^2 has 1 2-cell. The Euler characteristic will be $2 + n$, also the intersection form will be the $n \times n$ identity, so $\sigma(\#_n CP^2) = n$, so by the formula we get $c^2 = 5n + 4$. The characteristic class w_2 satisfies $Q_X(\alpha, \alpha) = w_2(\alpha_2)$, where $\alpha \in H^2(X, Z)$, and α_2 is the $\mod 2$ reduction, we call elements satisfying this identity, *characteristic elements*.

> Every H_2 element can be represented by embedded surfaces. $TX|_\Sigma = T\Sigma \oplus \nu\Sigma$, and use the product formula. $w_1(T\Sigma)w_1(\nu\Sigma) + w_2(T\Sigma) + w_2(\nu\Sigma)$, the first term is 0, since the spaces are orientable, the second is also 0, since the Euler characteristic is even, the third one is exactly the self intersection number.

We need a c such that $PD(c) = C \in H_2 < a_1, \dots, a_n >$ and satisfying $Q(a_i, a_j) = \delta_{ij}$. There is only one constraint on $C = \sum n_i a_i$, namely that all the n_i-s should be odd. Replace them by $2m_i - 1$, now we need $\sum(2m_i - 1)^2 = 5n + 4$. Reducing we see $4(n+1) \equiv 0 \mod 8$, ergo n should be odd.

In the other direction, we need to give an appropriate class. Take $\sum_1^m 3a_i + \sum_{m+1}^n a_i$, the square sum will be $9m + m - 1 = 10m - 1 = 5(2m-1) + 4$, so this will work for us.

> Gromov's h-principle: if the homotopic constraints for a geometric object are satisfied, then the geometric object also exists.

Y^3 a 3-manifold, presented as a Heegaard diagram $H = (\Sigma, \alpha, \beta, w)$, or maybe a 3-manifold with a knot inside, given by (H, α, β, z, w). To this we associate $Sym^g(\Sigma)$.

> $Sym^2(S^1) = \mu$, and $Sym^2(\Sigma_2) = T^4 \# C\bar{P}^2$

This symmetric power will be a Kähler manifold, whatever that may mean. The product of the αs will give a torus in this space, T_α, T_β, and a point gives a divisor $Sym^{g-1}(\Sigma) \times \{w\} := V_w$. The tori are g dimensional of course, and the divisor is $2g-1$, disjoint from the two tori.

$\hat{CF}, CF^-, CF^\infty, CF^+$ will be 4 versions of Flour homology.

1. $\hat{CF} = \oplus_{x \in T_\alpha \cap T_\beta} F < x >$
2. $CF^- = \oplus_{x \in T_\alpha \cap T_\beta} F[u] < x >$
3. $CF^\infty = \oplus_{x \in T_\alpha \cap T_\beta} F[u, u^{-1}] < x >$

We can get CF^∞ by $CF^- \otimes F[u, u^{-1}]$, in the other direction we need a filtration, and take u in $\deg u = -1$, and the filters of degree less than zero.

The final one sits in the exact sequence $0 \to CF^- \to CF^\infty \to CF^+ \to 0$. This has a strange base ring $F[u, u^{-1}]/F[u]$, every element is torsion.

Now we need the boundary maps. $\partial : \hat{CF} \to \hat{CF}$, where

$$\hat{\partial} x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y),\, \mu(\phi)=1,\, \phi \cap V_w = 0} \# \hat{M}(\phi) y.$$

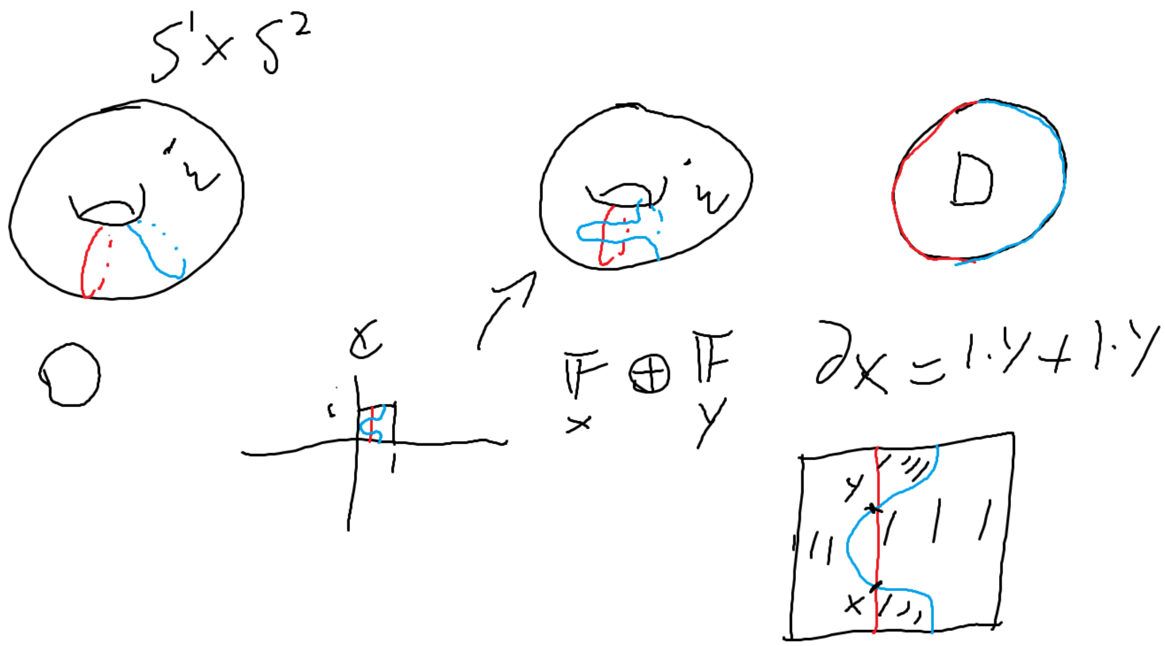
Here $M(\phi) = \{u : D \to Sym^g(\Sigma)\}$ with the appropriate boundary conditions, without the CR equations, so we need to pick a family of a.c.s-s on the symmetric power. This is still no good, we don't know if there are only finitely many homotopy types of maps from the disk to the symmetric power.

For CF^- we use the same setup. $\partial^- : CF^- \to CF^-$ defined by

$$\partial^- x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y),\, \mu(\phi)=1} \# \hat{M}(\phi) u^{n_w(\phi)} y.$$

Here $n_w(\phi) = \# \phi \cap V_w$. Other complications arise, we could get infinitely many intersection points, which is trouble also, or get infinitely many terms, but that could be solved by considering everything over $F[[u]]$.

HW#1: $C =< a, b >$, and $\partial a = (1 + u^n)b,\ \partial b = 0$. Compute homology over $F[u]$ and over $F[[u]]$.



Example, two different diagrams for the same manifold with different homologies ☹

$H = (\Sigma, \alpha, \beta, w)$ given. $\Sigma \setminus \alpha \setminus \beta$ has finitely many components, the closures of which are called elementary domains, denoted by $\mathcal{D}_i$.

Definition: A domain in H is a finite linear combination of elementary domains.

> 💡 To each ϕ we can associate such a domain, called the *shadow*. The coefficients are given by $n_{w_i}(\phi)$, where $w_i \in \mathcal{D}_i$.

At every point $p \in \alpha_i \cap \beta_j$ we can associate 4 numbers a_p, b_p, c_p, d_p, the multiplicities of the 4 domains which meet at this intersection point.

Definition: A *cornerless* domain $D = \sum n_i \mathcal{D}_i$ satisfying the equation $a_p + c_p = b_p + d_p$ for all p.

We claim, that every cornerless domain provides a 2 homology class. The boundary of a domain is a collection of α, β segments. Being cornerless is equivalent to having $\partial \mathcal{D}$ the union of complete α and β circles, after attachment of the handles, this boundary will be capped off, and we get a true embedded surface.

Definition: $\tilde{p}$ will denote the set of all cornerless domains. $n\Sigma$ will denote the subgroup $\sum n\mathcal{D}_i$.

We have a SES $0 \to n\Sigma \to \tilde{p} \to H_2(Y) \to 0$.

H_2 of a 3-manifold is always free, since it is equal to $Hom(H_1, Z)$ plus some other factor which depend on H_0, which is free, so this will give zero, from the universal coefficient theorem.

Definition: D is a *periodic* domain if it is cornerless, and $n_w(D) = 0$ ($D = \sum n_i D_i$, if $w \in D_j$, the $n_w(D) = n_j$). The set of these are denoted by p.

Definition: $H = (\Sigma, \alpha, \beta, w)$ is admissible, if each nonzero periodic domain has both positive and negative coefficients.

> 💡 Notice, this property distinguishes our two diagrams from before. In the first diagram all domains are cornerless, the periodic domain is the one which don't contain w, so the condition fails.
> On the second diagram we get $n(D_1 - D_2)$, so there it is satisfied.

$\tilde{p} = Z \oplus H_2(Y)$, so $p = H_2(Y)$, and if Y is a rational homology sphere, then all Heegaard diagrams of Y are admissible.

Lemma: If H is admissible, then there is a metric on Σ such that the area of each periodic domain is 0.

Claim: If $M(\phi) \neq \emptyset$, then $S(\phi)$ has only nonnegative coefficients.

From the assumption we get that the homotopy type has a 'holomorphic' representative, the coefficients of the shadow are intersection numbers, which since we are working with complex manifolds, are positive.

> ❗ This is of course not true, we need the same construction on almost complex manifolds, and almost complex submanifolds, also we have a path of almost complex structures.

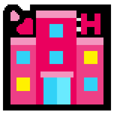
Claim: If H is admissible, and $x, y \in T_\alpha \cap T_\beta$, then there are only finitely many ϕ such that $S(\phi)$ has only nonnegative coefficients.

Pick a ϕ from x to y. For all the others consider $\phi - \psi$, and we want to argue, that $S(\phi - \psi_i)$ are periodic domains. $n_w = 0$, since we only consider those classes, which play in the definition for $\hat{\partial} x$. Their boundary will be a complete collection of α, β circles, to which we will return later.

Since they have both positive and negative coefficients, we want to prove, that with finitely many exceptions $S(\psi)$ has positive and negative coefficients also.

Assume there are infinitely many ψ_is, so at least the positive, or the negative coefficients go to $\pm\infty$, also both have to, since otherwise the signed area sum cannot stay zero. Moreover, we can assume, that there is one domain D, such that its coefficients go to ∞. Looking at the identity $S(\psi_i) = S(\phi) - S(\phi - \psi_i)$ we see, that $S(\psi_i)$ cannot have positive coefficients at the D coordinate.

So we get, that $\hat{\partial}$ is actually well defined, and also ∂^-, when we take it over $F[[u]]$, for the last one, we need another condition.



8.

Recall Σ, α, β, z gives you a 3-manifold, or if we have 2 points we get a 3-manifold together with a knot inside it.

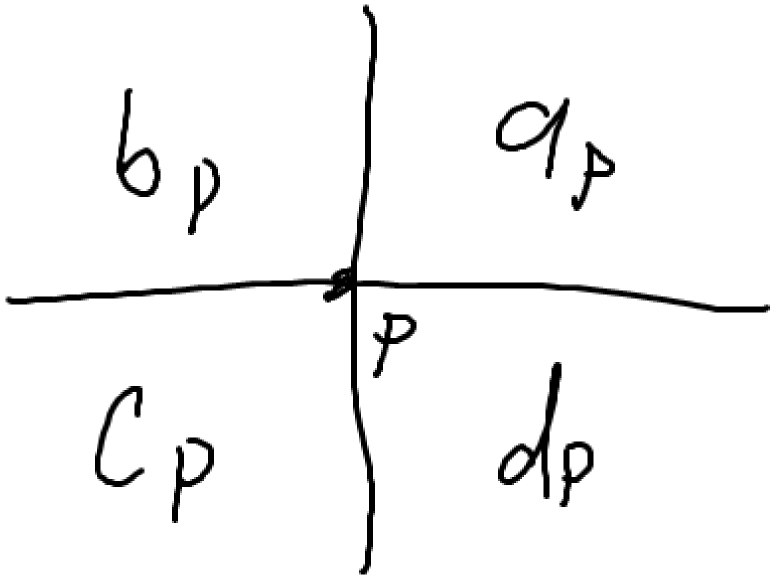
$\Sigma \setminus \alpha \setminus \beta = \cup \mathcal{D}_i$ gives you the elementary domains, this is loosely like a CW decomposition. A domain is a formal linear combination of these. We called a domain cornerless, if at every crossing (the domain specifies numbers for every elementary domain meeting at that crossing) the diagonal elements sum to the same number. These are the domains, which are bounded by complete α and β curves, these give us a real H_2 element. Σ is homologous to 0 in Y, so we want to get rid of the domains which have every elementary domain with the same multiplicity. A domain is periodic, if it is cornerless, and the multiplicity of the elementary domain containing w is 0 ($n_w(D) = 0$), this gives an isomorphism between periodic domains and the second homology.

If we have a map $u : D \to Sym^g(\Sigma)$ with the usual boundary conditions, one component going to T_α, the other to T_β gives rise to its shadow, defined by $S(u) = \sum n_{w_i} D_i$, where $n_{w_i} = \# u(D) \cap \{w_i\} \times Sym^{g-1}(\Sigma)$, the number of intersection points with the orbit of $w_i \in D_i$ in the symmetric product.

u represents a homotopy class ϕ of maps $D \to Sym^g(\Sigma)$, and the above shadow is independent of the choice of representative, and the choice of $w_i \in D_i$.

Definition: Suppose $x, y \in T_\alpha \cap T_\beta$ are two Heegaard states $D(x,y) := \{\sum n_i D_i | a_p + c_p = b_p + d_p + \chi(p \in x) - \chi(p \in y) \forall p \in \alpha_I \cap \beta_j \forall_{i,j}\}$.

Recall, that x, y are g-tuples of points in Σ.



The local picture with the multiplicities at an intersection point p

$D(x,x)$ are the cornerless domains. There is a map $D(x,y) \times D(y,z) \to D(x,z)$ by simple addition. This means also that $D(x,y)$ is an affine space for the group of cornerless domains ("torsor").

$x, y \in T_\alpha \cap T_\beta$ Heegaard states, we can associate to them an $\epsilon(x,y) \in H_1(Y, \mathbb{Z})$

ϵ is defined as follows. The Heegaard diagram corresponds to a Morse function, the index 1 critical points flow up to the α circles, and similarly for the index 2 points (after picking a metric). A Heegaard state in this picture corresponds to a flowline from an index 2 to an index 1 critical points, or rather a g-tuple of these, giving a bijection between the critical points. Call this γ_x, and do the same construction for y, and consider $\gamma_x - \gamma_y$, an embedded oriented 1-manifold, and so a homology class in $H_1(Y, Z)$.

From the diagram $\epsilon(x,y) = \sum \xi_i - \eta_i$, where ξ_i is a path from x_i to y_i on α_i, and similarly η_i connects x_i to $y_{\sigma(i)}$ on $\beta_{\sigma(i)}$. This depends on some choices on Σ, but in Y these choices (going around some α many many times for example) won't matter, since we factor out by them in $H_1(Y)$.

Theorem: $D(x,y) = 0$ iff $\epsilon(x,y) \neq 0$. If $\epsilon(x,y) = 0$, then $D(x,y) = \phi + D(x,x)$, where $\phi \in D(x,y)$, so it is a 1-dimensional affine space over the cornerless domains.

If $b_1(Y) = 0$, then $D(x, x) = Z \oplus H_1(Y) = Z$, and in this case... we will talk about this later.

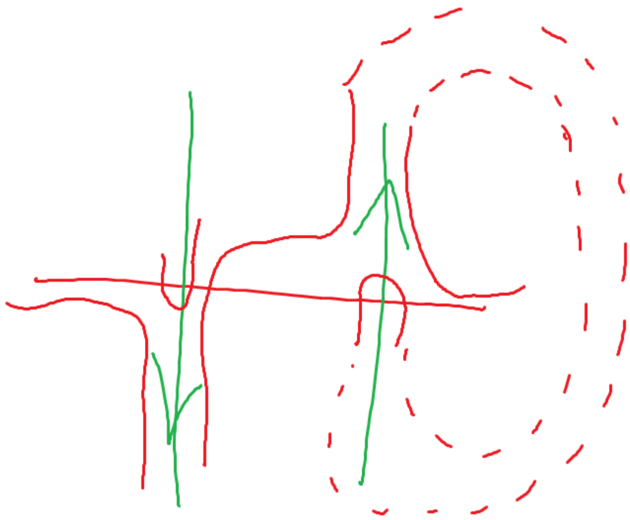
Recall, that we call a pointed Heegaard diagram *admissible*, if for every nontrivial periodic domain has both positive, and negative coefficients.

If $b_1(Y) = 0$, then any diagram is admissible. Also, any diagram can be isotoped to become admissible.

> The space of periodic domains never change under isotopy of a diagram, it only depends on the homology of Y, but we can make *more* elementary domains with isotopy, so at least we can hope, that with more degrees of freedom.

> ## Nobody understands how iphones work.

By a diffeomorphism, we make the αs look standard, the others will be arbitrary complicated horribleness. The αs have dual curves (meridian-longitude) call these δ. Take 2 copies of these, oriented differently, and rotate the α curve around and back, in both directions.



One spinning of an alpha around its dual curve.

<u>Theorem</u>: For a pointed Heegaard diagram, and a choice of dual curves, there is an N, the N-fold spinning of the diagram will become admissible. N depends on the number of intersection points, and b_1 of the manifold.

Now we can start considering $(\hat{CF}, \hat{\partial})$, and $(\mathbb{CF}, \partial)$ the hat theory (over F) and the boldface theory (over $F[[u]]$).

Definition: $x, y \in T_\alpha \cap T_\beta$ are equivalent iff $\epsilon(x, y) = 0$.

$(\hat{CF}, \hat{\partial})$ splits according to this equivalence relation, ergo can be written as $\oplus_{s \in spin^c(Y)} (\hat{CF}(H, s), \hat{\partial})$.

Fact: $x \sim y$ in the above sense iff they represent the same $spin^c$ structures. Remember that there is a map $s_w : T_\alpha \cap T_\beta \to spin^c(Y)$.

These homology groups will be graded as well. For the boldface theory $grad(u) = -2$. The grading itself will no be so simple, we might have grading by Z or Q, or some cyclic group.

To an intersection point $x \in T_\alpha \cap T_\beta$, and a point $w \in \Sigma$ we can associate a nowhere zero vector field (not just a $spin^c$ structure). In 3-mainfolds a vector field, and an oriented 2-plane bundle are the same, by orthogonal complement, which is actually a complex line bundle, and $TY = \mathcal{L} \oplus \epsilon^1$, this is called an almost contact structure on a 3-manifold.

Fact: $\forall Y^3 \exists X^4$ compact such that $\partial X = Y$.

If X is almost complex, then ∂X admits an almost contact structure, i.e. TY splits as a complex line bundle, and a trivial real line bundle. The construction is the following $\mathcal{L} = TY \cap JTY$, where J is the almost complex structure on X

Fact: $\forall (Y, \mathcal{L} \oplus \epsilon^1) \exists (X, J)$ such that $\partial(X, J) = (Y, \mathcal{L} \oplus \epsilon^1)$.

Start with x, get a $v_{x,w}$ vector field, take the orthogonal complement, and get some (X, J) 4-manifold for it. Take its first Chern class, and consider $\frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X))$, this doesn't make sense! $H^4(X) = 0$, since it has a boundary, also $[X] \in H_4(X, \partial X; Z)$.

$$H^2(X, \partial X) \to H^2(X) \to H^2(\partial X)$$

If c_1 comes from the relative cohomology, then we can make sense of everything written above and integrate on the fundamental class. For this we need, that $c_1|_{\partial X} = 0$. We can consider everything over Q, but now integrating against the fundamental class will give us some rational number.

This splits the $spin^c$ structures into 2 parts, those which have torsion c_1, and those which have non-torsion c_1, the above construction works for the first type.

Suppose we have a $spin^c$ structure with torsion c_1. Take X, J with boundary Y, and Z_1, J_1 with boundary $-Y$. Take another almost complex manifold X', J'. We can patch together X, J and Z_1, J_1 to get a closed a.c. manifold V, J_V. The signature, Euler characteristic and c_1^2 is additive. The first is a theorem of Novikov, the second is trivial, provided that 3-manifolds have 0 characteristic, and the third one is some magic with integrals. This means, that the number defined above makes sense, we get the same no matter what X, J we use, since by closing up with the same Z_1, J_1 we see, that $c_1^2 - 3\sigma - 2\chi = 0$ for the closed manifold by the Hirzebruch Hopf theorem.

$\hat{\partial} x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in D(x,y), n_w(\phi)=0, \mu(\phi)=1} \#\hat{M}(\phi) y$ is the magic formula and similarly $\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in D(x,y), \mu(\phi)=1} \#\hat{M}(\phi) u^{n_w(\phi)} y$ for the boldface theory.

Theorem: $\mu(\phi) = e(\phi) + p(\phi)$, where $e(\phi)$ is the Euler measure, and $p(\phi)$ is the point measure.

$e(\phi) = \sum n_i e(D_i)$, we pick a metric such that the α, β curves become geodesics, meeting at right angles. Now $e(D_i) = \frac{1}{2\pi} \int_{D_i} \text{curvature of g}$. For example a bigon on a sphere we get $\frac{1}{2}$, since 4 of these bigons give the complete sphere, and the integral will give the Euler characteristic. In general if D_i is a $2n$-gon, then $e(D_i) = 1 - \frac{n}{2}$.

For the point measure, we take the x_i, y_i intersection points, and average the a, b, c, d numbers at their points, the multiplicities given by D. $p(\phi) = \sum p_{x_i} + \sum p_{y_i}$.

Example: grid diagrams from last semester. The Euler measure will be zero, since the elementary domains will be all squares. The point measure will be 1 exactly when we have a big square, which have X, O-s in the corners, and no other point inside (since an inside point would contribute at least 2 to the sum).

For a doubly pointed diagram we get Y, K a manifold, and a knot. Up until now, we constructed an invariant of Y. What options do we have? We can change the boundary map in $\hat{CF}$, so that ϕ avoids not only the divisor of w but also the divisor of z, so $n_w(\phi) = n_z(\phi) = 0$. Another option, is to use z to define a filtration on $\hat{CF}$.

Lemma: $gr(x) - gr(y) = \mu(\phi) - 2n_w(\phi)$, for $\phi \in D(x, y)$, and gr the grading defined above for torsion $spin^c$ structure.

Definition: $A(x) = gr_w(x) - gr_z(x)$, where we use w and z to point the diagram for the torsion grading is called the Alexander grading.

$A(x) - A(y) = n_z(\phi) - n_w(\phi)$ from the lemma, where $\phi \in D(x,y)$. We have to either assume, that Y is a rational homology sphere, or the knot is 0 in homology.

A filtered chain complex has the form $V_1 \subset V_2 \subset \dots V$, and $\partial V_n \subset V_n$.

Fact: $\hat{CF}, \hat{\partial}, A$ is a filtered chain complex.

> If y is a component of ∂x, we need, that $A(y) \leq A(x)$, but this is clear, since if there is a holomorphic representative then by assumption on the boundary operator n_w is zero, and the holomorphic representative intersects everything positively, so by the above formula $A(x) - A(y) \geq 0$.

We can change the boundary map, to keep only the ones, which have boundary of the same grading. Formally, take $\oplus V_n / V_{n-1}$. This graded chain complex with ∂_A is called the associated grading. This corresponds exactly to the previous assumption, that not only $n_w = 0$, but n_z also.

Similarly we can consider these on the boldface theory, $(\mathbb{CF}, \partial, A)$ and we declare, that $A(u) = -1$, and can consider the associated graded chain complex, again the only change, is that we only consider ϕ-s, with $n_z(\phi) = 0$.

The A and M gradings in the definition of the Kauffmann states, and the construction of the Alexander polynomial, the Alexander and Maslow exactly correspond to the Alexander, and gr gradings we discussed today.

9.

Another take on the gradings.

$(\Sigma, \alpha, \beta, w)$ a diagram for an orineted closed 3-manifold Y. $T_\alpha \cap T_\beta \subset Symm^g(\Sigma)$. To every intersection point we can associate a homotopy class of oriented 2-plane fields, which is in turn equivalent to a nowhere vanishing vector field. There is an action of Z_2 on these by flipping the orientation and/or taking $-v$ instead of v. To a nowhere vanishing vector field, we can associate a $spin^c$ structure. In turn each $spin^c$ structure has a first Chern class in H^2, or equivalently in H_1.

Nowhere vanishing vetorfields are described by framed 1-manifolds in Y up to cobordism, this is a variant of the Pontrjagin-Thom construction.

A nowhere zero vector field can be described as a map $Y \to R^3 \setminus 0 = S^2$. Take the inverse image of the north pole, for example, this gives a 1-manifold, and the pullback of the tangent bundle gives a trivialisation of the normal bundle, ergo the framing.

We color the intersection points according to which $spin^c$ structure they represent. Note, that this construction depends on the choice of w. There is an(other) equivalence relation on the intersections, $x \sim y$ exactly when $\epsilon(x, y) = 0$, so the flowlines to y, then back to x is 0 in homology. From this we get the variants of CF. The claim is that the resulting homology is independent of the choice of w! Everything changes by replacing w, but the resulting homology will be the same.

One important fact, is that $\mathfrak{N}_3 = 0$. Another is, that (Y, ξ), a 3-manifold equipped with a 2-plane field, is the boundary of an almost complex 4-manifold (in the sense, that $\xi = TY \cap JTY$). The grading of X will be $(c_1^2(X, J) - 3\sigma(X, J) - 2\chi(X))/4$. If $c_1|_Y = 0 \in H^2(Y; Z)$ we get an integer for this grading, if this is zero only in rational homology, then $c_1 \in H^2(X, \partial X; Q)$, and we get a rational grading, but the differences are still intgers. This will be a relatively Z graded absolutely Q graded vectorspace.

$PDc_1 \in H_2(X, \partial X; Z)$ if $c_1 \in H_2(X; Z)$, and symmetrically if $c_1 \in H^2(X, \partial X; Z)$, then its Poincaré dual is in the absolute second homology of X. The sad thing is that we cannot take the square of c_1 in the first case, since the fundamental class doesn't exist.

Thirdly any two 2-plane fields on Y are almost complex cobordant in the previous sense. Writing out the difference of gradings, we see $\frac{1}{4}(c_1^2(X, J_1) - c_1^2(X, J_2))$, and can apply the difference of squares identity, since $c_1 \in H^2(X; Z)$, and even cohomology classes commute.

Assuming $c_1(X, J_1)|_{\partial X} = c_1(X, J_2)|_{\partial X}$, i.e. ξ_1 and ξ_2 represent the same $spin^c$ structures implies, that $K = c_1(X, J_1) - c_1(X, J_2)$ is compactly supported. From the long exact sequence of pairs on cohomology, we see that K comes from a $\bar{K} \in H^2(X, \partial X; Z)$.

What if $\bar{K} \& \bar{K}'$ map to the same K? $\bar{K} - \bar{K}' \mapsto 0$, so it comes from the previous term of the LES, $\delta(y) = \bar{K} - \bar{K}'$, so the difference will be $\delta(y) \smile (c_1(J_1) + c_1(J_2))$ (of $\bar{K} \smile c_1 J_1 + c_1 J_2$ and the primed version. δy lives on the boundary, so we can restrict everything there. The Chern classes will be tha same by assumption, we divide out by 2, so the ambiguity lies in $y \smile c_1(J_1)|_{\partial X}$. So we can take $Z/d(c_1(J_1))Z$, where d is the divisbility of $c_1(J_1)$, meaning the largest d, for which c_1 can be written as dx for another class $x \in H^2$. The difference in the choice of preimage $\bar{K}$ can give us $y \smile c_1 = y \smile dx = d(y \smile x)$ in the difference, so by looking at it in the factor, the different choices don't change anything $\mod d$.

In summary, if s is torsion, we get a relative Z graded, absolute Q graded *object*, if s is nontorsion we get a relative Z/dZ graded thing.

Examples

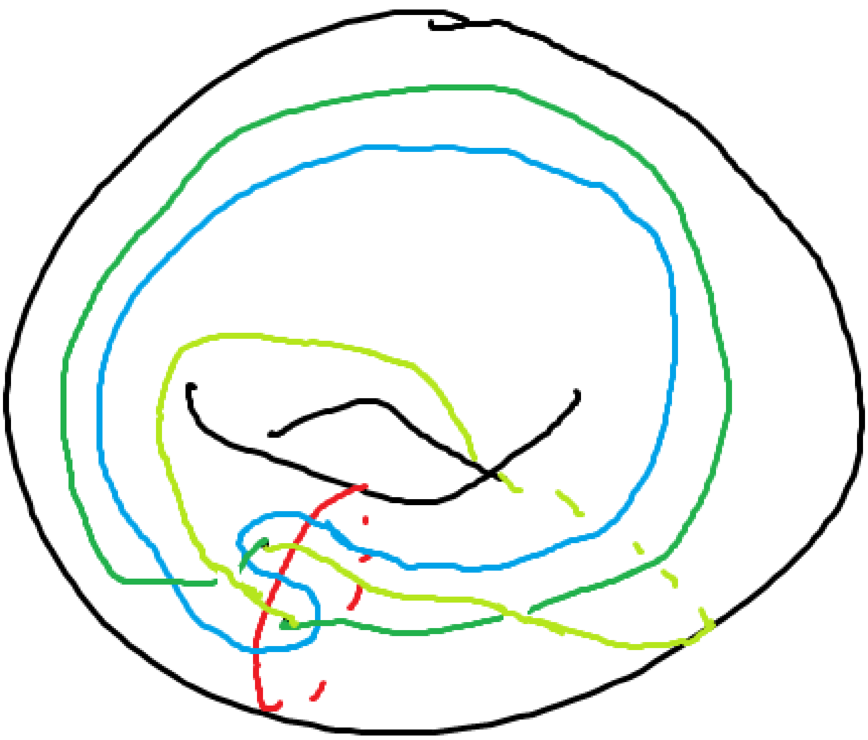
$Y = S^3$, or S^3 with a knot. Take a genus g Heegaard diagram, the symmetric product will be very complicated in general, so we look at $g = 1$.

Definition: A knot in S^3 is a $(1,1)$ knot if $\exists(T^2, \alpha, \beta, w, z)$ diagram which gives S^3, and the knot inside it.

The unknot has a genus 0 decomposition of course.

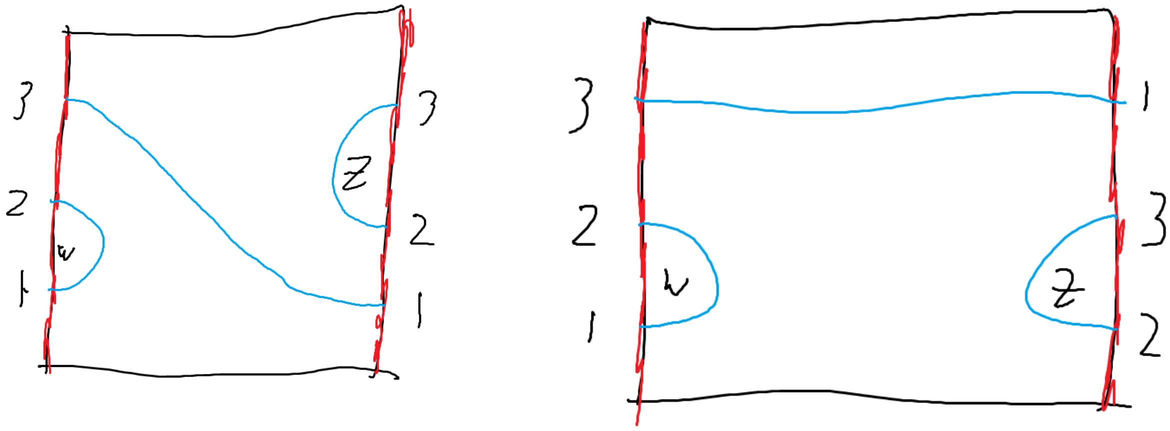
HW#1: Every torus knot is a $(1,1)$ knot.

One can also think about the fact, that every knot we can draw on a torus is a torus knot.

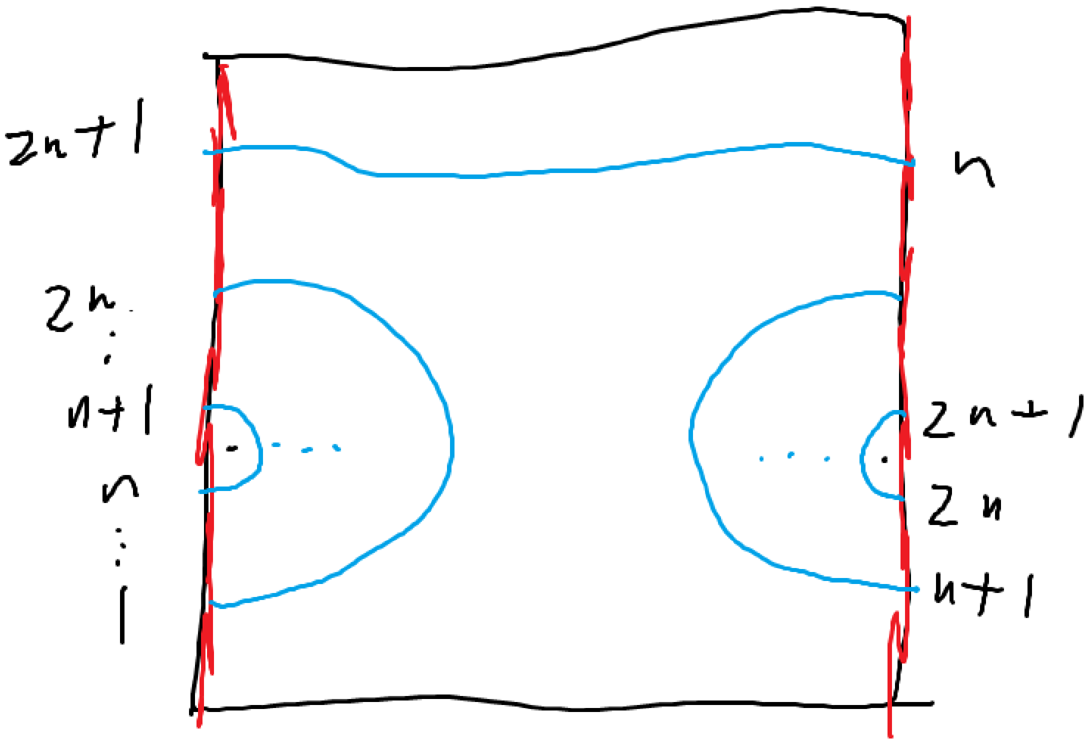


The two components of the trefoil

We claim, this represents the right handed trefoil, embedded in S^3. We draw it in another way, using the fundamental domain.

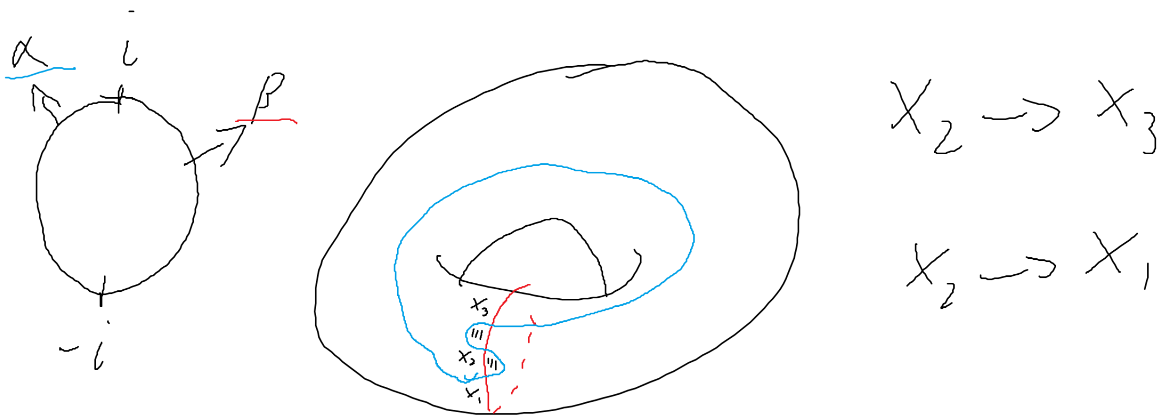


The numbers indicate that there is a shear on the fundamental domain, the identification is not the obvious one.



A diagram for the $T_{2,2n+1}$ Torus knot. The permutation of the numbering is circular.

We shall meditate more on the first picture. We have three intersection points of $T_\alpha \cap T_\beta$, and so $\hat{CF} = F^3$. We see discs from x_2 to x_3, and another one from x_2 to x_1, as shown in the picture.

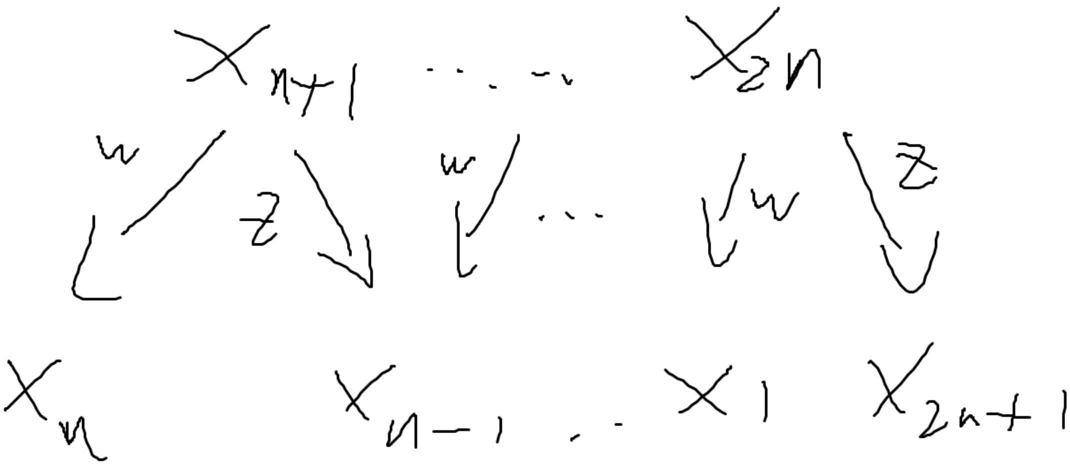


There are no other discs, as can be seen by drawing the universal cover, so $\partial x_1 = \partial x_3 = 0$, and for x_2 we have two holomorphic discs, one containing w once, and the other z once, going to x_1 and x_3 respectively.

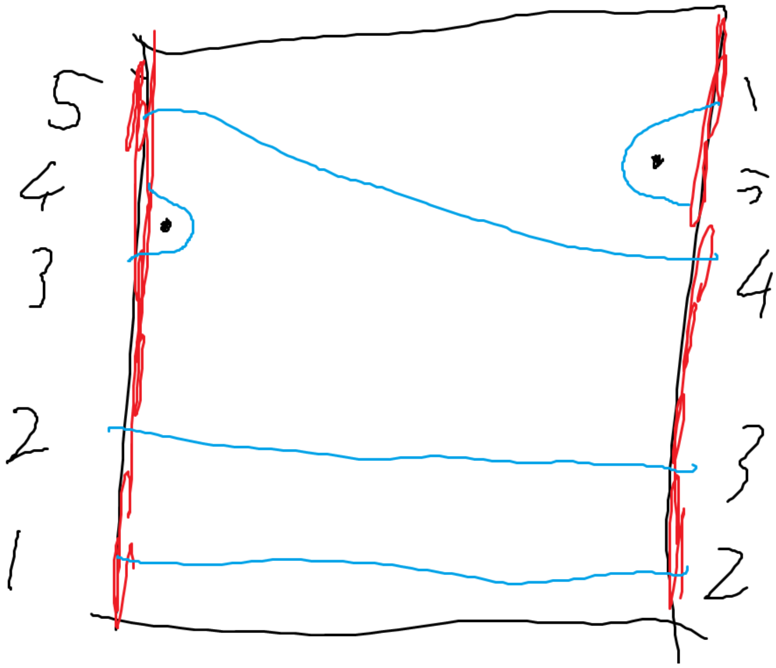
We can take Σ, α, β, z or Σ, α, β, w and look at $\hat{CF}$ or CF^-.

The latter will be $F[u]^3$ of course. $\hat{\partial} x = \sum_y \sum_\phi \# \hat{M}(\phi) y$, and $\partial^- x = \sum_y \sum_\phi \# \hat{M}(\phi) u^{n_w(\phi)} y$. For both ∂x_1 and ∂x_3 will be zero, since the sums are empty. For the ∂^- in one theory we see $\hat{\partial} x_2 = x_3$ since we need to avoid w, and so the homology will be generated by x_1. In the $-$ case $\partial x_2 = x_3 + u x_1$, we weigh by the intersection number with w. So we claim, that $HF^- = F[u] < x_1 >$. If we pick z to be the distinguished point, $\hat{\partial} x_2 = x_1$, and so x_3 generates the $\hat{HF}$ homology. For the knot variant if we look at the hat construction we have to avoid both points, so we have no holomorphic discs, no boundaries and so the chain complex degenerates to F^3. ∂_K^- is defined as before, but the intersection with the other point should be zero, so $\partial^- x_2 = u x_1$. The homology will become $F[u] < x_3 > \oplus F < x_1 >$.

Generally for the $T_{2,2n+1}$ diagram we will get



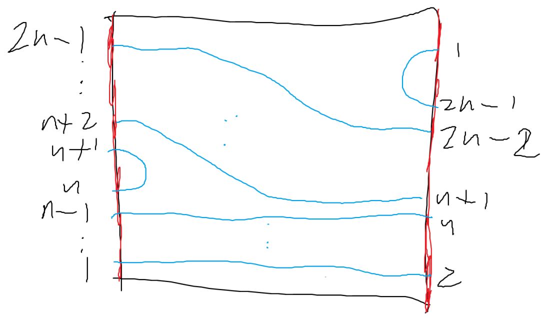
for the intersections of the holomorphic discs.



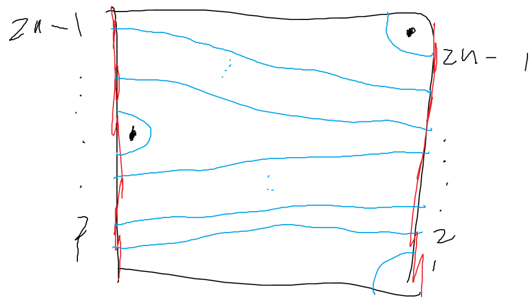
$T_{3,4}$

From covering space theory we know, that we can search for bigons in the cover of this diagram. We see an $x_4 \xrightarrow{z} x_3$ bigon, an $x_5 \xrightarrow{2z} x_2$ bigon, and symmetrically $x_5 \xrightarrow{w} x_1$ and $x_4 \xrightarrow{2w} x_2$.

For $C\hat{F}K, \hat{\partial}_K = 0$. $F[u] \oplus F^2 \oplus F$, a free part, a part of linear polynomials, and a part of constants, as an $F[u]$ module will be the HFK^-.



$T_{n,n+1}$



The same knot without the shear

HW#2: Compute the chain complex for the picture below.

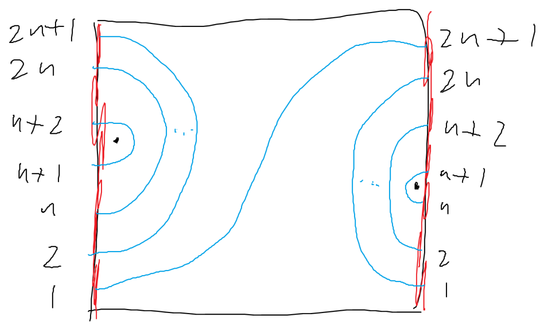


Diagram of a twist knots for the homework

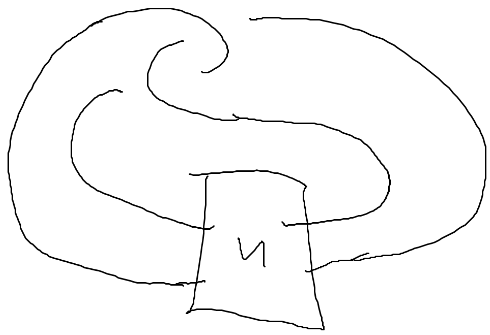


Diagram of a twist knot

10.

Y^3 closed oriented, represented by a Heegaard diagram, associated to it the symmetric power equipped with tori, and a submanifold corresponding to a point, also analytic choices, ergo a symplectic form, and an almost complex structure associated to it. From this we get 3 chain complexes, $\hat{CF}, CF^-, \mathbb{CF}$. We needed the admissibility assumption on the diagram to make sense (and *strong* admissibility for CF^-).

$$\hat{\partial} x := \sum_{y \in T_\alpha \cap T_\beta} \sum_{\mu(\phi)=1, n_w(\phi)=0} \# \hat{M}(\phi) y$$

$$\partial x := \sum_{y \in T_\alpha \cap T_\beta} \sum_{\mu(\phi)=1} \#\hat{M}(\phi) u^{n_w(\phi)} y$$

Theorem: These are chain complexes, their homologies are topological invariants of Y.

The $\hat{HF}(Y) = \oplus_{s \in spin^c(Y)} \hat{HF}(Y, s)$ splits as a sum on the spin^c structures of Y. If $c_1(Y, s)$ is torsion, then the corresponding homology will be relatively Z graded, absolutely Q graded finite dimensional vectorspace over F. On the other hand if $c_1(Y, s)$ is nontorsion, we only get a relative Z/dZ grading on the finite dimensional vector space $\hat{HF}(Y, s)$. Note, that the gradings are also invariants.

Fact: If Y is a rational homology sphere (denoted QHS^3) (ergo $b_1(Y) = 0$, or $|H_1(Y)| < \infty$), then

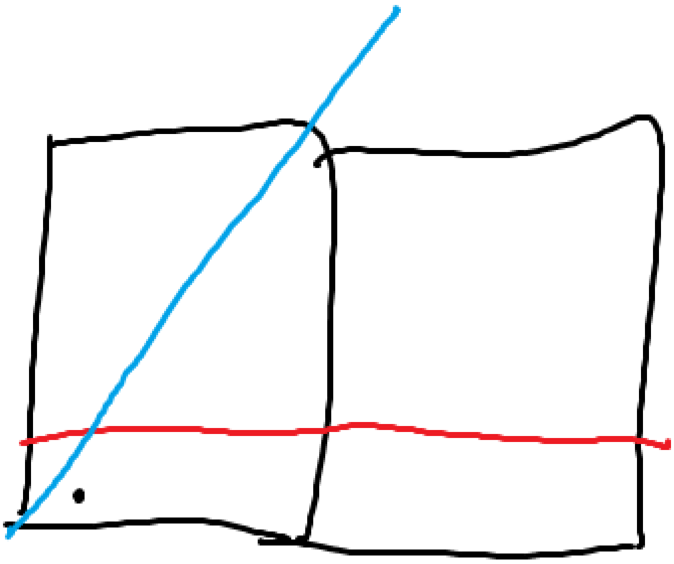
$$2 \not| \dim \hat{HF}(Y, s).$$

Definition: Y is a QHS^3 is an L-space, if $\dim \hat{HF}(Y, s) = 1 \forall s \in spin^c(Y)$ (L stands for lens spaces).

Construction: S^3 will be the unit complex numbers in C^2. It admits an action from $Z_p = Z/pZ$ by multiplication by a primitive p-th root of unity. $(z_1, z_2) \mapsto (\eta z_1, \eta^q z_2)$, where $\eta = e^{2\pi i/p}$, and $0 < q < p$ such that $(p, q) = 1$. We claim that this is a free action of Z_p on S^3, this is an easy check. We define $S^3/Z_p := L(p, q)$. The factormap is a p-fold cover, so $\pi_1(L(p, q)) = Z_p$.

💡 $L(1,1) = S^3$, $L(2,1) = RP^3$

HW#1: $L(p, q)$ admits a genus 1 Heegaard decomposition.



Heegaard diagram of a Lens space on a torus, the blue line has slope p/q.

There are no bigons in the covering space, so there are no nontrivial boundary maps in any version of Floer homology. This shows also, that each of the p intersectionpoints give different $spin^c$ structures. We already knew this, since $Z_p = \pi_1 = H_1 = H^2$, which parametrises exactly the $spin^c$ structures. By computation we get that $\hat{HF}(Y, s) = F$, so these are examples of L spaces.

> Lens spaces can be defined almost equivalently by saying that these are the spaces admittin a genus one Heegaard decomposition, but this also adds $S^1 \times S^2$ to the list, which is obviously not a quitoent of S^3, its universal cover is $R \times S^2$.

HF^- also splits as a sum over the $spin^c$ structures, it will be a finitely generated module over the polynomial ring (or the powerring for the boldface one). We also get the absolute Q relative Z grading for torsion first Chern class, and relative Z/dZ grading for nontorsion c_1. We also define the formal variable u to have grading -2 in the torsion case.

Fact: If Y is a QHS^3, then the rank of $HF^-(Y, s)$ is always 1.

From this fact we can derive the previous one by using the long exact sequence of Floer homology.

$$0 \to CF^- \xrightarrow{\times u} CF^- \to \hat{CF} \to 0$$

A relatively Z graded absolutely Q graded rank one module can be described as follows. $F[u] \oplus \bigoplus_i F[u]/p_i(u)$ the free part, and the u-torsion part. $p(u)$ can only be a monomial otherwise the grading wouldn't make sense. So we can encode these by a sequence of $(d, (d_1, n_1), \dots)$ numbers, where $d \in Q$ is the grading of the free part, and $d_i \in Q$ is the grading of the ith torsion part, and it is factored by u^{n_i}. These invariants are suficient to classify the lens spaces for example.

So now to each QHS^3 we can associate the rational number d. Heds up, the free part of a module is not well defined in general, we have to factor by the torsion part, or by some miracle we prove that it actually is well defined in our case. We take a third route, $d := max\{gr(x) | \nexists p \in F[u] : px = 0\}$, the maximal grading of a non-torsion element.

The homologies are isomorphism invariants, the chain complexes are invariants up to chain homotopy equivalence, which is a little stringer, but not that much.

$K \subset Y$ if we take a 3-manifold and a knot inside it, we take a doubly pointed Heegaard diagram, the symmetric product, and the whole story goes through as before.

$$\hat{\partial}_K x := \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi : \mu(\phi)=1, n_w(\phi)=0, n_z(\phi)=0} \#\hat{M}(\phi) y$$

$$\partial_K^- x := \sum_{y \in T_\alpha \cap T_\beta} \sum_{\mu(\phi)=1, n_z(\phi)=0} \#\hat{M}(\phi) u^{n_w(\phi)} y$$

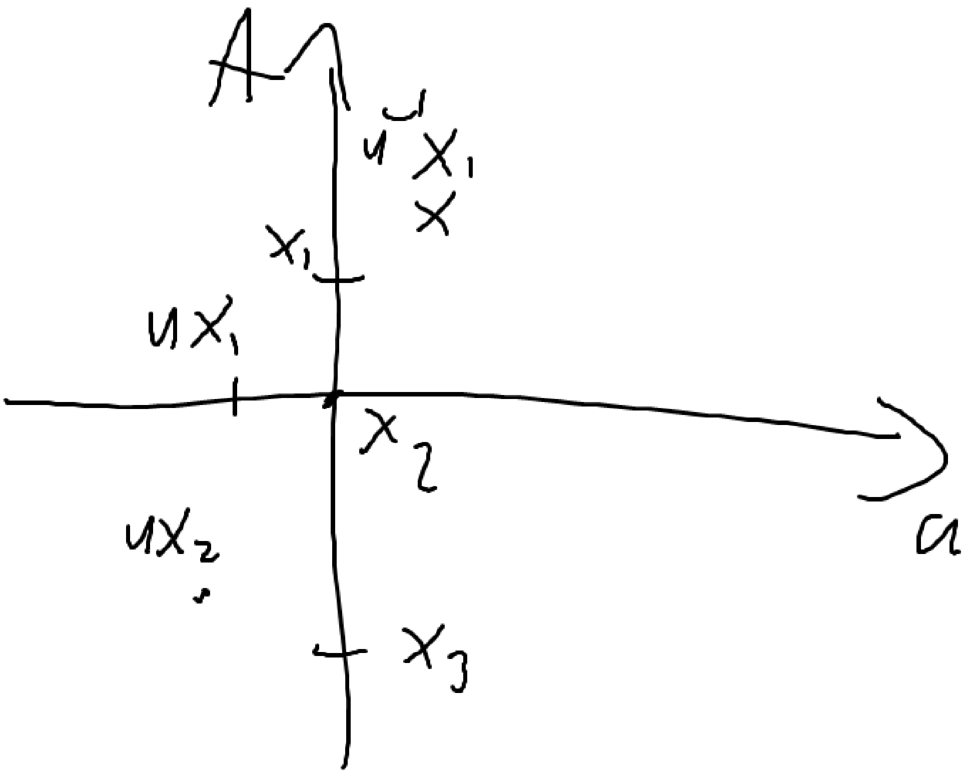
We claim that these are diffeomorphism invariants of the pair (Y, K).

Alternatively take $C\hat{F}K = \hat{CF}, \hat{\partial}, A$ where the last term is a filtration of Y, $A(x) = M_w(x) - M_z(x)$. We now package this information differently. Consider $CFK^\infty(Y, K, s) := \bigoplus_{x \in T_\alpha \cap T_\beta, s_w(x)=s} F[u, u^{-1}] < x >$. If V is a module over $F[u]$, we can consider $V \otimes_{F[u]} F[u, u^{-1}]$ a modules over the ring of Laurent polynomials. The boundary map stays the same, and since the discs are holomorphic, the intersection points are always positive. In the other direction the construction is not unique, if x generates over Laurent polynomials, so do ux or x/u. Fixing a filtration solves this problem, we can declare for example that $F[x] = \{x : A(x) \leq 0\}$.

Example $K \subset S^3$, CFK^∞ will be an absolutely Z-graded module, since there is only 1 spin^c structure. This modules is also generated by $u^n x$ for all n, and $x \in$

$T_\alpha \cap T_\beta$ over the base field F. We declare $a(u^n x) = -n$, and $A(u^n x) = A(x) - n$.

For the trefoil x_1 has $(A, M) = (1, 0)$, x_2 has $(0, -1)$ and x_3 will get $(-1, -2)$.



Picturing the gradings on the plane.

For each generator draw an arrow to each of the components of its boundary. The same diagram works for CF^-, we only consider the left half-plane, for $\hat{CF}$ we only look at the y axis, for CFK^- only consider the horizontal arrows, and so on.

For each (K, r) knot in S^3 and rational number pair we associate another manifold by Dehn surgery.

For a 3-manifold Y take $Y \times I$, and the knot in $Y \times \{1\}$. We can identify a neighbourhood of the knot with a 2-handle. This gives a cobordism between Y and something else, namely $(Y \setminus int\ \nu(K)) \cup (D^2 \times S^1)$, this requires that the knot is framed.

> 💡 In russian this is called perestroika, and not surgery, which is a much better name lol.

Generally, take (Y, K) closed manifold and a knot. $Y \setminus int\ \nu(K)$, and glue to it $S^1 \times D^2$ along the boundary, which is also T^2, the torus. A self-difeomorphism $\phi : T^2 \to T^2$ induces an isomorphism on H_1, ergo an element of $SL_2(Z)$. One generator of $H_1(T^2)$ will be the meridian (the generator, which dies in the tubular neighbourhood of K), the other generator will not be canonical in general, it exists iff the knot is 0 in homology. The meridian is the generator of $ker\ (\partial(Y \setminus int\ \nu(K)) \to \nu(K))$, for the other obvious map $ker\ (\partial(Y \setminus int\ \nu(K)) \to Y \setminus int\ \nu(K))$ we dont know how big this group is. Look at everything in 3d now. $S^1 \times D^2$ is a 0 handle, attached with a 1-handle. Turn it upside down, so it is a 3-handle with a 2-handle attached. To specify attaching this space, we only need to give a circle γ in the boundary torus of $Y \setminus \nu(K)$.

Actually any two curves on a torus representing the same homolog class are actually isotopic, so wo only need to specify $[\gamma] = p[\mu] + q[\lambda]$. We need an orientation, what happens, if we take the other? $-[\gamma] = -p[\mu] - q[\lambda]$, these numbers will be relatively prime, since only primitive elements are representable by connected submanifolds. So for $Y, K, \lambda, p/q$ we get $Y_{p/q}(K)$, the manifold obtained by Dehn surgery.

HW*: $Y_{p/q}(K)$ is given as a 4-d 2-handle attachment along K in Y iff $p/q \in Z$.

> 💡 This was Dehns counterexample to the Poincaré conjecture, $H_1(S^3_{p/q}(K); Z) = Z/pZ$.

Example: $S^3_{-p/q}(\circ) = L(p, q)$

We can do the same procedure along links, instead of knots. $L = (K_1, \dots, K_n)$, and $r = (r_1, r_2, \dots, r_n)$, where $r_i \in Q$. To this we associate by repeated surgery $S^3_r(L)$, a 3-manifold.

Theorem (Lickorish, Wallace): For every closed, oriented Y, there is r, L such that $S^3_r(L) = Y$ for homeo, or diffeo. One can take $r \in Z^n$.

From this we can conclude, that every 3-manifold is the boundary of a 4-manifold for example.

Theorem (Ozsváth-Szabó, 2006): Suppose $(CFK^\infty(K), \partial, A, a)$ determines $H\hat{F}^-(S^3_{p/q}(K))$ through an algebraic construction.

Theorem (Ozsváth-Manolescu): The same as above for links.

> The paper was writen in 2008, submitted in 2012, not accepted yet! 250+ pages, and so on.

Setup: Suppose we have a framed knot in Y, we can look at $Y_f(K)$ and $Y_{f+\mu}(K)$, and there is an exact triangle of cobordisms connecting these manifolds in this order. We can apply $\hat{HF}$ to this, the cobordisms induce maps on the corresponding Floer homology.

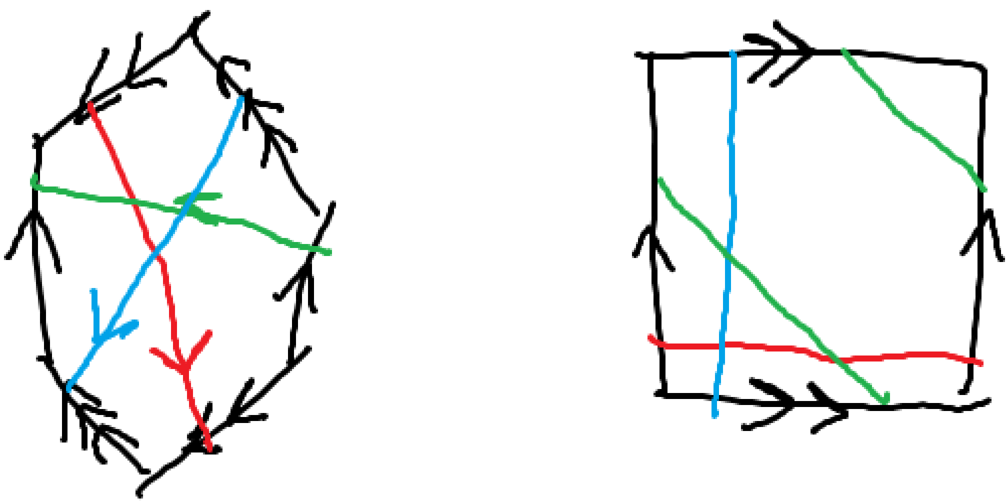
Theorem: The triangle between the $\hat{HF}$s we get in this way is exact.

12.

Surgery exact triangle

Definition: Suppose M^3 compact oriented 3-manifold with $\partial M = T^2$. The 3 simple cloesd curves α, β, γ form a triad, if they intersect each other exactly pairwise. Order these curves cyclicly, so we can fix orientations such that each intersection is negative.

M, α, β, γ determines three 3-manifolds Y_α and so on.



Example of triad curves on a torus in two different presentations.

We produce these manifolds by Dehn surgery so that the homology generator, which gets killen when we fill the torus gets mapped to the given curve, the boundary torus is like the neighbourhood of a knot, and these curves determine the framing.

Example: $K \subset S^3$. We get back S^3, or any $S^3_n(K)$, and the third one is determined to be the sum of the other two, so $S^3_{n+1}(K)$, and this generalises to $K_f \subset Y$,

$Y_f(K)$ and $Y_{f+\mu}(K)$, the manifold itself, surgery along it, and surgery along it+the meridian.

If we have $p_1 q_2 - p_2 q_1 = \pm 1$ we get $S^2_{p_1/q_1}(K)$, $S^3_{p_2/q_2}(K)$, and finally $S^3_{((p_1+p_2)/(q_1+q_2))}(K)$.

Theorem (Surgery Exact Triangle): If $Y_\alpha, \ldots$ form a surgery triad, there are maps F_i $i = 1, 2, 3$ such that the $\hat{HF}(Y_\alpha) \xrightarrow{F_1} \hat{HF}(Y_\beta) \xrightarrow{F_2} \hat{HF}(Y_\gamma) \xrightarrow{F_3} \hat{HF}(Y_\alpha)$ triangle is exact. The same is true for $\mathbb{HF}$, but not for HF^-.

Corollary: Suppose Y_1, Y_2 are L-spaces, and $|H_1(Y_3; Z)| = |H_1(Y_1)| + |H_1(Y_2)|$, and they form a triad, then Y^3 is also an L-space.

> We know that $\dim \hat{HF}(Y_3) \geq |H_1(Y_3)$ from general theory, since H_1 parameterises the spin^c structures. In the exact triangle we have $\hat{HF}(Y_1) = F^{|H_1(Y_1)|}$, and the same for Y_2, these are vectorspaces, so no torsion or anything. If the first map is the zero map, the exact triangle unfolds into a short exact sequence, and we see, that $\hat{HF}(Y_3)$ splits as the sum of the $\hat{HF}$-s of the first two, if the map is nonzero, the dimension will be even smaller, but it cannot be smaller from the general fact, so that map is always zero, and the homologies add together.

Corollary^2: Suppose $K \subset S^3$, and $S^3_n(K)$ so that $n > 0$ and this surgery space is an L-space. Then $S^3_m(K)$ is also an L-space for all $m > n$.

> S^3 is an L-space, the n-surgery is also by assumption, so we can use the previous corrolary.

Example: $S^3_5(\text{right trefoil})$ is a lens space.

One possible way of sliding handles, the other one gives a +5 right trefoil

<u>Fact</u>: $S^3_n(T_{p,q})$ is an L-space if $n = pq - 1$.

<u>Setup</u>: C_i, ∂_i $i = 1, 2, 3$ chain complexes (considered to be finite dimensional vector spaces over F, but it works in general as well). $f_i : C_i, \partial_i \to C_{i+1}, \partial_{i+1}$ cyclically chain maps, and also $h_i : C_i \to C_{i-1}$ also cyclically, satisfying $f_{i+1} \circ f_i = \partial_{i+2} \circ h_i + h_i \circ \partial_i$, i.e. $f_{i+1} \circ f_i$ is chain homotopic to the zero map. Furthermore $\phi_i := h_{i+1} \circ f_i + f_{i+2} \circ h_i$ an endomorphism of C_i such that they are chain homotopic to id_{C_i}.

<u>Theorem</u>: Under the above assumptions $H(C_i)$ together with $H(f_i)$ forms an exact triangle.

<u>Lemma</u>: ϕ_i is a chain map, i.e. $\partial_i \circ \phi_i - \phi_i \circ \partial_i = 0$ (we can write + instead of -, since we are working mod 2).

$$\partial_i \circ (h_{i+1} \circ f_i + f_{i+2} \circ h_i) + (h_{i+1} \circ f_i + f_{i+2} \circ h_i) \circ \partial_i =$$
$$\partial_i \circ h_{i+1} \circ f_i + f_{i+2} \circ \partial_{i+2} \circ h_i + h_{i+1} \circ \partial_{i+1} \circ f_i + f_{i+2} \circ h_i \circ \partial_i =$$

The second and fourth terms give tha chain homotopy between the double composition and the zero map, so we get $f_{i+2} \circ (f_{i+1} \circ f_i) + (f_{i+2} \circ f_{i+1}) \circ f_i = 0$.

Now we need that $im(Hf_i) = ker(Hf_{i+1})$, we know one containment, namely that $im \subset ker$, by the existence of h_i.

$ker \subset im$. Let $b \in C_{i+1}$ such that $f_{i+1}(b) = \partial c$ for some c, and $\partial b = 0$. Let $a = h_{i+1}(b) + f_{i+2}(c)$, this should live in C_i, we want to show that its boundary is 0, and $[f_i(a)] = [b]$.

$\partial a = h_i \partial_{i+1} b + f_{i+2} f_{i+1} b + f_{i+2} \partial_{i+2} c$, since b is a cycle, the first term is zero, and $\partial c = f_{i+1}(b)$ by definition, thus the last 2 terms cancel mod 2.

What is

$$f_i(a) = f_i(h_{i+1} b + f_{i+2} c) = \phi_{i+1} b + h_{i+2} f_{i+1} b + f_i f_{i+2} c = \\ \phi_{i+1} b + h_{i+2} \partial c + f_i f_{i+2} c = \phi_{i+1} + \partial h_i c$$

so $[f_i(a)] = \phi_{i+1}(b) = [b]$, since ϕ is chain homotopic to the identity.

Theorem': Same assumptions for $C_i, \partial_i, h_i, f_i$, and ϕ_i is defined in the same way, but instead of homotopy with the identity, but a *quasi-isomorphism*.

Definition: $C_1, \partial_1, C_2, \partial_2$ a map F is a quasi isomorphism, if $H(F)$ is an isomorphism.

The theorem is true by similar reasoning to the above.

Definition: D_1, D_2 are quasi isomorphic, if there is a D_3, and $F_{12} : D_1 \to D_3$ and $F_{23} : D_2 \to D_3$ quasi isomorphisms.

Maps in Heegaard-Floer theory

The boundary map was defined by counting(equivalence classes of) holomorphic discs in the symmetric power of a Heegaard diagram, we need to look at the Maslow index bubbling and all sots of problematic things.

How do we interpret surgery with Heegaard diagrams? $K \subset Y$, we delete a neogihbourhood of the knot, and decompose the rest, the last handle will be capping of the neighbourhood. We see thus Σ, α, β a surface, a collection of g circles, and a collection of $g-1$(!) circles, and take the last circle to be the meridian of νK, call it μ. $Y = \Sigma, \alpha, \beta \cup \mu, w$, now $Y_f(K) = \Sigma, \alpha, \beta \cup \lambda, w$, a different circle, from now on we call the original collection of curves β, and the one after the surgery γ. Now we will have $Symm^g, T_\alpha, T_\beta, T_\gamma$, and we perturbed γ so that these tori are transverse, and the perturbation is chosen to be admissible. Consider holomorphic triangles. Pick the three third roots of unity on the boundary of the unit disk, pick pairwise intersection points of the 3 tori, and look at maps, where the corresponding arcs on the unit circle map to one of the curves. Look at $m_{\alpha\beta\gamma} : \hat{C}F(\alpha\beta) \otimes \hat{C}F(\beta\gamma) \to \hat{C}F(\alpha, \gamma)$ defined by $x \otimes y \mapsto \sum_z \sum_{\psi \in w(x,y,z); \mu=0} \# M z$, we dont need to factor out M, since specifying 3 points gives a map uniquely, so there is no action.

Look at $\Sigma \times$triangle$\cup I \times U_\alpha$, we attach the interval times handles onto the edges of the triangles. We get a smooth 4-manifold with 3 disjoint boundary components, which will be $Y_{\alpha\beta}, Y_{\gamma\alpha}, Y_{\beta\gamma}$.

$\alpha, \beta \cup \mu$ gives back Y, and $\alpha, \beta \cup \lambda$ gives the surgery $Y_f(K)$, where the framing is represented by λ, but what is $\beta \cup \mu, \beta \cup \lambda$? We claim, that this is $\#_{g-1} S^1 \times S^2$, since these two sets of curves are "the same" up to homology in the first $g-1$ coordinates, and in the last one we have 2 curves, meeting exactly in 1 points, and we've seen that two parallel curves give $S^1 \times S^2$ summand in the diagram. The Laudenbach-P...? theorem says that every self diffeomorphism of the boundary of this space extends inside, so we can glue this in, in a unique way to get a cobordism between $Y_{\alpha\beta}$ and $-Y_{\gamma\alpha}$. Actually, this cobordism is the handle attachment cobordism. We could keep on doing this, multiply by the 3-simplex, and attach triangle times the handles to get a 5-manifold with boundary.

Theorem (Gay-Kirby 2012): Every (closed oriented) 4-manifold admits a presentation like this. i.e. $\exists\ \Sigma, \alpha, \beta, \gamma$ such that $X_{\alpha\beta\gamma}$ union 3 copies of 1-handlebodies is X.

They looked at functions $X \to R^2$ generic, and look at the critical points, like we did with Morse functions. They called this a trisection of X.

$\hat{C}F(\# S^1 \times S^2)$ has 2^{g-1} generators, the last guy has only 1 intersecion, on the other curves we can pick either of the 2 intersections. There is a top, and a bottom generator. We calculated this before for only one summand.

$f_{Y_{\alpha\beta}, Y_{\alpha,\gamma}}(x) = m_{\alpha\beta\gamma}(x \otimes t_{top}) \in \hat{C}F(\alpha\gamma)$ will be the definition of the map.

We see a triangle formed by these maps, between the different surgeries, we need that these f-s are chain maps.

Claim: Consider $\mu = 1$ triangles in $Symm^g(\Sigma), T_\alpha, T_\beta, T_\gamma$. How can this degenerate? It can degenerate along a circle inside, but the dimension is not large enough for sphere bubbles to form, or along a single boundary component, but those will cancel, or it can degenerate into a triangle and a bigon. We get 3 terms, the last one will vanish since $\partial t_{top} = 0$.

This turns Heegaard Floer theory into a functor from the cobordism category of 3-manifolds to the category of vector spaces, and linear maps.

We have the chain maps, need the chain homotopies also. We create a map $m_{\alpha\beta\gamma\delta}$: $\hat{C}F\alpha\beta \otimes \hat{C}F\beta\gamma \otimes \hat{C}F\gamma\delta \to \hat{C}F\alpha\delta$. Look for holomorphic rectangles in $Symm^g\Sigma$ with the boundary going arcwise to one of the tori given by the curve families. $x \otimes y \otimes z \mapsto \sum_v \sum_{\psi \in w(x,y,z,v); \mu=-1} \#Mv$. The last two terms give

$\#_{g-1} S^1 \times S^2$, since they differ in one spot, so we can take again the corresponding top generator, and define the map $h_{\dots} : m_{\dots}(x \otimes t_{top} \otimes t_{top})$. The problem is, that for a quadrilateral, we are overdetermined. In the Riemann mapping theorem we can prescribe only 3 points. So we have to look at "every rectangle" $[0, t] \times [0, 1] \times (0, \infty)$, and maps from this space, now we can expect to find a discrete number of soultions. How can a rectangle degenerate? boundary bubbles pair up, inside bubbles cannot exists, we can go to neighbouring edges, and degenerate to a bigon and a rectangle, or we can go across in either direction into two triangles, these will be the maps, between which we realized the chain homotopy.

13.

Donaldson diagonalizability

Theorem (Donaldson, 1982): X smooth closed oriented 4-manifold with Q_X negative definite. Then $Q_X \sim -I = n < -1 >$.

Suppose X as above, maybe not negative definite, then Q_X can be definite or indefinite. In the latter case it can be $m < 1 > \oplus n < -1 >$, when the form is odd, when it is even, we get $nE_8 \oplus \ lH$, where $H = [0, 1; 1, 0]$. This shows for example that the signature is always divisible by 8.

Rokhlin's theorem says that the signature of a spin manifold X is always divisible by 16. Size, signature and type characterize the indefinite forms. The definite case is much more complicated number theory, but the above theorem shows that the part in topology is simple.

Corollary: X_1, X_2 simply connected closed oriented smooth 4-manifolds, then X_1 is homeomorphic to X_2 iff the Euler characteristic, the signature and the type of the intersection form coincide.

Applications

Let E be the E_8 plumbing (presented by circles linked as in the E_8 graph, all framed with -2). This gives a manifold with boundary, which is the Poincaré homology sphere. Said differently $\Sigma(2, 3, 5) := \{(z_1, z_2, z_3) \in C^3 | z_1^2 + z_2^3 + z_3^5 = 0\}$. Question: is there an X^4 such that ∂X is the same, and X is a homology disk? Suppose there is such an X, and glue it together with E with an orientation reversing diffeomorphism of the boundary. Since the boundary is a homology sphere, from Mayer Vietoris, the homologies form a direct sum, since the other part doesn't have homology, we see that $Q_{E \cup X} = E_8$.

Look at now the boundary connected sum of E_8, Rokhlin is not enough, the signature is divisible by 16, but by Donaldson, this doesn't bound a homology disc either. Corollary: the n-times boundary connected sum doesn't bound either.

Question: Does ∂E bound a positive definite X? Glue it together again along the boundary, to get a negative definite form $E_8 \oplus Q_{\bar{X}} = n < -1 >$, but this is impossible.

Lemma: Take E_6 (the same graph, but with a shorter tail, only 5 in the main line not 7), this intersection form doesn't embed into any diagonal intersection form.

We have to understand the vectors of square -2, and we try to pair up the basis vectors with the ones in E_6.

Again, by the same principle the n-fold boundary connected sum of Poincaré homology spheres cannot bound a positive definite manifold.

If we have two 3-manifolds, we can consider their connected sum. This has a natural identity element, S^3. Do we have inverses? If we have nontrivial fundamental group, this cannot happen, free product cannot kill a group.

Suppose Y_1, Y_2 $\mathbb{Z}HS$s, i.e. $H_*(Y_1; Z) = H_*(S^3; Z)$. For example $S^3_{\pm\frac{1}{q}}(K)$ are examples, or their connected sums. We say that $Y_1 \sim Y_2$ iff there is a cobordism W from Y_1 to Y_2, ergo $\partial W = -Y_1 \sqcup Y_2$ such that W is a homology cobordism, ergo $H_*(W, Y_1; Z) = H_*(W, Y_2; Z) = 0$. Its an easy check that this is actually an equivalence relation. The inverse will be $-Y$, so the reverse orientation, since obviously $Y \sqcup -Y$ is cobordant to S^3.

Result: $\Theta^3_{\mathbb{Z}}$ will denote this resulting group. We can make any number of changes, the other one which is studied is $\Theta^3_{\mathbb{Q}}$, there is an obvious map from the former to the latter by changing coefficients. Another interesting case is when we take $spin^c$ structures on the manifolds, and the cobordisms $\Theta^{3,spin^c}_{\mathbb{Q}}$.

There is a surjective map $\Theta^3_Z \xrightarrow{\mu} Z_2$ named after Rokhlin. We use that every 3-manifold Y is a spin boundary. Now map $Y \mapsto \sigma(X)/8 \mod 2$, where $\partial X = Y$. This map descends to the equivalence classes, since an equivalent guy glues a space with no homology to X. Suppose we have X_1, X_2 both bounded by Y. Let Z be bounded by $-Y$, look at $\sigma(X_1 \cup Z)$ and $\sigma(X_2 \cup Z)$, these will be closed spin 4-manifolds, glued along a ZHS, so the homologies add together, thus the signatures add, and they will have the same parity, thus this homomorphism is well-defined.

We see that this is onto, since $P = \partial E \mapsto 1$. This means $[P] \neq 0 \in \Theta^3_Z$, we also checked earlier that no multiple of P can be trivial also. The trivial elements of this group are those, which bound a homology disc (ergo are cobordant to S^3). Freedman's theorem says, that every topological ZHS^3 bounds a ZHD^4, so in the topological case everything is trivial. This gives a Z subgroup in Θ^3_Z. This Z is a direct summand also. The current state of the art is $\Theta^3_Z = Z^\infty \oplus A$ for some abelian group A.

Theorem: Every equivalence class can be represented by a hyperbolic 3-manifold.

Theorem: Not every equivalence class can be represented by a Siefert fibered 3-manifold.

One can consider the subgroup generated by all Seifert fibered manifolds, which is still a proper subgroup.

Theorem (Lisca): $< [L] | L \text{ lens space} > \leq \Theta^3_Q$ is described.

Having the above Z as a direct summand, is equivalent to having a homomorphism onto Z. We still don't know if the Rokhlin map can be lifted to Z.

Theorem (Galewski-Stern '70): Every (!) compact manifold of dimension ≥ 5 is homeomorphic to a simplicial complex iff $\exists a \in \Theta^3_Z$ such that $2a \neq 0 \& \mu(a) = 1 \mod 2$.

Theorem (Manolescu 2013): There is no such element.

> He defined a $\beta : \Theta^3_Z \to Z$, not a homomorphism, but satisfying $\beta(-a) = -\beta(a)$, and $\beta \equiv \mu \mod 2$.

Take the above E with the reverse orientation, close it topologically with a disc by Freedman, and multiply it by S^1 or something, to get an example of a non-triangulable 5-manifold.

The above relations on a is exactly the question as to weather Θ^3_Z splits under the short exact sequence given by the Rokhlin homomorphism.

$$0 \to \ker \mu \to \Theta^3_Z \xrightarrow{\mu} Z_2 \to 0$$

Let Y be a ZHS^3, associate to it $\hat{HF}(Y)$ finite dimensional vector space, it is absolutely Z-graded. This isn't enough, we need HF^-, a finitely generated module over $F[u]$, absolutely Z graded, since we are working with integral homology spheres. Last time we discussed, that this will have rank one, as a module, $F[u] \oplus \bigoplus F[u]/u^{n_i}$, we need to describe which grading do the generators live in, $(d, (g_1, n_1), ..., (g_k, n_k))$ plus the grading of the free part d, and we focus on this only.

This gives a map $\Theta^3_Z \to Z$, where $Y \mapsto d(Y)$. Firstly $d(S^3) = 0$. Next what happens to $d(Y_1 \# Y_2)$?

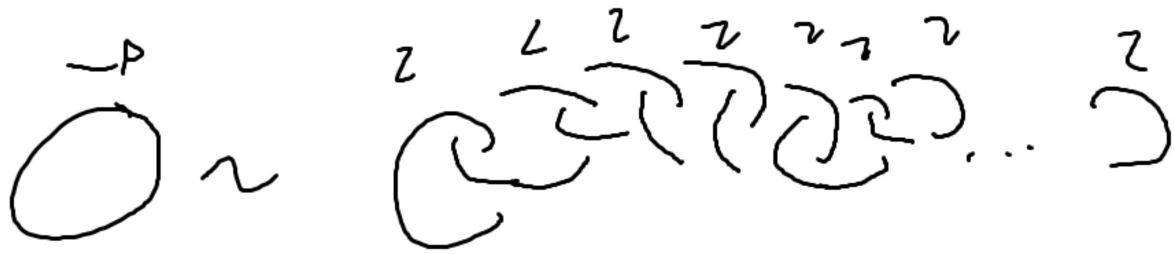
Theorem: $CF^-(Y_1 \# Y_2)$ will be chain homotopic to $CF^-(Y_1) \otimes_{F[u]} CF^-(Y_2)$.

The generators are clear, the boundary map is complicated. The tensor product for the homologies gives some extra trash from the Tor functor, but those will all be torsion, thus we get $d(Y_1 \# Y_2) = d(Y_1) + d(Y_2)$, and $d(\bar{Y}) = -d(Y)$, so all is well, except, that we don't know if this is well defined on the integral homology sphere homology cobordism group.

Theorem: Suppose Y_1, Y_2 are two ZHS^3-s and W is a negative definite cobordism $Y_1 \to Y_2$. Then $4d(Y_1) + \max\{Q_W(x,x) | x \in \mathrm{char(W)}\} + b_2(W) \leq 4d(Y_2)$.

This already gives what we want, since in a homology cobordism there is no second homology, and we get that $d(Y_1) = d(Y_2)$, and d is a homomorphism to Z. To finish one can calculate $HF^-(P) = F_{(-2)}[u]$, and $\frac{1}{2}d$ will be onto Z.

Consider $L(p,1)$, this is a 3-manifold, or a 4-manifold which has boundary this space, a complex line bundle with first Chern class $-p$. We can ask, which $L(p,1)$ bounds a QHD^4? We can cap off the X_p which has boundary this lens space by this QHD^4, b_2 will be 1, the intersection form is negative definite, so $< -1 >$. $a^2 = -p$, if we write this from the generator, we get $a^2 = n^2 g^2 = -n^2$ so p has to be a square. We claim 1,4 is ok, and nothing else. We present the lens space by a different diagram



two 4-manifolds with the same boundary, there are $p - 1$ circles.

The 4-manifold is different, but the boundary is the same. We connected sum with $C\bar{P}^2$, and slide over it all the way, the circles goes off, we see a connected sum with CP^2, we can throw it away since it doesn't do anything on the boundary, and so on, in the end we see a single circle with $-p$ on it.

Our closed off manifold is positive definite, so it should be diagonalizable, we see $p - 1$ twos on and around the main diagonal of the intersection form, and this will not work for big p. So in the $p = 4$ case we don't have a contradiction, but we still need a construction.

Take CP^2, a CP^1 inside it. Take a tubular neighborhood, its complement is a disk. Take a degree two curve, $x^2 + y^2 + z^2$, it has self intersection $+4$, a tubular neighborhood, the complement has no homology, so we get what we wanted, $L(4,1)$ rationally bounds. Indeed $L(p^2, pq-1)$ where $(p,q) = 1, p > q$ also bounds a QHD^4.

Now back to the inequality. Suppose we know the inequality, and somebody gives us a negative definite 4-manifold X. Delete 2-points to get a cobordism W from S^3 to S^3. $Q_X = Q_W$ is clear.

The inequality gives us that $\max\{Q_X(x,x) | x\ char\} + b_2(X) \leq 0$. The characteristic cohomology classes correspond to $spin^c$ structures.

<u>Theorem</u> (Elkies): Suppose Q a symmetric unimodular bilinear form on Z^n. If Q is negative definite, then $0 \leq \max Q(x,x) | x\ characteristic + n$, and equality holds iff Q is diagonalizable.

From this we are done, since there are two inequalities with the same ends.

<u>Observation</u>: Q on Z^n is diagonal iff there are $2n$ vectors with $Q(a,a) = -1$.

> One direction is trivial, in the other, if we have a, b with $Q(a,a) = Q(b,b) = -1$ then either $a = \pm b$ or a is orthogonal to b.

Let $L = (Z^n, Q)$ a lattice. $\theta_L := \sum_{v \in L} e^{\pi i Q(v,v) z}$ is called a θ series, which is convergent on the upper half plane one needs only to approximate how many lattice points of length n there are, this grows slower than a polynomial, and we are in business.

Consider $R(z) = \theta_Q(z)/\theta_{I^n}(z)$, so compare our lattice to the trivial one. The numerator will decompose as $\theta_I(z)^n$, this doesn't have zeros on the upper half plane, so R is holomorphic. $(P)SL_2(Z)$ acts on $\mathbb{H}$, the upper half plane. The group is generated by two elements, translation by one T, and taking the negative reciprocal S. The two relations are $S^2 = 1, (ST)^3 = 1$.

Our theta functions are invariant under T^2, and so the ratios also don't change. R is also invariant under S. The quotient will be S^2 with some singular points, i will be an orbifold singularity, it doesn't matter some why, $i\infty$, and ± 1 will also be problematic. Using the equality assumption one can show that our function will stay bounded even in a neighborhood of ± 1.

How do we prove the inequality itself? Take a homology cobordism W between Y_1, Y_2.

Claim: For an ZHS^3 $HF^\infty(Y, s) = F[u, u^{-1}]$, the triangle-counting map induced by the cobordism can be refined according to $spin^c$ structures.

> 💡 Each triangle in the symmetric product give a $spin^c$ structure on the 4-manifold.

These maps have a well-defined degree shift, which is $\frac{1}{4}(c_1^2(s) - 3\sigma(W) - 2\chi(W))$. Take $\alpha \in \ char(W)$, and $s \in spin^c(W)$ with $c_1(s) = \alpha$. Then we have a map $F_{W,s} : HF^\infty(Y_1, s|_{Y_1}) \to HF^\infty(Y_2, s|_{Y_2})$. W being negative definite implies that this map is an isomorphism.

You slice up the cobordism, and hope that in each slice we get an isomorphism. The problem is that we cannot take every slice to be a ZHS^3, so we need to work hard. Attaching 1,3 handles is pretty standard, for 2-handles one needs to use the surgery exact triangle. This needs to be connected to the d invariant by the SES of the chain groups of the CF theories.