Difftop

14:15 —

öf. kompakt sokaságok
1D — $\{S^1\}$
2D — $\{S^2, A_p, A'_q\}$

leképezéseket vizsgánk a sokaságból valami ismert térbe (P, $\mathbb{R}^n$)

homotópia erejéig 1 db $S^1 \to \mathbb{R}^n$ leképezés van

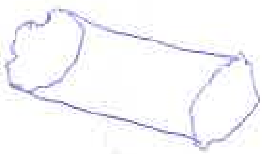
ez túl durva, legyen beágyazás, és beágyazáson keresztüli homotópia

$S^1 \to \mathbb{R}^n$-be $n=1$-re nincs is
(képe nyílt, de nem kompakt)

$n=2$-re 2 db (◯ , ◯)
Jordan-görbetétel

$n=3$-ra csomóelmélet

$n \geq 4$-re 1 db



két hurok valahogy homotóp, ez ad egy nemsemt, ennek kellő simítás után csak transzverzális kettős pontjai lesznek.

az egyik görbével megkerüljük a másikat.

látjuk hogy a csomóelmélet is túl durva finom

$S^1 \to \mathbb{R}^2$ immerzió erejéig
keresztüli homotópia
$\mathrm{Imm}(S^1, \mathbb{R}^2)$ reguláris

$\boxed{\text{HF}_1}$ $\{S^1 \to \mathbb{R}^2$ immerziók$\}/\sim$ csoport.

összekötjük

$df = 0$

<u>Def.:</u> $f, g: M \to P$ immerziók regulárisan homotópok, ha $\exists H: M \times I \to P$ sima homotópia, amire $\forall t \in I$ $p \mapsto H(p,t)$ immerzió.

a $p \mapsto \dfrac{(df)_p}{\|(df)_p\|}$ egy $S^1 \to S^1$ leképezés, ennek a

foka detektálja a reguláris homotópia osztályokat?

nem igazán, a std ◯ beágyazás az egységelem, ennek a foka 1

$\boxed{\text{HF}_2}$ $f \mapsto \deg \dfrac{df}{\|df\|} - 1$ egy szürjektív homomorfizmus

$\mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathbb{Z}$

(*) ez izomorfizmus.

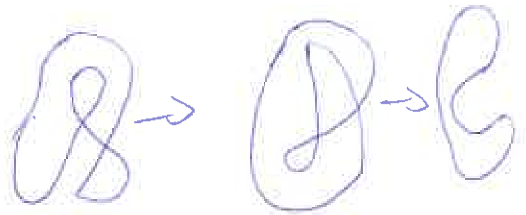
$S^1 \xrightarrow[\text{immerzió}]{} \mathbb{R}^3$ immerziók$/_{\text{reg. hom.}} = 0$, bármely két immerzió reg. homotóp.

Felületek $\mathbb{R}^3$-ba?

$F^2 \hookrightarrow \mathbb{R}^3$ A_p ✓ A'_q ✗

immerzió megy

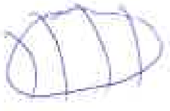
klein kancsó a
Stiner tengelyén elvágva
$\Leftarrow$ dsg $= 1 \Rightarrow$ nem homotóp
a std. beágyazással



ez egy Möbius-szalag szép
peremmel, ehhez ragasztunk egy D^2-t ✓.
a homotópia képe egy henger, ezt
odaragasztjuk, ennek a pereme már jó
(
a foksz ám nem jó ; a normális egy
$F^2 \to S^2$ leképezést ad, de ez konstans.

Tétel (Whitney): $\forall$ zárt M^n-re $\exists M \hookrightarrow \mathbb{R}^N$
beágyazás. ($\exists$ $M \hookrightarrow \mathbb{R}^{2n+1}$ és a beágyazások
sűrűek a leképezések között) ($\nexists M \looparrowright \mathbb{R}^n$ imm.)
és az imm. sűrűek, $\exists M \hookrightarrow \mathbb{R}^{2n}$, $\exists M \looparrowright \mathbb{R}^{2n-1}$,
de ezek már nem sűrűek)

Biz: M-et lefedjük koord. környezetekkel
$M = \bigcup U_j$ térképek, $\varphi_j : U_j \to \mathbb{R}^n$



$(\varphi_1, \ldots, \varphi_R)$-t szeretnénk

ehhez φ_j-ket kiterjesztjük $\overline{\varphi_j}$ az egész M-re
vesszük $V_j \subset U_j$-ket, $\bigcup V_j = M$, V_j-n a kiterjesztés
úgy lecsengjen ahogy U_j-ből kiérünk.

$(\overline{\varphi}_1, \ldots, \overline{\varphi}_R) : M \to \mathbb{R}^{R \cdot n}$

kellően besűrítjük az U_j-ket, hogy két
pontot elválasszanak, és a φ_j-k képei
legyenek diszjunktak, és a 0-t se vegyék fel,
a derivált max rangú valamely koordinátában
ezért az egész leképezés is az.

—

csökkentsük a dimenziót:

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, dashed] & \mathbb{R}^{N+1} \arrow[d, "\pi"] \\ & \mathbb{R}^N \end{tikzcd}$$

babráljuk meg a vetítősíkot!

v nem jó vetítő irány ha
– több szörös pontot csinál
($\exists p, q \in M$: $i(p) - i(q) \parallel v$)
– lehet érintő helyzetben, ezzel a rangs csökkenne
$\exists p$: v érinti $i(M)$-ben.

$M \times M \setminus \{(p,p) : p \in M\} \to S^N : (p,q) \to \frac{i(p) - i(q)}{\|i(p) - i(q)\|}$, ha $2n < N$,
akkor nem lehet szürjektív

{$v \in S^N$ | v érinti $i(M)$-et} $\to M$ ("hol?")
f

$S^{n-1} \to *$

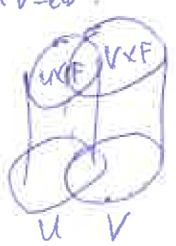
def: $p: E \to B$ lokálisan triviális fibrálás (nyaláb), F fibrum
ha $\forall b \in B$ $\exists U \ni b$ környezet melyre:

$$\begin{array}{ccccc} U\times F & \cong & p^{-1}(U) & \subset & E \\ \downarrow & & \downarrow & & \downarrow p \\ U & = & U & \subset & B \end{array}$$

megj.: ha F diszkrét, akkor $|F|$ rétegszámú fedést kapjuk vissza.

ezzel csinálunk difható sokaságot abból az érintőnyaláb szerű dologból.

átmeneti fv-ek:



$$\begin{tikzcd}
U\times F = p^{-1}(U) \subset E \supset p^{-1}(V) = V\times F \\
U \subset B \supset V \\
U\cap V \\
(U\cap V)\times F
\end{tikzcd}$$

$\varphi_{UV}: (U\cap V)\times F \to (U\cap V)\times F$
„fibrumtartó"
gyakorlatilag egy $U\cap V \to \mathrm{Homeo}(F)$ leképezés

elég az átmeneteket megadni
$\varphi_{UV} = \varphi_{VU}^{-1}$; $\varphi_{UW} = \varphi_{VW}\circ\varphi_{UV}$ ahol értelmes

$\boxed{HF_3}$ TM vektornyaláb $\mathbb{R}^2$ fibrumú nyaláb $O(n)$ struktúracsoporttal

$\boxed{HF_4}$ $\exists ?$ $\mathbb{R}P^n \searrow \mathbb{R}P^{n-k}$ retrakció?

$\boxed{HF_5}$ $[\mathbb{R}P^n, S^1] = ?$

$\boxed{HF_6}$ bármely két $F^2 \hookrightarrow \mathbb{R}^3$ immerziónak u.a. a foka. (F^2: zárt, irányítható)

$\boxed{HF_6}$ adjunk példát ö.f. X, Y terekre és $f: X\to Y$, hogy $f_* = 0: \pi_n X \to \pi_n Y$ $\forall n$, de $f \not\simeq 0$.

$\boxed{HF_7}$ F_1, F_2 zárt, öf felület, egyik sem $S^2 \vee \mathbb{R}P^2$
$\Longrightarrow [F_1, F_2] \xleftrightarrow{1:1} \mathrm{Hom}(\pi_1(F_1), \pi_1(F_2)) / \mathrm{Inn}(\pi_1(F_2))$

*folyt.: a vetítő irány ne legyen párhuzamos egy érintővektorral se.

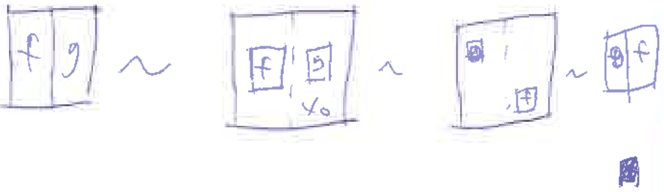
TM egyenessénbjei $2n-1$ d. sokaság, nem tudja lefedni S^N-et, tehát van megfelelő vetítő irány amíg $N > \max(2n, 2n-1)$ (ezzel $2n+1$-ig jönnénk. Az injektivitást elengedve még eggyel lejjebb léphetünk. Imm.)

Ha az eredeti $(\overline{\varphi}_1, \dots \overline{\varphi}_k) \mapsto (f, \overline{\varphi}_1, \dots, \overline{\varphi}_k)$ hozzávesszük f-et koordinátának, vetítünk, ha f irányában nem tudunk, akkor is van egy kellően közeli vetítő irány az $(1,0,\dots,0)$-hoz, ezzel a sűrűségi állítást is beláttuk.
$\square$

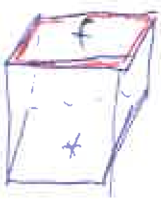
Def.: (n. homotopikus csoport)

Áll. $n>1$-re kommutatív

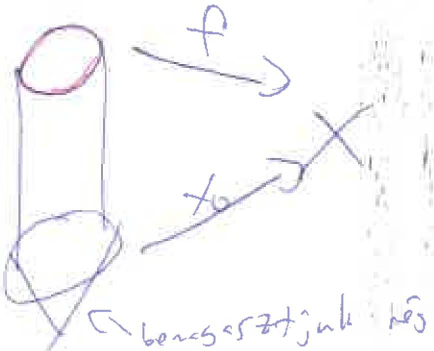
Biz.:



Pontosan akkor reprezentálja a nullelemet egy leképezés, ha az kiterjed egy eggyel nagyobb tömör gömbre/kockára



Első oldal f
$\forall$ másik oldal konstans x_0



← beragasztjuk még egy S^n-el, ami x_0-ba megy

Vissza:

összehúzzuk a kedvenc pontunkba a D^{n+1}-et, ez ad egy jó homotópiát

$f \sim x_0$.

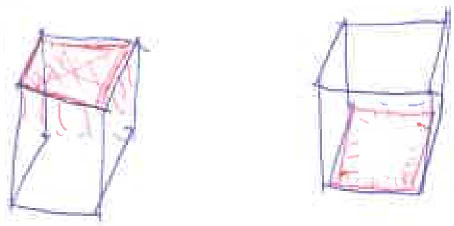
bázispontváltás egy $\pi_1(X,x_0)$ hatást ad.

Def.: $x_0 \in A \subset X$ $\pi_n(X,A,x_0)$ relatív homo. csoport $= \{f:(D^n, S^{n-1}, S^{n-1}_+) \to (X,A,x_0)\}/_{\text{homotópia}}$

vagy $([0,1]^n, [0,1]^{n-1}\times\{0\}, *) \to (X,A,x_0)$

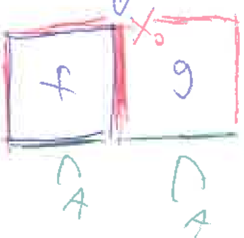
vagy $([0,1]^n, \partial[0,1]^n \setminus [0,1]^{n-1}\times\{0\}, *) \to (X,A,x_0)$

Ez persze mind u.a., az egészt palástot az alsó lap nélkül átvihetjük az alsó lapba



vagy $(D^n, S^{n-1}, *) \to (X,A,x_0)$
összeragad!

művelet:



asszoc., invert, egység, stb ✓
⇓
csoport, ha $n>1$, $n\geq 1$-re

$\pi_1(X,A,x_0)$ az A-ból kezdődő, x_0-ban végződő utakat számolja, ezeket nem lehet értelmesen összefűzni.

$\boxed{f\,g}$ $n \geq 3$-ra kommutatív $\pi_n(X,A,x_0)$.

Def.: $\pi_n(X,A,x_0)$ elemeinek a reprezentánsait relatív szferoidnak nevezzük.

Tétel: $x_0 \in A \subset X$ – erre van egy hosszú egzakt sorozat.

Biz.: $\pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{id_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0)$

① ② ③

Minden szferoid relatív szferoid is.

A ∂ megszorítja a relatív szferoidot az A-ba képződő részére

①: $n+1$-golyót képeztünk X-be, a belsejét elfelejtettük, majd beágyaztuk X-be, ergo 0-homotóp.

visszafelé teljesen hasonlóan, ha valaki

$\pi_n(A, x_0)$-ből a 0-t reprezentálja, a

homotópia ad egy relatív szferoidot.

② ikx $\boxed{A}$, ezt tekintsük X-ben, majd a

relatív homotópia csoportban. Itt ez

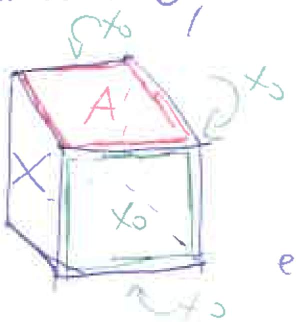
0-homotóp, mert x_0 $\boxed{A}$ x_0 betolhatjuk x_0-ba

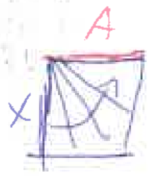
a képet A-n belül.

ez is egy rel. szferoid

ha egy $\pi_n(X, x_0)$-beli elem relatív

szferoidként 0,





homotópia

egy

A-ba képző f-el

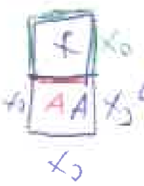
ez a visszairány a

② félig egzaktsághoz.

③ a kompozíció nyilván 0, mert egy

π_n X-beli elemet szorítunk meg az

anyag is x_0-ba menő peremére



$\hookleftarrow \partial f \in \pi_n(A)$ a 0-t reprezentálja,

az egy homotópia, ez az egész egy

rendes szferoid. ✓

$\boxed{HF_{10}}$ $x_0 \in A \subset B \subset X$

$\to \pi_n(B, A) \to \pi_n(X, A) \to \pi_n(X, B) \to$

$\pi_{n-1}(B, A) \to$ hosszú egzakt.

Példa: $\pi_3(D^2, S^1) = 0$, de

$\pi_3(D^2/S^1 = S^2) \neq 0$

$\|$

$\mathbb{Z}$

$\boxed{HF_{11}}$ változat

$[(F_1, *)(F_2, *)] \overset{1-1}{\longleftrightarrow} \mathrm{Hom}(\pi_1((F_1,*)), \pi_1((F_2,*)))$

$\boxed{HF_{12}}$ $S^1 \to S^3 \to \mathbb{C}P^1$ Hopf fibrálás lok. triv.

nyaláb?

$\boxed{HF_{13}}$ Hopf fibrálás nem nullhomotóp (indirekt, $\mathbb{C}S^3 \to S^2$)

$\boxed{HF_{14}}$ $H_i(A_p, \mathbb{Z}) = ?$ $H_i(A_q, \mathbb{Z})$

$\boxed{HF_{15}}$ $H_i(\mathbb{C}X, \mathbb{Z}) = ?$

cone over X

$\boxed{HF_{16}}$* f^*E depends only on the homotopy class of f.

(B compact)

Def.: $f^*E \to E$

$$\begin{array}{ccc} f^*E & \to & E \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

$f^*E = \{(e, x) \mid p(e) = f(x)\}$

$E \times X$

f^*E a pullbackje E-nek f-el. Ha p egy

direkt szorzat vetítése, akkor f^*E is direkt

szorzat.

köv.: lok.triv. nyaláb visszahúzottja

l.t.ny.

*16-os háziban izomorfizmus

thm: $E \xrightarrow{F} B$ a fibre bundle $\Rightarrow \exists$ LES

$\to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to$,

lemma*: $\pi_n(B) = \pi_n(E, F)$ for any Serre-fibration $f: E \to B$, $F = f^{-1}(x)$

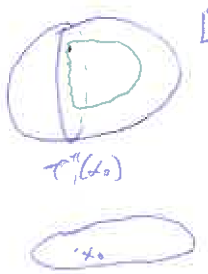
Def.: $f: E \to B$ is a Serre-fibration, if

$\forall K$ finite CW complexre a következő diagram kommutatív, és létezik az átlós $\tilde{H}$ leképezés

$$\begin{tikzcd} K \times \{0\} \arrow[r, "h"] \arrow[d, hook] & E \arrow[d, "f"] \\ K \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

lemma: fibre bundles are Serre,*

***Proof:**



E rel. spheroid S

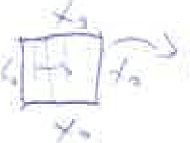
$p \circ S$ is a spheroid in $\pi_n(B)$

this is a map $\pi_n(E, f^{-1}(x_0)) \to \pi_n(B)$,

we claim its bijective.

Surjectivity: consider a spheroid γ in $\pi_n(B)$

, we can consider it to be a homotopy from the constant map to ~~...~~ x_0 const. map.



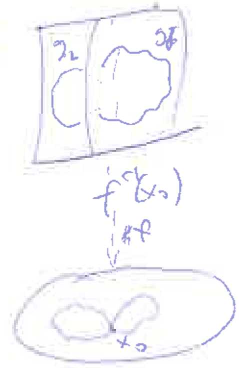
$\to B$, take a lift of this homotopy!

$G: [0,1]^n \to E$, and $G(0, \cdot) = \text{const}$, because we pick $h \equiv \text{const}$ for the lift. Because its a lift, $f(G(1,\partial)) = x_0$. This gives a rel. spheroid

$[0,1] \times S^{n-1}$, because ~~as the spheroid~~ spheroid is a sphere, we factor out almost every dimension.

factor by the 0 edge, this gives a relative spheroid.

injective:



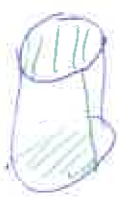
and $f \circ g_1 \sim f \circ g_2$ $H: [0,1] \times S^n \to B$, we factor by the upper face, to get $\bar{H}: [0,1] \times D^n \to B$, we lift this to $\tilde{\bar{H}}: [0,1] \times D^n \to E$, the lower face is g_1, the upper face is a lift of $f \circ g_2$,

lemma: $g_2, g_3: D^n \to E$; $f \circ g_2 = f \circ g_3$

$* \mapsto e_0$

$\partial D^n \mapsto f^{-1}(x_0)$

we have the constant homotopy 'downstairs'



the upper face is $f \circ g_3$, the lower is $f \circ g_2$, this part can be thought of as the lower face, it is null-homotopic, we lift the green part to $g_2 \cup -g_3 \cup *$, this means, that $g_2 \sim g_3$ rel $f^{-1}(x_0)$. □

* **Proof:** Cover B, and E by trivializing neighbourhood. We have $H: [0,1] \times K \to B$, $h: K \times \{0\} \to E$

If we are in a trivialising patch, this lifting means, a continuous map to the fibre

$$\begin{tikzcd} K \times \{0\} \arrow[r, "h"] \arrow[d, hook] & B \times F \\ K \times I \arrow[r, "H"] & B \end{tikzcd}$$

need to extend $\pi_1 (K \times \{0\}) \xrightarrow{h} B \times F \xrightarrow{\pi_2} F$ to $K \times [0,1]$, this is trivial.

induction on dim K: on $sk_0 K$ we need to lift paths from an endpoint finitely many times

from $sk_{j-1} K \to sk_j K$

$\exists\, sk_{j-1} K \times I \to E$ from induction we subdivide K until the triangles are mapped inside trivial neighbourhoods, there we set a cylinder, we can extend the homotopy to the inside by projecting the sides to the inside. $\square$

$\boxed{HF_{14}}$ Show that $P(X,x_0) \xrightarrow{f} X$ is a Serre.
$P(X,x_0) = \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}$, and the map f is $\gamma \mapsto \gamma(1)$.

remark: $f^{-1}(x_0) = \Omega(X, x_0)$, the loops based at x_0.

From the long exact sequence we get

$$\to \pi_n(\Omega(X,x_0)) \to \underbrace{\pi_n(P(X,x_0))}_{\sim 0} \to \pi_n(X)$$

~ 0, we can contract the loops

$\Rightarrow \pi_{n+1}(X) = \pi_n(\Omega X)$.

remark: from the Hopf fibration, we see, that $\forall n \geq 3$ $\pi_n(S^3) = \pi_n(S^2)$, in particular $\pi_3(S^2) = \mathbb{Z}$

$\boxed{HF_{17}}$ $\pi_n(S^7) = \pi_{n-1}(S^6) \oplus \pi_n(S^7)$

$\pi_n(S^{15}) = \pi_{n-1}(S^8) \oplus \pi_n(S^{15})$

Jet spaces

For tangent spaces, we considered curves at a point, they were equivalent if they were the same up to first order. Now we do the same up to k-th order equivalence. The curves become $\leq \deg k$ polynomials locally.

$\mathcal{J}^k_{(0,x_0)}(\mathbb{R}, M) = \{\gamma \in C^\infty(\mathbb{R}, M) \mid \gamma(0) = x_0\} / k\text{-th order agreement}$

$\mathcal{J}^k(M)$ ← this is a fibration, but not a vector bundle as the transition maps are not linear.

$$\begin{tikzcd} \mathcal{J}^k(M) \arrow[d, "P_n^{\leq k} \simeq \{\deg \leq k \text{ poly}\}"] \\ M \end{tikzcd}$$

$\boxed{HF_{21}}$

$$\begin{tikzcd} S^3 \arrow[r, "h"] & S^2 \arrow[d, "k"] \\ S^3 \arrow[u, "k"] & S^2 \end{tikzcd}$$

mi a kompozíció foka?

$\boxed{HF_{22}}$ $X_\varphi = D^4 \cup_\varphi S^2$. Ha φ a Hopf leképezés, akkor $X_\varphi \simeq \mathbb{CP}^2$, ha φ konstans, akkor pedig $X_\varphi \simeq S^4 \vee S^2$. Ebből lássuk, hogy Hopf $\not\simeq 0$ homológiailag.

$\boxed{HF_{23}}$ $S(A_p) \sim S^3 \vee (\bigvee^{2p} S^2)$

Szuszpenzió

Theorem (Whitehead): $f: X \to Y$ X, Y connected CW complexes if $\forall n$ $f_{*n}: \pi_n X \to \pi_n Y$ is an isomorphism, then f is a homotopy equivalence.

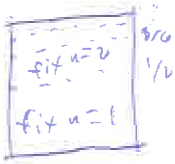
$\boxed{HW_{24}}$ Construct connected CW complexes X, Y s.t. $\forall n$ $\pi_n X \simeq \pi_n Y$, but $X \not\simeq Y$.

Proof: assume f is an embedding of a subcomplex: $X \subset Y$. Look at the long exact sequence

$$\pi_n X \overset{\text{iso}}{\xrightarrow{\sim}} \pi_n Y \to \overset{=0}{\pi_n(Y, X)} \to \pi_n(X) \overset{\text{iso}}{\xrightarrow{\sim}}$$

f_* are isom. $(\Rightarrow) \pi_n(Y,X)=0 \quad \forall n.$

take e^n minimal cell in $Y\setminus X$, $\partial e^n \subset X$. Every rel. spheroid is Θ, so e^n can be moved into X, this will leave $\partial e^n \subset X$ fixed, this homotopy extends to Y. If we have ∞-ly many cells, we can do the whole $sk_n(Y)$ at the same time, since it is also a subcomplex.



we put Y on top, this will be continuous because of the CW topology

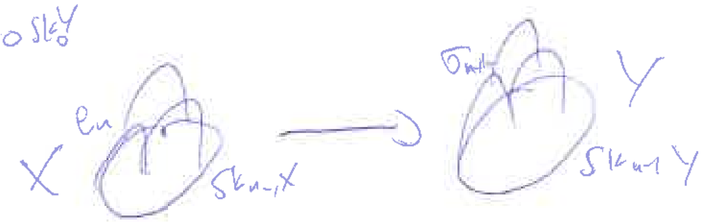
In the general case, let f be a cellular map. $f: X \to Y$

$$Y \cup Cyl(f) \simeq$$

So we are good, if f is cellular.

Claim: $f: X \to Y$ is homotopic to a cellular map.

induction on cell dim. $f(Sk_n^X)$ can be homotoped to Sk_n^Y



$e^n \subset X$ maps to Y, its image meets only finitely many cell interiors. $f(e^n) \cap \sigma^{n+k} \neq \emptyset$

if the image doesn't cover the whole cell, we can blow it down to the boundary, and so f no longer intersects the cell σ^{n+k}

if the image is the cell, then we approximate f on the cell by a smooth map, that can't be surjective. With this, we get that $f(e^n) \subset Sk_n Y$. For infinitely many cells in a dimension, this can be done all at the same time independently of one another, finally we fix dimensions one by one, like before. (the homotopy will be constant after some time at each point, thus continuous)

Jet spaces cont.

M^m, N^n $\quad J^r_{p,q} = \{\text{smooth } f(M,p) \to (N,q)\} / \deg^r \text{ tangency}$

$p \in M$, $q \in N$

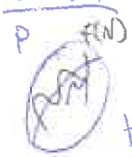
$J^r(M,N) = \bigcup_{p,q} J^r_{p,q}$, $f \in C^\infty(M,N)$, $j^r f$;

$\downarrow$

$M \times N \to M$

$(\ldots) j^r f: M \to J^r(M,N)$, s.t. this is a section

Def.: $M \subset P$ submanifold $f: N \to P$ f is transverse to M ($f \pitchfork M$), if $\forall p \in M \cap f(N)$ $T_p(M) \oplus \mathrm{Im}\, df_x = T_p P$, $p = f(x)$



Corollary: $f^{-1}(M) \subset N$ is a submanifold if $f \pitchfork M$

theorem: $\{f \in C^r(M,P) \mid f \pitchfork M\}$ is dense in $C^r(N,P)$, dense open if W is closed, G_δ otherwise.

Proof: if $P = \mathbb{R}^n$, $M = \mathbb{R}^m \times \{0\}$

$f \pitchfork M \iff p_2 \circ f$ has 0 as a regular value

$$\mathbb{R}^m \times \mathbb{R}^n \to 0 \times \mathbb{R}^n$$

generally we do the same on each chart, take bump functions, and we see that there is an open set of possible modifications of f s.t. its transversal, we glue them together by bump functions s.t. the bump has smaller derivative than the open sets before,

$\boxed{HF_{25}}$ $H^*(\Sigma X) = ?$

redukált szuszpenzió

$\boxed{HF_{26}}$

Szisofon 3 féle nevű rész: lokális, lok max és nyereg. hány darab lehet ezekből

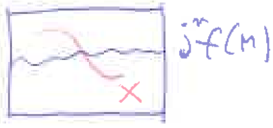
$\boxed{HF_{27}}$ $\pi_m(X \vee_p D^{n+1}) = \begin{cases} \pi_m(X) & m < n \\ \pi_m(X)/\langle [\varphi] \rangle & m = n \end{cases}$

$[\varphi]$ orbitja a π_1 hatás alatt

Tétel (Jet-transzverzalitás): $X \subset J^r(M,N)$ részsokaság, $\{f \in C^r(M,N) \mid j^r f \pitchfork X\}$ sűrű nyílt, ha M kpt, G_δ különben

Proof: $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ is easier. The r-jet space can be identified with the deg. $\le r$ polynomials.

$J^r(M,N) \hookrightarrow M \times \mathcal{P}^r_{m,n}$, let $f \in C^r$ be arbitrary,



we push f with a polynomial P

$j^r(f+p) = j^r f + j^r p$

we can shift arbitrarily in the $\mathcal{P}^r_{m,n}$ direction, so we can get transversal intersections.

On general M, we cover it by charts, and repeat the previous argument.

we want a regular value for $z \mapsto z - j^r f$ close to 0, and we can find that. $\square$

Corollary: generic $f: \mathbb{R}^n \to \mathbb{R}$ have only nondegenerate critical points

Proof: $X = \{2\text{-jets } \mathbb{R}^n \to \mathbb{R},$ linear part $= 0$, quad part is degenerate$\}$

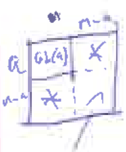
this is almost a manifold, fix the rank of the quad part, and its kernel, and the nondegen part.

This is a finite union of, codim $\geq n+1$ elements $\iff$

$\nexists j^2 f \pitchfork X \iff j^2 f \cap X = \emptyset$

def: $\Sigma^r = \{A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n) \mid \mathrm{rk}(A) = \underbrace{\min\{m,n\} - r}_{a}\}$, this is a subspace of $J^1(\mathbb{R}^m, \mathbb{R}^n)$
$M \quad N$

we want $\mathrm{codim}_{\mathrm{Hom}(\mathbb{R}^m,\mathbb{R}^n)} \Sigma^r$



upper $a \times a$ square nondegen, generates every column of the matrix

no choice $a^2 + a(m-a) + a(n-a) = a(m+n-a)$ is the whole dimension the whole space is mn dimensional, so the codim is $mn - a(m+n-a)$

assume $n = m+k$, so $a = m - r$

$m(m+k) - (m-r)(2m+k-m+r) = m^2+mk-m^2 \ldots$

$= r(k+r)$

remark: Σ_i points are not an embedded submanifold! also, an embedded submanifold of $\dim < \operatorname{codim} \Sigma_i$ can evade all of the Σ_i points of the map.

$\boxed{HF_{18}}$ Show, that $f: M^n \to \mathbb{R}^{2n}$ is generically an immersion.

def.: f is self-transverse, if $\forall p, q$ ($p \neq q$) $: f(p) = f(q) \Rightarrow$ $f: M \to N$

$\Rightarrow \operatorname{im} T_p f + \operatorname{im} T_q f = T_{f(p)} N$

Claim: any immersion can be arbitrarily approximated by self-transverse immersions (C^2 immersion, M cpct)

Proof: $\Delta_f := \{(p,q) \in M \times M \setminus \text{diagonal} : f(p) = f(q)\}$, because the immersions are local embeddings, Δ_f does not intersect a neighbourhood of the diagonal.

$M = \bigcup U_j \leftarrow$ charts, and $f|_{U_j}$ embedding $\forall j$

we can deform $f|_{U_i}$ so its transverse to $f|_{U_j}$, locally

$x \mapsto Ax + o(\|x\|)$ is the map, we bound uniformly the first and second derivative, so that $\|Ax - Ay\| \geq \frac{1}{\|A^{-1}\|}\|x-y\|$ stays true in the chart, guaranteeing $f|_{U_i}$ staying an embedding throughout. $\square$

def.: $f: M \to \mathbb{R}$ is Morse, if $\forall$ critical points are nondegenerate.

remark: From the jet transversality theorem, a generic $M \to \mathbb{R}$ map is Morse. (dense in $C^2(M, \mathbb{R})$)

f Morse $\Rightarrow \operatorname{Crit}(f)$ is discrete

claim: $p \in \operatorname{Crit}(f)$, f Morse $\Rightarrow$ in some local coordinates about p: $f(x_1, \ldots, x_n) = f(0) + x^T f''(\ldots) x$

proof.: Start from some coordinates, inductively get $f = c \pm x_1^2 \pm x_2^2 \pm \ldots \pm x_m^2 \pm f(x_{m+1}, \ldots, x_n)$.

We pick the coordinates s.t. $f''(p) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$

$\boxed{HF_{28}}$ $X * Y = X \times Y \times I / \{x \times Y \times 0\}, \{X \times y \times 1\}$

„formális konvex kombináció"

$H_*(X * Y) = ?$

$\boxed{HF_{29}}$ $T^2 \to \mathbb{R}^3$ nem homotópiával kifordíthatók.

theorem (Freudenthal): $S: \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is surjective if $n \geq k+1$, and bijective if $n \geq k+2$. *

def.: $\pi^S(k) = \lim_{n \to \infty} \pi_{n+k}(S^n)$ makes sense

*2) $\ker S: \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$ is generated by $[\mathrm{id}_{S^n}, \mathrm{id}_{S^n}]$

def. (Whitehead product): $f: S^a \to X$, $g: S^b \to X$, we define

$[f, g]: S^{a+b-1} \to X$

$\boxed{D^{a+b}}$ $S^{a+b-1} = S^{a-1} \times D^b \cup D^a \times S^{b-1}$

$(x,y) \mapsto g(y)$ | $S^b = D^b / \partial D^b$, the boundary goes to a point

$(x,y) \mapsto f(x)$

Pontrjagin construction

$f: S^{n+k} \to S^n$???

$k = 0$: deg

$k > 0$, we get closed submanifolds of dim k.

$(df)_q^{-1}(v) \wedge (T_q f^{-1}(p))^{\perp} = f^*(v)$

we set a continuous choice of n normal frame for $f^{-1}(p)$.

$M^k \hookrightarrow S^{n+k}$ with n independent normals.

In the other direction:

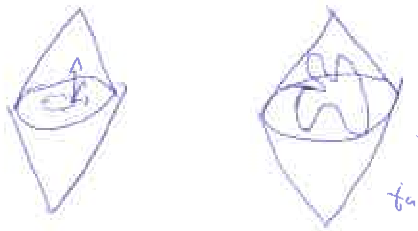
make a tubular neighbourhood. Map each tubular section to the standard D^n, with the normals giving the isomorphism.

we factor out D^n by ∂D^n, so this map on the tubular neighbourhood is defined, and its constant on the boundary, this extends to the all of S^{n+k}, giving a map to S^n.

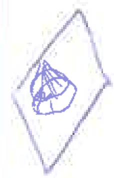
Those two constructions are inverses to each other up to homotopy of functions and framed cobordism of ~~on~~ submanifolds.

this means that $\pi_{n+k}(S^n) \longleftrightarrow \mathrm{Emb}^{fr}(k, S^{n+k})$, this is the Pontrjagin construction.

$$
\begin{array}{ccc}
S: \pi_{n+k}(S^n) & \to & \pi_{n+k+1}(S^{n+1}) \\
\updownarrow & & \updownarrow \\
\mathrm{Emb}^{fr}(k, S^{n+k}) & \to & \mathrm{Emb}^{fr}(k, S^{n+k+1})
\end{array}
$$



double point projection directions form a 2k-dim subspace. tangent directions are a 2k-1 dim. subspace so if $n \geq k$, a generic projection is good for us, since we only have in $S^{n+k} \times [0,1]/\sim$ an S^{n+k} choice of projecting direction. The framing might not be good, so we want an embed a cone over M^k.



we flow along the favorite normal vectors, and then pull M into a point, we delete the top of the cone, so its a manifold, and we approximate this map by an embedding, pull the manifold up, and project down, so the k+1. st vectors point up, thus the map is surjective for $n > k$.

Injection goes similarly, we do this project for a framed cobordism, s.t. the framing becomes good, and the cobordism gets mapped to the equatorial S^{n+k+1}, so this works for $n > k+1$. $\square$

$\boxed{HW_{30}}$ give a Morse function on $\mathbb{RP}^2$ with 3 crit. points, is there another surface with a fn. like this?

$\boxed{HW_{31}}$ $f: M^n \hookrightarrow \mathbb{R}^N$, prove that for almost all lines in $\mathbb{R}^N$, the projection to that line composed with f is Morse.

$\boxed{HW_{32}}$ $f: X \to Y$ induces isomorphisms on all π_n, (X, Y: cw complex, connected) it does so on H_n as well.

$\boxed{HW_{33}}$ If the target space is a top. group, the whitehead product is 0.

$\boxed{HW_{34}}$ a) $S^{2k} \not\simeq$ top. group (hom. equiv)

b) S^{2k} is not a retract of a top. group.

$\boxed{\text{Fé19}}$ M^n parallelizálható ⟺ M-en n sehol sem eltűnő vektormező.

$\boxed{\text{Fé20}}$ $S^{n_1}\times S^{n_2}\times \dots \times S^{n_k}$ mikor parallelizálható, ha $k \geq 2$, $n_i > 0$.

$\boxed{\text{Fé21}}$ $\mathbb{C}P^3 \supset X = \{[z_0;\dots;z_3] \mid \sum_0^3 z_i^4 = 0\}$ ez sokaság implicit fv. tétel miatt. ~~[illegible]~~ Morse függvény / homológia

ált. Freudenthal tétel: X k-öf. $\Rightarrow S: \pi_n(X) \to \pi_{n+1}(SX)$ izo. $n < 2k+1$, epi $n \le 2k+2$-re.

faktorizációs tétel: ha A m-öf., (X,A) n-öf. akkor $\pi_r(X,A) \to \pi_r(X/A)$ izo. $r \le n+m$, epi: $r \le n+m+1$-re

ebből a tétel biz.: $SX = CX/X$, a pár HES-ből

$\pi_{n+1}(CX,X) = \pi_n(X)$, X k-öf $\Rightarrow m=k$, (CX,X) $k+1$ öf. ✓

kivágási tétel: $X = A\cup B$ nyíltak, $(A, A\cap B)$ n-öf $\Rightarrow$ $(B, A\cap B)$ m-öf.

⇓

akkor $\pi_r(A, A\cap B) \to \pi_r(A\cup B, B)$ izo. $m+n-1 \geq r$, epi $m+n \geq r$-re.

kiv. ⟹ fakt. biz: $X/A \sim X \cup CA$ (X cw!)

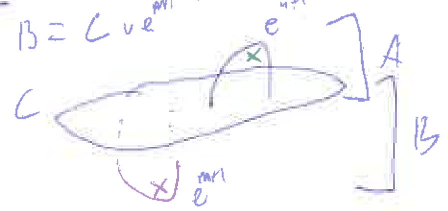
$\pi_r(X\cup CA, CA) = \pi_r(X \cup CA) = \pi_r(X/A)$

$A = X$ $B = CA$ val faktorizációs tétel szerint

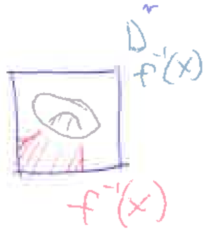
(X,A) n-öf, (CA,A) $m+1$-öf

kivágás biz.:

lemma: $A = C \cup e^{n+1}$-re igaz a kivágási tétel
$B = C \cup e^{m+1}$ $(C = A\cap B)$



Biz: $D^r \to C \cup e^{n+1} \cup e^{m+1}$, a perem [illegible] C-be, eleve elkerüli e^{n+1}-et.

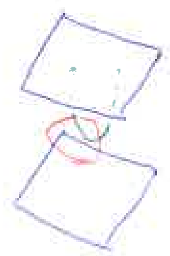


e^{n+1} egy pontjának őse zárt részsokaság (sima appr., stb)

$f^{-1}(x)$ $n+1$ kodimenziós, $f^{-1}(x)$ $m+1$ kodim egy homotópia még eggyel csökkenti a kodimenziót ha $n+m < r$. A négyzet oldalát feltoljuk a generikus irány ↑ //// szakaszok mentén, ezek elkerülik ○-et, ez egy homotópia $(A\cup B, B)$-ben, és a feltolás után egy $(A, A\cap B)$ reprezentánst kapunk (egy-egy pontot elkerül, lefújjuk). *

Ha vannak kisebb cellák, azokat a pár összefüggősége miatt lefújhatjuk, ebből kapunk egy vele homotóp ekvivalens teret (sőt def. retraktum), ebben induktívan az előző állítást alkalmazzuk. Ha ∞-sok cella van, egy adott szferoid véges sokat metsz. ✓

* injektivitás hiányzik még.
ismét feltoljuk a D-t úgy a kocka alsó lapját, ezzel elkerülve azt.



Def.: $\pi_n^S(X) = \lim_k \pi_{n+k}(S^k X)$.

$k \geq n+1$-re már stabilizálódik

ez egy homológiaelmélet! lol

nem sima, mert a pontból a gömböket kapjuk ...

emlék: $[X, S^{N-n}] \longleftrightarrow \mathrm{Cob}^{fr}_{N-n}(\emptyset, X)$

N-n kobordizmus X-ben

Pontrjagin–Thom konstrukció: $[X^N, T\xi^n]$

$\updownarrow$ 1:1

$\left\{ \xi \xrightarrow{\mathbb{R}^n} B \right.$ vektornyaláb $\mathrm{Cob}^{\xi}(N-n, X)$

fibrumként egy közelép

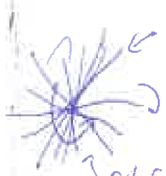
$T\xi = D\xi / S\xi \longleftarrow$ az összes pontot egy pontba behúzzuk.

$\{ i: M \hookrightarrow X,\ j: M \to B,\ \nu_{M \subset X} \cong j^* \xi \}$

példa (bábi: D) $\xi^n = \mathbb{R}^n \to \cdot$

$\xi^1 = \mathbb{R}P^\infty$ feletti tautologikus vonalnyaláb

$\mathbb{R}P^1$ felett Möbius szalag



egymásik $\mathbb{R}P^1$ pontjai

fel fordult

hc $\mathbb{R}P^1$-en közlemosyánk.

ebből növő unióként.

$\mathbb{R}P^\infty$ klass $\mathbb{Z}$. fér, elég az 1-vázon megegyeznie

két lekérzések: $[X \to \mathbb{R}P^\infty] \longleftrightarrow \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2)$

$\boxed{HF_{39}}$ $T\begin{pmatrix} \gamma^1 \\ \downarrow \\ \mathbb{R}P^n \end{pmatrix} = \mathbb{R}P^{n+1}$

$\boxed{HF_{37}}$ metszési index kobordizmus invariáns.

$M^a, N^b \hookrightarrow P^{a+b}$, $W^{a+1}, Z^{b+1} \hookrightarrow P \times [0,1]$

kobordizmus M, M' ; kob. N, N'

$\boxed{HF_{36}}$ $\pi_i(V_{n,k}) = \begin{cases} 0 & i < n-k \\ \text{ciklikus} & i = n-k \end{cases}$

Compression theorem

Theorem: $(P^n, \nu) \hookrightarrow Q^q \times \mathbb{R}^1$, $p < q \Rightarrow$

P cpt, ν normal vectorfield (not in $\mathrm{Im}\,TP^n$)

$\exists$ isotopy of $Q^q \times \mathbb{R}$ s.t. the image of ν becomes $\partial/\partial s$ ← s coordinate of $\mathbb{R}$. $\square$

remark: $p = q$ case, a sphere in $\mathbb{R}^3$ would give an immersion of S^2 to $\mathbb{R}^2$ ↯

Proof: $\partial/\partial s$ points "up", we want to turn the tangent vectors. This cannot work, iff the image of ν is straight down, we fix this: normalize ν, so locally $\nu: U \to S^q \subset \mathbb{R}^{q+1}$, from the dimension criterion we know, we can avoid the south pole, and with a small deformation $\mathrm{Im}\,\nu$ is still outside of $\mathrm{Im}\,TP$.

Now we rotate $\mathrm{Im}\,\nu$.

Problem if we intersect TP. Only happens, if we rotate at least $\frac{\pi}{2}$ degrees, if we deform ~~table~~ ν to be strictly normal. This makes ν into the upper halfspace.

Now extend ν to a weakly upward pointing field on the whole $Q^q \times \mathbb{R}$ ($\bar{\nu}$ s.t. $\bar{\nu}(s) > \varepsilon$) we do a linear combination with $\partial/\partial s$, so it's $\partial/\partial s$ on the boundary of the tubular neighbourhood given by ν, and extend to a constant $\partial/\partial s$ outside. Take the flow $\frac{d}{dt}\Phi_t = \bar{\nu}$

P^n is compact, its $S \in \mathbb{R}^1$ coordinate is bounded, we let the flow go for at least $\frac{S_2 - S_1}{\varepsilon}$ time

P ends up in the $\bar{\nu} \equiv \partial/\partial s$ region, at time 0 $\nu = \bar{\nu}$, at time $t > \frac{S_2 - S_1}{\varepsilon}$ $\nu \equiv \partial/\partial s$, this is our isotopy. $\square$

Remark: The time to run this flow can be arbitrarily low, this gives arbitrarily C^0-small isotopies.

We could also have multiple vector fields, then $(P^p, \nu_1, \dots, \nu_k) \hookrightarrow Q^q \times \mathbb{R}^k$, $p+k+1 < q$ works also.

To prove this from the basic version, we need an analog for immersions, it works, but with regular homotopy, not isotopy.

Cor.: The higher homotopy groups of spheres are actually represented by immersed oriented hypersurfaces in $\mathbb{R}^{k+1}$ ($S^{n+k} \to S^n$, we successively straighten the vectors, until we get only one normal, which gives the orientation).

Theorem (Hirsch): $\mathrm{Imm}(P^p, Q^q)$ is weak hom. equiv. to $\mathrm{Mono}(TP, TQ)$.
fiberwise monomorphism

Proof (The epi. case):
We have an $f: P \to Q$ continous.

$$TP \rightarrowtail TQ \text{ over } f.$$

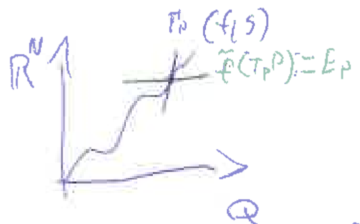
We need to deform this to get an immersion. Assume f smooth, pick $g: P \hookrightarrow \mathbb{R}^N$

we want to move

let $T_p^{\#}$:= the image of T_pP under $(f,g) \subseteq \mathbb{R}^N \times Q$, we also have

$\tilde{f}: T_pP \to T_{f(p)}Q$

$$T_p^{\dagger} \angle \overset{\cong}{=} T_pP \overset{\cong}{\to} E_p$$



$T_p^{\dagger} \cong E_p$, assume orthogonally.

extend this to a
$$T(Q\times\mathbb{R}^N) \overset{\Phi}{\to} T(Q\times\mathbb{R}^N) \text{ isomorphism.}$$

$e_1, \dots, e_N$ gives the orthocomplement of E_p, the vector fields $\Phi e_1, \dots, \Phi e_N$ are weak orthogonal to P. Now we compress, it straightens out the Φe_i-s, and so moves T_p to E_p, so now the projection onto Q gives an immersion. □

The inj., we take the homotopy in $\mathrm{Mono}(T(P\times I), TQ)$ this gives an immersion $P\times I \hookrightarrow Q$, we can take this constant on the boundary. For the bigger ones, we need a parametric version of the compression theorem.

$\boxed{H_w^{40}}$ M^{2k} is stabily parallelizable $\Rightarrow$ parallelizable
$$TM \oplus \varepsilon^m = \varepsilon^{k+m}$$
sum of vector bundles
trivial bundle.

$\boxed{H_w^{41}}$ Σ a surface, then all elements in $H_1(\Sigma; \mathbb{Z}_2)$; $H(\Sigma, \mathbb{Z})$ are representable by closed curves.

$\boxed{H^{42}}$ Adjunk példát $\tilde{f}_1, \tilde{f}_2\ \Phi \to \mathbb{R}^2$ izotópia erejéig különböző leképezésre, melyek u.a. a powerne.

Morse theory

$f: M \to \mathbb{R}$, M closed

claim: p critical point of $f \Rightarrow \exists$ some chart around p where $f(x_1,\dots,x_n) = c - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2$.

Proof:

lemma: $f(0)=0 \Rightarrow f(\underline{x}) = \sum x_i F_i(\underline{x})$ with F_i smooth.

pr: $f(\underline{x}) = \int_0^1 \frac{\partial f(t\underline{x})}{\partial t}\, dt = \sum_{i=1}^{n} \int_0^1 x_i \partial_i f(t\underline{x})\, dt$ □

since p is critical, $F_i(0)=0$ as well, so $f(\underline{x}) = \sum x_i x_j H_{ij}(\underline{x})$, we can symmetrize this, so $H_{ij} = H_{ji}$, $H_{ij}(0) = \partial_i \partial_j f(0)$, and Gram-Schmidt. □

$M^c := \{p \in M \mid f(p) \leq c\}$. for c small $\Rightarrow M^c = \emptyset$

c large $\Rightarrow M^c = M$.

theorem: if f has no crit. values in $[a,b]$, then M^a is a deformation retract of M^b, even diffeomorphic to it.

proof: choose a metric on M, grad f, or rather the normalized version $\frac{\text{grad} f}{\|\text{grad} f\|^2} = V$ makes the flow move level sets to level sets. $Vf = \left\langle \frac{\text{grad} f}{\|\text{grad} f\|^2}, \text{grad} f \right\rangle = 1$.

Take the flow ϕ_t^{-V} on $f^{-1}([a-\varepsilon, b+\varepsilon])$.

on M^a: id, on $M^b \setminus M^a$: ϕ_t for $t \in [0, f(p)-a]$.

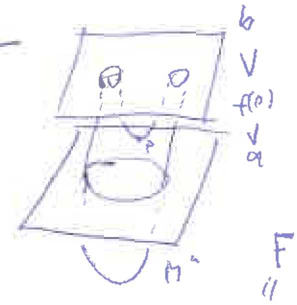
This is a def. retract.

Now take $\lambda: M \to [0,1]$ smooth s.t. $\lambda|_{M \setminus M^{a-\varepsilon}} \equiv 1$ and $\lambda|_{M^{a-\varepsilon}} \equiv 0$, and $\phi_{b-a}^{-\lambda V}$ gives a diffeomorphism. □

Theorem: If f has a single crit. point in $M^b \setminus M^{a-\varepsilon}$ of index j, then $M^b \simeq M^a \cup D^j$.

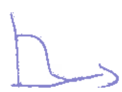
(diffeom. to $M^a \cup_{\partial D^j \times D^{n-j}} D^j \times D^{n-j}$).

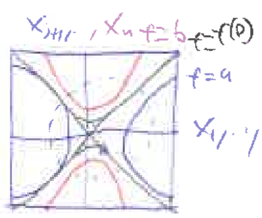
Proof:



F $\shortparallel$

$f \rightsquigarrow f - \mu(\xi + \eta)$, where $f = c - \xi + \eta$ around p (in U) and μ is a bump function





we want: $M_F^b = M_f^b$

$M_F^a = M_f^a$ and

$M_F^{f(p)} \setminus M_f^{f(p)} \subset U$, F has no other critical points in $(f(p), b)$, it will have only p.

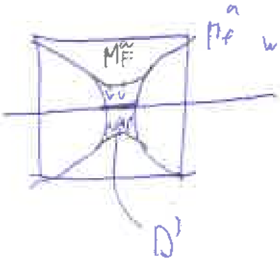
μ will be monotone decreasing, $0 \geq \mu' > -1$

$\mu'(0) = 0$

$dF = df - \mu'(\xi + \eta)(d\xi + d\eta)$ //

$M_f^a \cap \{x_1 = q_1, \dots, x_j = q_j\} = \{\eta \leq \sum q_i^2 - (f(p)-a)\}$

F is monotone, sol. symmetric, so the intersection with M_F^a will be a larger ball.



M_F^a we deform the new level set to the old one, and the D^1 disk

$\square$

~~lemma~~: $p, q \in \mathrm{crit}(f)$, $\mathrm{ind}\, p < \mathrm{ind}\, q$, $f(p) \geq f(q)$

$\Rightarrow \exists g: M \to \mathbb{R}$ Morse s.t. $\mathrm{crit}(f) = \mathrm{crit}(g)$, the indicies stay the same, but $g(p) < g(q)$.

Proof: $r \in \mathrm{Crit}(f) \Rightarrow C_r = \{x \in M \mid$ gradient flow takes x to r at $\pm\infty$ time $\}^*$. If $C_p^- \cap C_q^+ = \emptyset$ ~

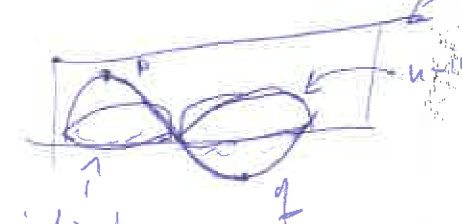
C_r^- C_r^+

* iterated closure, and limit of the ~~downwards~~ descending manifold.

– we can push f down on ~~a neighbourhood~~ the whole descending manifold, or "start" along flow lines, and smooth it together with f.

If we have a flow line between p and q

level set of f,



ind $q-1$

ind $p-1$

dimension difference $n-2-(\mathrm{ind}\, q - \mathrm{ind}\, p) \leq n-2$, so generically they shouldn't meet.

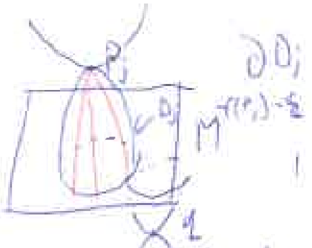
Take a tubular neighbourhood given by flow lines of the ascending manifold of q in the level set, ~~modify~~ ~~small~~ deform this tubular part a bit, so they are disjoint and change the metric, so the new vector field is the gradient.

$\boxed{HF_{43}}$ $\oplus \varepsilon'$: $\mathrm{Vect}_k(M) \to \mathrm{Vect}_{k+1}(M)$ surj. ha $k \geq n$, ~~bij~~ bij. ha $k \geq n+1$

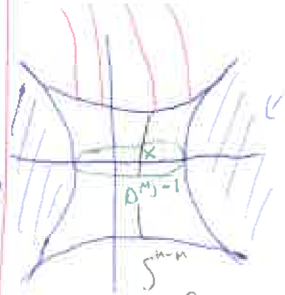
$f: M \to \mathbb{R}$, p_i crit points of index m_i.

$\leadsto D^{m_i}$ cells. Cellular homology $C_k = \langle p_i \mid m_i = k \rangle$

what is $\partial p_j = ?$



the cells here are crit. points of index $m_j - 1$

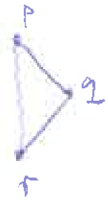


$M^{f(q)-\varepsilon}$

the flowlines form a descending sphere

$\subset$ boundary of the handle

$\partial^2 = 0$!



compactify $M(p,r)$, the flow lines cancel.

Cor: Poincaré duality :)

Theorem: $\partial W^{n+1} = M_0^n \sqcup M_1^n$, s.t. $M_0 \hookrightarrow W$, $M_1 \hookrightarrow W$ are homotopy equivalences, $\dim W \geq 6$, $\pi_1(W) = \pi_1(M_0) = \pi_1(M_1) = 1$

$\Rightarrow W$ is diffeomorphic to $M_1 \times I$

~~Proof~~

Prop: $M_0 \hookrightarrow W$ and $M_1 \hookrightarrow W$ are homotopy equivalences and all are simply connected $\Leftrightarrow \forall i\ \pi_i(W, M_0) = \pi_i(W, M_1) = 0$, and also

$\Leftrightarrow H_i(W, M_0) = H_i(W, M_1) = 0$

Proof: Whitehead's and Hurewicz's theorems give the first and second implications. □

Proof (h-cob): pick $f: W \to [0,1]$ Morse, $f(M_0)=0$, $f(M_1)=1$, and in a neighbourhood near M_0, M_1 we have an embedding s.t. $M_0 \times I \hookrightarrow W$ commutes.

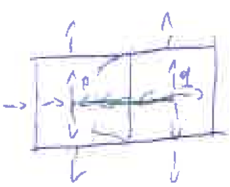
$$\begin{tikzcd} M_0 \times I \arrow[r, hook] \arrow[dr, "pr_2"'] & W \arrow[d, "f"] \\ & {[0,1]} \end{tikzcd}$$

We can construct this with smooth extensions and ~~good~~ Morse approximations.

Assume f weakly self-indexing $\operatorname{ind} p < \operatorname{ind} q \Rightarrow f(p) < f(q)$

index and ind 1 can be canceled with the boundary, we flow down from the critical point to the 0-level

Lemma:  if $\operatorname{ind} p = \operatorname{ind} q + 1$, and there is a single (generic) gradient flow line from p to q, and no other critical values between $f(p)$ and $f(q)$, then these two critical points can be canceled.



the perpendicular flow can be canceled, we won't prove this.

Pr: $\dots \to C_j \to C_{j-1} \to \dots$ is an acyclic complex
$\Rightarrow$ it is the sum of $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

Proof: $0 \to \ker \partial_j \to C_j \to \operatorname{Im} \partial_j = \ker \partial_{j-1} \to 0$

$\langle b_1, \dots, b_n \rangle = \ker \partial_j$

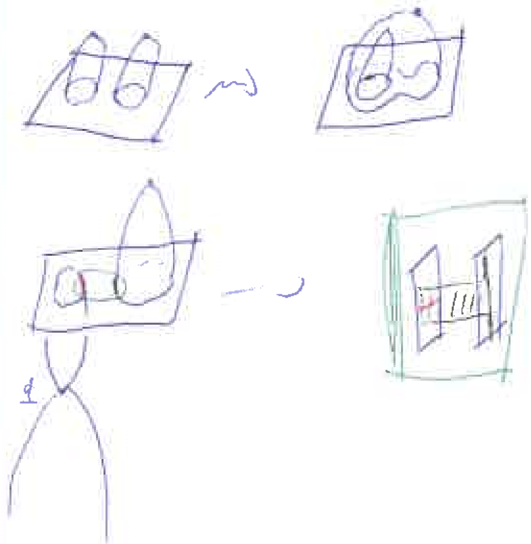
$\langle a_1, \dots, a_s \rangle = \operatorname{Im} \partial_j = \ker \partial_{j-1}$

So every step is the sum of

$(0 \to \mathbb{Z} \to \mathbb{Z} \to 0) \oplus (0 \to \mathbb{Z} \to \mathbb{Z} \to 0)[i]$. □

We mimic this change of basis on the manifold by changing f.

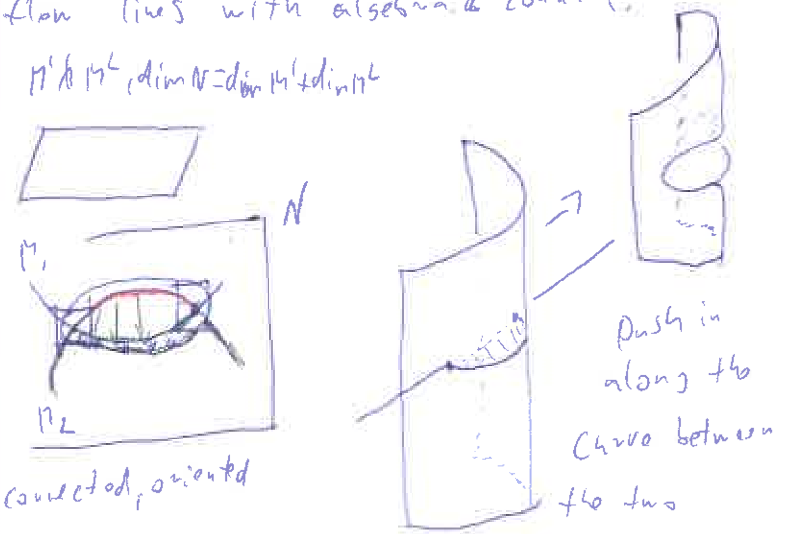
Handle slide $= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ change.



Lemma: Sign switches, basis permutations and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate every basis.

Problem still, is we could have multiple flow lines with algebraic count 1.

$M^k \pitchfork M^l$, $\dim N = \dim M^k + \dim M^l$

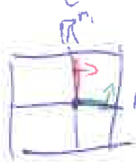


connected, oriented

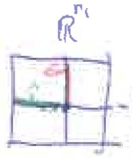
We need an embedded disc*

* inside the complement of $M^1 \cup M^2$, two curves between the two intersection points one in M^1, the other in M^2. We need to suppose $\pi_1(N) = 0$, we know nothing about these curves, also the disc has to avoid M^1, M^2, so $\operatorname{codim} M^1 \geq 3$, also the disc has to be embedded, so $\dim N \geq 5$.

We want to embed this disc so it has boundary the two curves. Pick a vectorfield along normal to M^1, M^2 the curves.



$\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, and

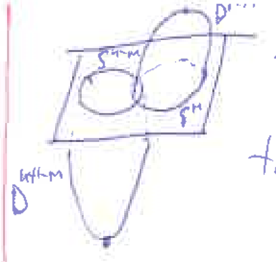


$\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$ at the two intersection points. rank of the normal bundle $\geq 2 \Rightarrow$ we can extend this to the whole cycle, embed the disc using the normal bundle, the normal bundles to $\cdot$, $\cdot$ are glued together to give the trivial bundle over S^1,

so $TD^2 \oplus \varepsilon^{m_1+m_2-2} \cong \varepsilon^{m_1+m_2}$

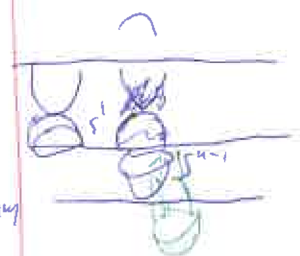
$\nu_{M^1 \subset N}$ $\nu_{M^2 \subset N}$

lastly we move the $\cdot$ curve along the disc, to the outside of $\cdot$, this extends to an isotopy of the space, and after this trick, we see no intersection.



D^{n+1} S^{n-1} S^{n} D^{m+m}

this trick works ~~onto~~ from ind $q \geq 3$ to ind $p \leq n-3$



3
2
1

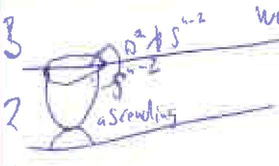
$\pi_1(W^{1\frac{1}{2}} \setminus S^{n-1}) \overset{?}{=} 1$

actually, cut out a neighbourhood of p, and its $\widehat{\iota_{\nu}}$ manifolds,

we flow down $\circ$ to level $\frac{1}{2}$, we only cut out S^0 ~~descending points~~, so we are in

$$\pi_1(W^{=\frac{1}{2}} \setminus S^0) = \pi_1(W^{=\frac{1}{2}}) = \pi_1(M_0) = 1$$

we flow back, and we are done, so we can put a disc ~~onto~~ between a 1,2 index crit. point as well. ($n \geq 3$)



3
2
1

$D^2 \wedge S^{n-2}$ S^{n-2} ascending

we want $\partial D^2 \cong *$ in $W^{=2\frac{1}{2}} \setminus S^{n-2}$, which "can" in principle be done, so we have an S^2 intersecting the ascending S^{n-2}. We add a new crit point pair of index 3 and 4, where the descending sphere of the index 3 is precisely the S^2 we constructed.

So we have geometric multiplicity one we eliminate every thing $\Rightarrow$ we get a cylinder.

Corr. (Reeb): $f: M \to \mathbb{R}$ Morse with 2 crit. points $\Rightarrow M \overset{\sim}{=} S^n \Leftarrow H_*(M) = H_*(S^n)$ atso.
$n \geq 6$

homeomorphic.

Proof:  ~~abstract~~ cut out a ball, get a space, with the homology of a cylinder.

$\pi_1(M) = 1$

Corr.: $M^n, \partial M^n$, $n \geq 6$ has homology $\cong H_*(D^n)$

$\Rightarrow M^n$ diffeomorphic to D^n | $H_*(M, \partial M) \cong H_*(D, \partial D)$

Proof:  pick a point, cut it out.

We are left with a homology cylinder $\Rightarrow$ its diffeomorphic to a cylinder, we glue it back and get D^n with a diffeomorphism.