Aldim sok

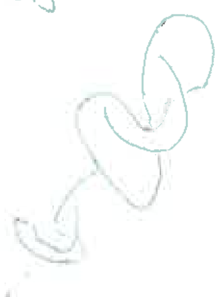
stipsicz.andras@renyi.hu

Knots

$S^1 \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$

$f: S^1 \to \mathbb{R}^3$ smooth regular embedding

no wild knots



Two knots K_1 & K_2 are isotopic iff $\exists\, C: S^1 \times [0,1] \to \mathbb{R}^3 \times [0,1]$

2) $C|_{S^1\times\{0\}} = K_1$, 3) $C|_{S^1\times\{1\}} = K_2$, 4) C is a regular smooth embedding, and

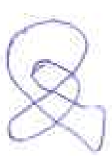
1) $C(S^1 \times \{t\}) \subset \mathbb{R}^3 \times \{t\}$

This is an equivalence relation; the classes are called knot types.

A link is a multicomponent knot.

$L: \coprod_1^N S^1 \to \mathbb{R}^3$

Consider the projection $\mathbb{R}^3 \to \mathbb{R}^2$ on K.



A generic projection will give only double points, with transverse intersections.

We also record which strand was above/below; this leads to knot diagrams. These determine the knot up to type.

Def.: two knots are concordant, if $\exists: C: S^1 \times [0,1] \to \mathbb{R}^3 \times [0,1]$, satisfying 2.–4).

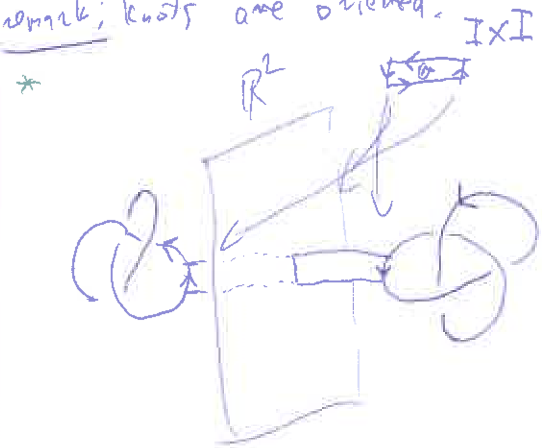
Def.: For a knot K, we define the mirror $m(K)$ by applying an orientation reversing diffeomorphism of $\mathbb{R}^3$ ($(x,y,z) \mapsto (x,y,-z)$)

ex.:

understrand(—) overstrand

Def.: K_1, K_2 two knots s.t. they can be separated by a plane in $\mathbb{R}^3$, we define their connected sum. *

remark: knots are oriented!

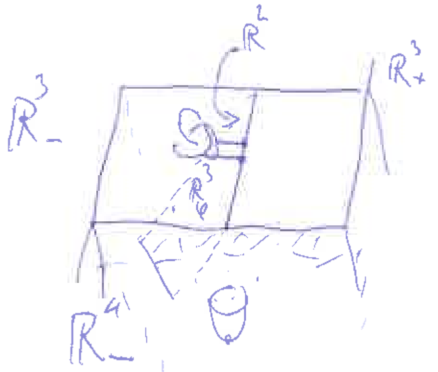


the band must meet the plane in an interval

$K_1, K_2 \longmapsto K_1 \# K_2$

↑ the knot type is well def.

Prop: $K \# m(-K)$ is concordant with the unknot



Polar coords (x, y)

we swing the punctured knot around, and on the lowest point, we see the unknot.

Remark: (knot types, $\#$, U) is a semigroup

Fact: $K \neq U$, then for any other knot K', $K \# K'$ will not be isotopic to the unknot

$g(K) \geq 0$ & $g(K) = 0 \iff K = U$ & $g(K_1 \# K_2) = g(K_1) + g(K_2)$

↑ genus, we'll see it later

Consider $\mathcal{C} = \{$concordance classes of knots$\}$, this is a group under connected sum, unknot id. element, the inverse $m(-K)$.

This group is Abelian (HW)

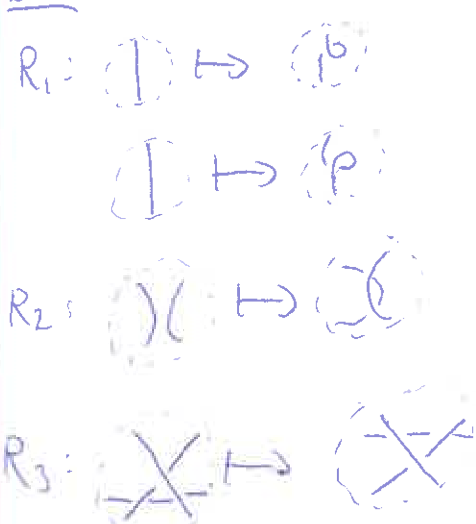
Not finitely generated :(but countable

figure 8 knot, this is symmetric.

$K \# m(-K) = U$
$\|$
$K \# K$

2-torsion!

Def.: Reidemeister moves



Theorem (Reidemeister): ⌐⌐

- If we apply R_1–R_3 to a knot diagram, we get isotopic knot types. ✓
- If D_1 & D_2 are diagrams of isotopic knots, then D_1 can be transformed into D_2 by a finite sequence of Reid. moves and planar isotopy.

Proof: hints

$$i: S^1 \times [0,1] \hookrightarrow \mathbb{R}^3 \times [0,1] \quad \text{isotopy}$$
$$\downarrow \text{proj}$$
$$g \dashrightarrow \mathbb{R}^2 \times [0,1] \ *$$

Theorem (Whitney) $g: \mathbb{R}^2 \to \mathbb{R}^3$ a generic function looks like either $(x,y) \mapsto (x,y,0)$ or $(x,y) \mapsto (x^2, xy, y^{*})$ ← "Whitney umbrella" the rank of J_g can be either

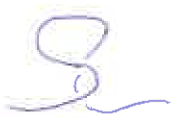
0, 1 or ②, and 0 is not a generic situation.

(x^2+y, y^2) $(x, y, 0)$

* We can have

- double ⟷ R_2
- triple ⟷ R_3
- singular points ⟷ R_1

Remark: One type of R_1 + $R_{2,3}$ ⟹ the other R_1



Def. (Alexander polynomial): *

Def: D is a diagram, $p \in D$ fix, not a crossing

a Kauffman state is a bijection between the crossings of D ($= Cr(D)$) and the set of domains which don't have p on their boundary s.t., i $K: Cr(D) \leftrightarrow \mathcal{D}_p(D)$ (bijective) s.t. $K(c)$ has c on its boundary.

Remark: $|Cr(D)| = |\mathcal{D}_p(D)|$ always. HF

* $\Delta_K(t) = \sum_{K \text{ kauffman state}} (-1)^{A(K)} t^{d(K)}$

there is also a link version $\in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ and a multivariable version $\in \mathbb{Z}[t_1^{\pm 1/2}, t_2^{\pm 1/2}, \dots, t_n^{\pm 1/2}]$

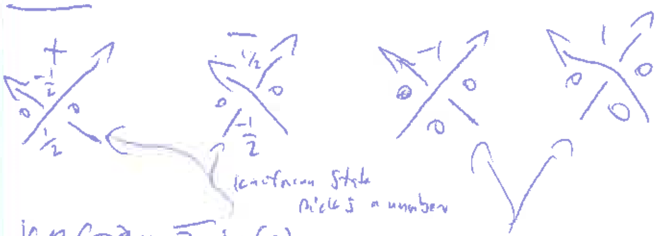
HF | D → graph Γ_B, Prove that Kauffman states

checkerboard color the domains (no domains of the same color have common boundary)

vertices are the black domains crossings are the edges (analogously Γ_W)

are in bijection with spanning trees of Γ_B, or Γ_W.

remark: crossings have an induced orientation



$A(K) = \sum_{c \in Cr(D)} K(c)$ $\quad M(K) = \sum_{c \in Cr(D)} K(c)$

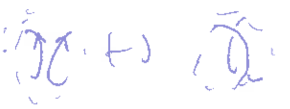
Def: $\Delta_K(t) = \sum_{K \text{ Kauffman}} (-1)^{M(K)} t^{A(K)}$

Prop: Δ_K is independent from (D, p)

Reidemeister moves, and moving p along the knot.

R_1: 

this gives 1 extra crossing and domain.
Every Kauffman state has to map the new crossing to the new domain, since it's the only neighbour of the new domain, it's easy to see, that A, and M is unchanged.

R_2:  2 new crossings, 1 new domain, and the middle crossing before is split in two.

$(-1)^i t^{\frac{-1}{2}} + (-1)^i t^{\frac{1}{2}} - 0$ ✓

6, 1 cancel $A: \frac{1}{2} - \frac{1}{2}$ $M: -1+0$

4, 3 cancel (similar)

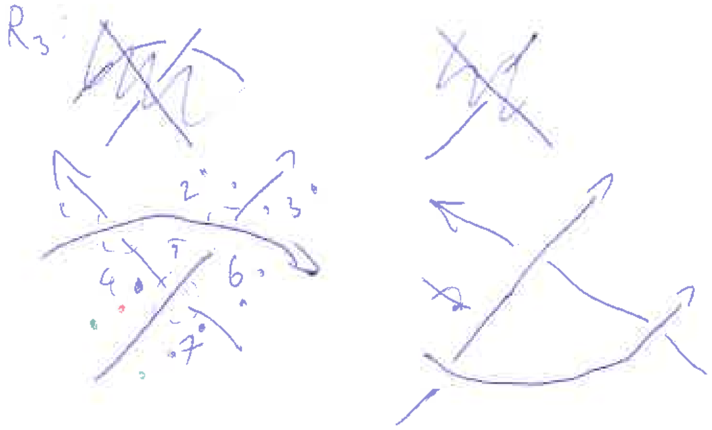
5, 2 give the before contribution

we deal with the identically oriented case

 the middle has a crossing above or below in the

Original Kauffman state, so if we want to pick 2 or 5, our choice is determined, and the A, and M is 0.

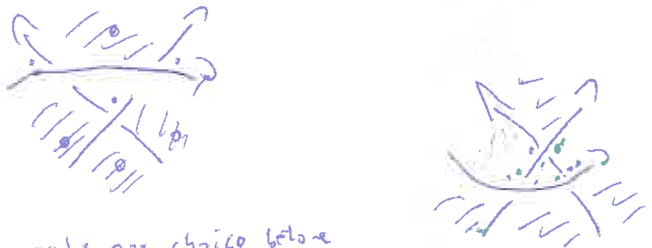
R_3:



three cases (...) depending on which domains are already occupied from outside choices.

only one choice! "4-0"

"3-1"



only one choice below

3 choices here

•	•	•
A-1	0	0
M-2	-1	0

2 options

$A(\cdot)=A(\cdot)$

2 options

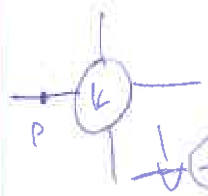
remark: changing the $(-1)^M$ to q^M (formal variable) doesn't lead to an invariant. The unknot and it's varieties with R_{1-3} write it.

$\Delta_0(t)=1$, $\Delta_{trefoil}(t)=t-1+t^{-1}$, for both handed versions, the Alexander poly. can't distinguish K from $m(K)$.

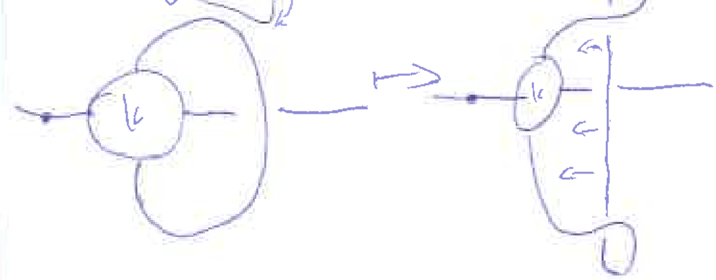
moving the point p:

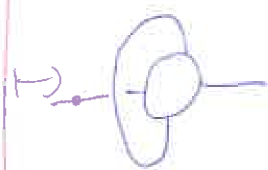
Project the crossing to infinity ($\mathbb{R}^2 \subset S^2$)

⇓



perturb the vertical strand





HW1: Extend the notion of Kauffmann states to link diagrams,

Invariance under R_{1-3} works the same, to move p we need a relation

Suppose we have

D_0 D_+ D_-

we claim, that $\Delta_{D_+}(t) - \Delta_{D_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{D_0}(t)$

(Skein relation)

and from this, induction.

↳ Suppose in D $\exists P \in D$ s.t. if we start traversing the knot, when we get to a crossing first, it will be an under crossing, it's the unknot

Def.: $K \subset S^3$, $S^3 \setminus K \mapsto \pi_1(S^3 \setminus K)$

$H_1(S^3 \setminus K, \mathbb{Z}) = \mathbb{Z}$

$\tilde{X} \leftarrow \pi_1(\tilde{X}) = [\pi_1(X), \pi_1(X)]$

$\downarrow$ ← cover map

X

$H_1(\tilde{X}, \mathbb{Z}) = \pi_1' / \pi_1''$ is a $\mathbb{Z}[t, t^{-1}]$ module

$\Delta_K(t)$ is derived from this module

$H_1(\tilde{X}, \mathbb{Z}) = \mathbb{Z}[t, t^{-1}] / I = (\Delta_K(t))$

Def.: $K \subset S^3$ is a slice knot, if it is concordant to the unknot, ergo

$\exists (D^2, \partial D^2) \xhookrightarrow{C^\infty} (D^4, \partial D^4)$ s.t. $\partial D^2 = K$.

We saw, that $K \# m(-K)$ is slice.

Also, start with the n-component unlink; and n-1 bands, and attach

the band to two of the circles, in a way that the end product is connected

This will be slice!

Fill the circles in, the bands intersect it in lines.

we push those singularities out in 4-space

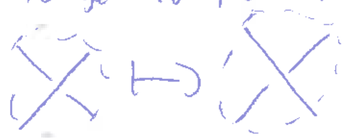
These are called Ribbon knots

Conjecture: Is there a slice knot, that is not Ribbon?

Complexity of knots

Def.: unknotting number is the minimal # of unknotting moves to get to the unknot.

Unknotting move:



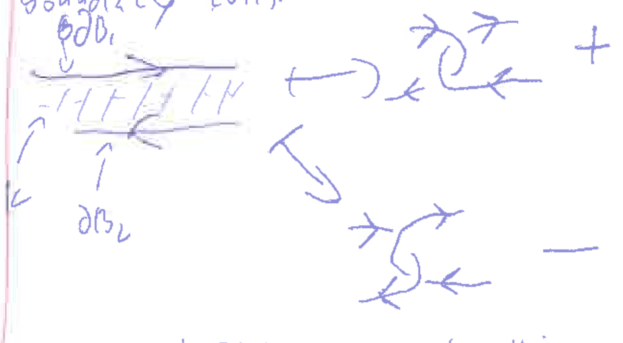
from the prev. lemma, we see that $u(K) \leq \frac{c(D)}{2}$

$u(K) = 0$ is the unknot

$u(K) = 1$ we don't know

$K \mapsto W_n^{\pm}(K) =$ pos/neg. whitehead doubler

we put a band on K, s.t. the middle of the band is K with some full twists, we take the one, which has linking # n,* of the 2 boundary components of the band



Claim: $W_n^{\pm}(K)$ has unknotting number ≤ 1.

*this is the # of full twists of the band.

Def (Linking #): $\vec{K_1}, \vec{K_2}$ disjoint knots, take their diagram, as a link diagram. We have double points, either K_1 meets itself, or K_2, or K_1 meets K_2.

, the linking # is

$\frac{1}{2} \cdot \sum_{c \in D_1 \cap D_2}$ of the signs $= lk(K_1, K_2) \in \mathbb{Z}$.

HW 1) show that $lk(K_1,K_2) \in \mathbb{Z}$ & symmetric.

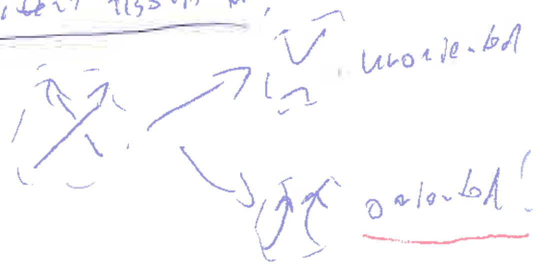
2) Consider $[K_1] \in H_1(S^3 \setminus K_2) \cong \mathbb{Z}$, show that $lk(K_1,K_2) = m \cdot [K_2] \in H_1(S^3 \setminus K_2)$?

Fact: $\forall K \subset S^3$ can be presented as the boundary of an orientable embedded surface.

$g(K) = \min\{g(\Sigma) \mid \Sigma \subset S^3$ embedded surface, orientable, $\partial\Sigma = K\}$

Seifert surface: We checkerboard color the diagram, and glue D^2's onto the black regions, this won't be orientable!

Seifert algorithm:



We resolve each crossing, get a union of oriented circles. To every circle, we associate the # of circles containing it. We lift each circle to z coordinate equal to this number, and we restore the crossings with bands. ✓

Remark: $g(K) = 0 \iff K = \text{unknot}$

We can bound this $g(K)$ from above.

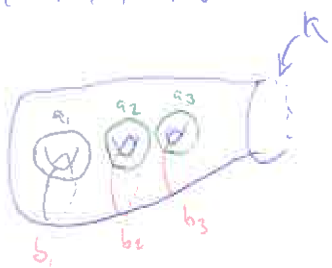
From below?

$K \subset S^3$ & $\Sigma \rightsquigarrow S \in M_{2g \times 2g}(\mathbb{Z})$

$g(\Sigma) = g$

Consider $H_1(\Sigma)$, $a_1, b_1, \ldots, a_g, b_g$ a basis of this $\mathbb{Z}^{2g}$

Abstractly $\Sigma =$



We have to perturb the a_i/b_i, so they are disjoint, now, if $e_i = a_i$, or b_i in a list, e_i^+, (ordered) $1 \leq i \leq 2g$ to

$S_{ij} = lk(e_i, e_j^+)$

linking number.

Def. $\Delta_K(t) = \det(t^{\frac{1}{2}} S - t^{\frac{1}{2}} S^T)$.

this is basis independent, linearly independence from Σ is harder

Cor: $g(K) \leq \deg \Delta_K(t)$

it's also symmetric $\Delta_K(t) = \Delta_K(t^{-1})$

Fact: $\exists$ nontrivial knots with $\Delta_K \equiv 1$

HW 2) $\Delta_{Wh_0^{\pm}(K)} \equiv 1 \quad \forall K$

Theorem (Fox–Milnor): If K is slice, then $\Delta_K(t) = f(t) \cdot f(t^{-1})$ for some polyn. f.

Hw 3) Show that the fig-8 is not slice.

Def.: $\sigma(K)$ (knot signature) is the $tr(S+S^T)$.

Thm.: $\frac{1}{2}|\sigma(K)| \leq u(K)$

Theorem (Reidemeister-Singer): If Σ_1 and Σ_2 are two Seifert surfaces of K, then Σ_1 and Σ_2 admit isotopic stabilizations.

Def.:  , $\gamma: [0,1] \to \mathbb{R}^3$,

$\gamma(0), \gamma(1) \in \Sigma$, $\gamma|_{(0,1)} \notin \Sigma$ and γ comes into Σ from the same side on both ends.

Take a tubular neighbourhood of γ, and glue this handle onto Σ, this is Σ_γ the stabilisation of Σ by γ, it is orientable from the side condition of γ.

Proof: Step 1: $\forall$ any Seifert surface has a sequence of stabilisations, such that the result is the Σ, which we get from a diagram D of the knot by Seifert's algorithm.

Step 2: if D_1 and D_2 differ by an R_{1-3} move, then the algorithmic surfaces are isotopic, or differ by a stabilisation.

(Appendix B, 3. section)

we work with this presentation,

$\Sigma_g \to \mathbb{R}^3 \to \mathbb{R}^2$ we project down

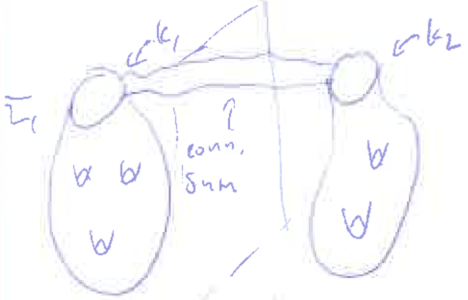
profit? :D

$K_1 \# K_2 \neq$ unknot

Theorem: $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Def.: K is prime, if $K = K_1 \# K_2 \Rightarrow K_1 = U$ or $K_2 = U$

Proof: $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ is clear.

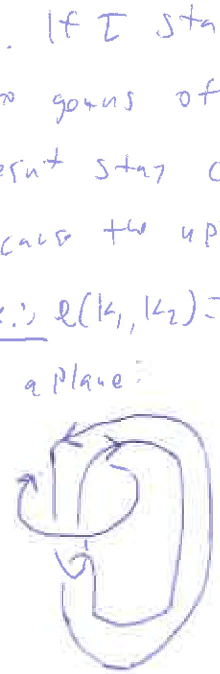


now

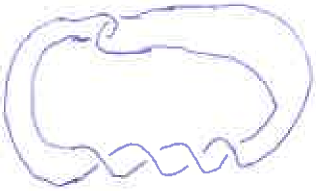
minimal for $g(K_1 \# K_2)$ Σ, we perturb Σ, so the separating plane intersects it transversely, in a 1-d manifold, an interval, and circles.

If we have a circle in the intersection, we delete a tubular neighbourhood, and close it up. If Σ stays connected, we decreased the genus of the minimal surface ↯ if it doesn't stay connected, we can discard it, because the upper part is the only needed.

Ex.: $\ell(K_1, K_2) = 0$, but K_1, K_2 can't be separated by a plane: whitehead link, $\Delta \neq 0$



· $W_n^{\pm}(0)$ – twist knots



· Alternating knots: K is alternating, if it admits a diagram s.t. when traversing the diagram we see the crossings in an alternating manner over–under–over–... ex. trefoil

· Torus knots

goes around – q times and – p times, knot if $(p,q)=1$

Def.: K knot in S^3; $Cr(K)$ = minimal # of crossings

$u(K)$ = unknotting # $g(K)$ = Seifert genus

$\sigma(K)$ = signature of $S+S^T$, $\Delta_K = a_0 + \sum a_i(t^i + t^{-i})$

HW1: $\frac{1}{2}|\sigma(K)| \leq u(K)$

HW2: $\sigma(\text{[illegible]}) = ?$ $\sigma(\text{[illegible]}) = ?$

HW3: $\Delta_{T_{2,n}} = ?$ $2 \nmid n$, $n > 0$

HW4: $\Delta_{K_1 \# K_2} = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$

Extra: if D is a diagram with a unique Kauffmann state $\Rightarrow K = 0$

Theorem: If K is alternating $\Rightarrow g(K) = \deg \Delta_K$

Def.: $g_4(K) = \min\{g(\Sigma) \mid (\Sigma, \partial\Sigma) \overset{C^\infty}{\hookrightarrow} (D^4, S^3)$ s.t. Σ is compact oriented, $\partial\Sigma = K\}$, g_4^{top} similarly with C^0 embeddings.

It's clear, that K slice $\Leftrightarrow g_4(K) = 0$, K is top. slice, if $g_4^{top}(K) = 0$,

Theorem (Freedman): $\Delta_K(t) = 1 \Rightarrow g_4^{top}(K) = 0$.

(Feller): $g_4^{top}(K) \leq \deg \Delta_K$.

Jones polynomial

(unnormalized ver.)

$\langle \cdot \rangle$: (Diagrams up to planar isotopy) $\to \mathbb{Z}[q, q^{-1}]$

1) $\langle \emptyset \rangle = 1$ 2) $\langle L \cup O \rangle = \langle L \rangle \cdot (q + q^{-1})$

3) $\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$
0-res. 1-res.

to construct $\langle \cdot \rangle$, do the following for D

① number the crossings

② apply a resolution at each crossing (eg. pick a full resolution)

③ for a full resolution consider $(-q)^r \cdot (q+q^{-1})^k$ where r = # 1-res., k = # circles

④ $\langle D \rangle = \sum_{res} (-q)^r \cdot (q+q^{-1})^k$

Proof: $\langle \text{[illegible]} \rangle = \langle \text{[illegible]} \rangle - q \langle \text{[illegible]} \rangle =$

$= (q+q^{-1}) \langle \frown \rangle - q \langle \frown \rangle = q^{-1} \langle \frown \rangle$

$\langle \text{[illegible]} \rangle =$, $\langle \text{[illegible]} \rangle = \ldots$, $\langle \text{[illegible]} \rangle = \ldots$

Def. D a diagram of K, n_- negative crossings, n_+ pos. crossings. Let $\hat{V}_K(q) = (-1)^{n_-} \cdot q^{n_+ - 2n_-} \cdot \langle D \rangle$ be the unnormalized Jones polynomial.

Def.: $V_L(q) = \dfrac{\hat{V}_L(q)}{q+q^{-1}}$, $V_{K_1 \# K_2} = V_{K_1} \cdot V_{K_2}$

Conjecture: $V_K(q) \equiv 1 \Rightarrow K = 0$.

Skein rel.: $q^{-2} V_{L_+} - q^2 V_{L_-} = (q^{-1} - q) V_{L_0}(q)$

Def.: a knot K is fibered, if $\exists \varphi: S^3 \setminus K \overset{C^\infty}{\to} S^1$ s.t. 1) $d\varphi \neq 0$ proper
2) $\varphi^{-1}(t)$ is a Seifert surface of K

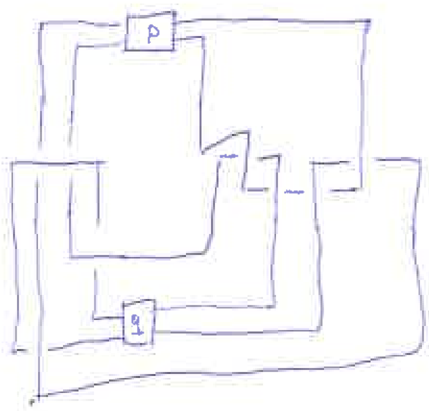
can be built from $[0,1] \times \Sigma /$ monodromy

Fact: $T_{p,q}$ are fibered.

Theorem: If K is fibered $\Rightarrow \Delta_K$ is monic, i.e. $a_d = \pm 1$.

$$\Delta_{T_{p,q}} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} \cdot t^{-\frac{(p-1)(q-1)}{2}}$$

Ex.:



Kanenobu knots p,q full twists. all have the same Δ and V.

$\boxed{Hw^1}$ $g_4(K) \le u(K)$ $\quad \left(\frac{1}{2}|\sigma(K)| \le g_4(K)\right)$

$$g_4^{top}(K) \le \deg \Delta_K \quad g_4(K) \le g_3(K)$$

$\boxed{Hw^2}$ Compute ~~knot~~ Jones poly. of

$\boxed{Hw^3}$ Find a relation between V_K & $V_{m(K)}$

Ex.: Jones $= q^2 + q^6 - q^8$

Vector spaces over $\mathbb{F}_2$

Khovanov homology

Suppose W finite dim. over $\mathbb{F}_2$. A $\mathbb{Z}$ grading is a decomposition of W into subspaces

$W = \bigoplus_{i \in \mathbb{Z}} W_i$ ← this has homogeneous elements of degree i, other points may not lie in any W_i, it doesn't have a grading, the origin is in every subspace.

A bigrading of W is a decomp.

$$W = \bigoplus_{a,b \in \mathbb{Z}} W_{a,b}$$

A chain complex is an ex. of this

$C_* = \bigoplus C_i$, $\partial : C_* \to C_{*-1}$, thus

$H(C_*, \partial) = \ker \partial / \operatorname{im} \partial$ also graded.

$\chi(C_*, \partial) = \sum (-1)^i \operatorname{rk} H_i(C^*, \partial) \in \mathbb{Z}$

Suppose W bigraded $(= \bigoplus W_{a,b}$, $\partial : W \to W$ a map with $\partial \circ \partial = 0$, also $\partial : W_{a,b} \to W_{a,b+1}$. The resulting homology will be bigraded, and χ will be a polynomial.

$W \xrightarrow{\dim} n \in \mathbb{Z}$, if $W = \bigoplus W_a$ we can make the graded dimension $\sum \dim W_a \cdot q^a \in \mathbb{Z}[q]$, similarly for bigraded vector spaces.

Suppose W graded vect. space, $W\{\ell\}$ is a graded vect. space with $(W\{\ell\})_a = W_{a-\ell}$
grade a, grade $a-\ell$

$\boxed{Hw^4}$ relate grad. dim W to grad. dim $W\{\ell\}$.

Def.: (W, ∂) with W vect. space, $\partial \in \operatorname{End}(W)$ is a chain complex if $\partial \circ \partial = 0$.

$H_*(W, \partial) = \ker \partial / \operatorname{im} \partial$, $v \in \operatorname{im} \partial$ is called a boundary, $v \in \ker \partial$ is a cycle, $v \in W$ is a chain.

If W is graded, then (W, ∂) is a graded chain complex if $\partial(W_a) \subset W_{a-1}$.

Def.: $f : (W_1, \partial_1) \xrightarrow{\text{hom}} (W_2, \partial_2)$ is a chain map, if $f \partial_1 = \partial_2 f$.

These maps induce an $H(f) : H_*(W_1, \partial_1) \to H_*(W_2, \partial_2)$

Graded chain map, if $f((W_1)_a) \subset (W_2)_{a+c}$ for all a, some c

Def.: $f, g : (W_1, \partial_1) \to (W_2, \partial_2)$ are chain homotopic, if $\exists h : W_1 \to W_2$ s.t. $f - g = \partial_2 h + h \partial_1$

lemma: f, g chain homotopic $\Rightarrow H(f) = H(g)$

Def.: $f : (W_1, \partial_1) \to (W_2, \partial_2)$ is a chain hom. equiv., if $\exists g : (W_2, \partial_2) \to (W_1, \partial_1)$ s.t. $g \circ f \sim id_{W_1}$

$f \circ g \sim id_{W_2}$

* Now the construction, $V = \mathbb{F}_{(-1)} \oplus \mathbb{F}_{(1)}$

we tensor V with itself for every circle in the resolved diagram. $V \otimes V$ is graded with

$gr(v \otimes w) = gr(v) + gr(w)$.

α a resolution of $D \longrightarrow V^{\otimes k(\alpha)}\{r(\alpha)\}$ (# of circles: $k(\alpha)$)

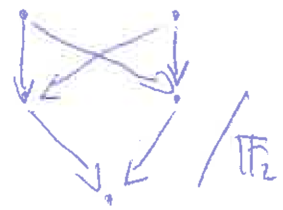
$[\![D]\!] = \bigoplus_{\alpha \text{ resolution of } D} V^{\otimes k(\alpha)}\{r(\alpha)\}$, and the # of 1-resolutions give another grading, so this will be bigraded. *

H_W^1 a)

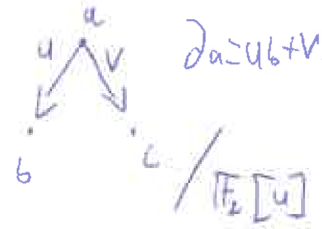
means $\partial a = b + c$, $\partial b = \partial c = 0$

$V = \mathbb{F}_2\langle a, b, c \rangle$

b)



H_W^2



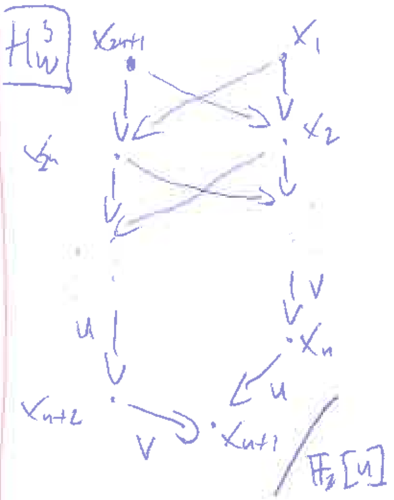
$\partial a = ub + vc$ a) $V = 0$ b) $V = 1$

c) $V = u$ d) $V = u^n$ $n \in \mathbb{N}$

compute homology!

($\mathbb{F}$-os $\mathbb{F}[u,v]$ Glott)

H_W^3



a) $V = 0$

b) $V = 1$

remark: $V_{m(K)}(q) = V_K(q^{-1})$ the proof works for Δ_K also, but $\Delta_K(t) = \Delta_{m(K)}(t)$ so it never sees a knot from it's mirror.

Def.: $V_K(q) \in \mathbb{Z}[q, q^{-1}]$

$M(V_K(q))$ = max exponent in $V_K(q)$

$m(V_K(q))$ = min — " —

$B(V_K(q)) = M - m$ the breadth of $V_K(q)$.

Theorem: If K admits an n crossing diagram, then $n \geq \frac{1}{2} B(V_K)$.

If D is an n-crossing alternating diagram of K, then $n = \frac{1}{2} B(V_K)$.

* discussed again

* $Ch = [\![D]\!][-n_-]\{n_+ - 2n_-\}$

(1-resolution grading shift; internal "quantum" grading shift)

we see now that $\chi(Ch) = \chi(H_K(Ch, \partial)) = \tilde{V}$ since the homological grading will give the alternating signs.

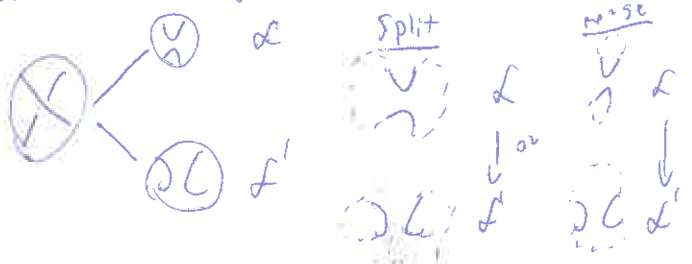
We now define $\partial : Ch \to Ch$, so $\partial|_\alpha : V^{\otimes k} \to Ch = \bigoplus V^{\otimes k(\alpha)}$

∂ will map to a subspace, where the index α' is bigger than α by exactly one $0 \to 1$ change, ("neighbours" on the cube*)

$\partial_\alpha : V^{\otimes k(\alpha)} \to \bigoplus_{\alpha' = \alpha \text{ after a } 0\to1 \text{ change}} V^{\otimes k(\alpha')}$ (so we have

to define $\partial_\alpha^{\alpha'} : V^{\otimes k(\alpha)} \to V^{\otimes k(\alpha')}$

Consider the crossing where α and α' differ



either a circle splits or 2 circles merge

$V_1 \otimes V_2 \otimes \dots \otimes V_{k(\alpha)} \to V_1 \otimes \dots \otimes V_k \otimes V_{k'} \otimes \dots \otimes V_{k(\alpha')}$

we need a map $V \to V \otimes V$, and similarly

$V_1 \otimes V_2 \xrightarrow{m} V_3$ (bilinear pairing) $\quad V_1 \xrightarrow{\Delta} V_2 \otimes V_3$ (co-multiplication)

$m: V_+^1 \otimes V_+^2 \mapsto V_+^3$ this makes $V \cong \mathbb{F}_2[X]/(X^2)$

$V_+^1 \otimes V_-^2 \mapsto V_-^3$

$V_-^1 \otimes V_+^2 \mapsto V_-^3$ $\quad 1 \leftrightarrow V_+$

$V_-^1 \otimes V_-^2 \mapsto 0$ $\quad X \leftrightarrow V_-$

$\Delta: V_-^1 \mapsto V_-^2 \otimes V_-^3$, $V_+^1 \mapsto V_+^2 \otimes V_-^3 + V_-^2 \otimes V_+^3$

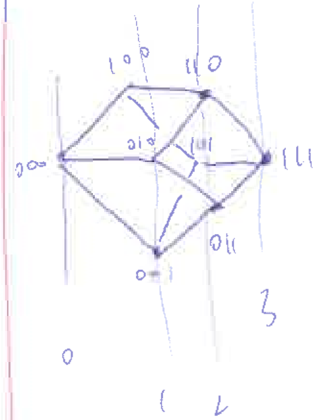
the grading is preserved by m and Δ

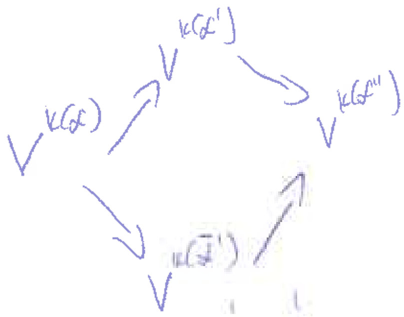
~~Z+~~ $2 + r(\alpha) \mapsto (+r(\alpha) + 1)$, $\quad 0 \xrightarrow{r(\alpha)} r(\alpha)+1-1$

and so on.

$D \rightsquigarrow (C^n(D), \partial_D)$ 1) is this a chain complex? 2) $H_*(C^n(D))$ is a knot invariant? 3) is this interesting?? :D

$\partial \circ \partial = 0$?





$\alpha = (0, \dots 0, \dots 0, \dots)$

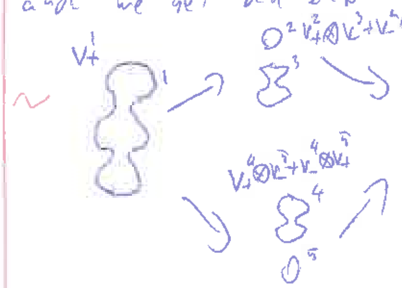
$\alpha' = (\dots 1, \dots 0, \dots)$

$\alpha'' = (\dots 1, \dots 1, \dots)$

$\bar{\alpha}' = (\dots 0, \dots 1, \dots)$

$\rightsquigarrow$ gives the same result, since the 2 multiplications are done disjointly

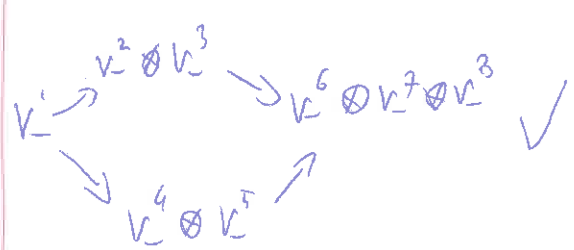
and we get an element squared, which is 0.

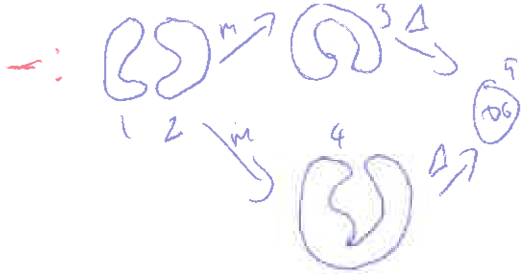


$0^6 \; V_+^6 \otimes (V_-^7 \otimes V_-^8) + V_-^6 \otimes (V_+^7 \otimes V_-^8 + V_-^7 \otimes V_+^8)$

$0^8 (V_+^6 \otimes V_-^7 + V_-^6 \otimes V_+^7) \otimes V_-^8 + V_-^6 \otimes V_-^7 \otimes V_+^8$

and everything cancels ✓





$$v_+^1 \otimes v_+^2 \to v_+^3 \to v_+^7 \otimes v_-^6 + v_-^5 \otimes v_+^6$$
$$\searrow \quad v_+^4 \quad \nearrow$$

HW 9: Calculate — !

Lemma: $0 \to C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \to 0$ SES of chain complexes induces an exact triangle on homology:

$$H(C_1) \xrightarrow{H(f)} H(C_2)$$
$$\delta \nwarrow \quad \swarrow H(\beta)$$
$$H(C_3)$$

Proof: $c \in C_3$, $\partial c = 0 \Rightarrow \exists\, b \in C_2$: $\beta b = c$. $\beta(\partial b) = \partial \beta b = \partial c = 0$

$\exists a$: $\alpha a = \partial b$ since $\partial b \in \ker \beta$

$\alpha(\partial a) = \partial \alpha a = \partial\partial b = 0 \Rightarrow \partial a = 0$

$\delta: c \mapsto a$

since α injective. □

* $f: C_1 \to C_2$ chain map. $\mathrm{Cone}(f) = (C_1 \oplus C_2, \begin{pmatrix} \partial_1 & 0 \\ f & \partial_2 \end{pmatrix})$ is a chain complex.

$$0 \to C_2 \to \begin{pmatrix} C_1 \\ \downarrow f \\ C_2 \end{pmatrix} \to C_1 \to 0$$

$(c_2 \mapsto 0, c_2)$

$$H(C_1) \xrightarrow{H(f)} H(C_2)$$
$$\nwarrow \quad \swarrow$$
$$H(\mathrm{Cone}(f))$$

Khovanov again :D

1) $[[\emptyset]] = 0 \to \mathbb{F}_2 \to 0$

2) $[[O \sqcup L]] = V \otimes [[L]]$
$= \mathbb{F}_{(-1)} \oplus \mathbb{F}_{(1)}$

3) $[[\times]] = 0 \to [[\asymp]] \xrightarrow{d} [[)(]] \to 0$

d defined by m and Δ.

* again, chain complex since we work in $\mathbb{F}_2$, and f is a chain map.

Convention: C_1, C_2 $\mathbb{Z}$ graded vspaces/modules, then $\mathrm{Cone}(f)_{(d)} = (C_1)_{d-1} \oplus (C_2)_d$.

This fixes the problem of $H(f)$ dropping or preserving the grading (since the connecting hom. should drop it, but a chain map can preserve it).

Independence cont.

D_1, D_2 diagram of L, so they are connected by R_{1-3} moves, we need invariance under those.

Suppose C a chain complex, $C' \leq C$ a subcomplex.

This induces a LES $H(C') \to H(C)$.

$$\nwarrow \quad \swarrow$$
$$H(C/C')$$

Observe: Suppose $C' \leq C$ and

1) $H(C') = 0 \Rightarrow H(C) = H(C/C')$

or

2) $H(C/C') = 0 \Rightarrow H(C) = H(C')$ □

R_1: $[[\text{(kink)}]] = \mathrm{Cone}([[\text{(circle)}]] \xrightarrow{m} [[\Omega]])\{1\}$

n crossings, 2^n resolutions; 2^{n-1} res; 2^{n-1} res

we need $H[[\Omega]] = H[[\sim]]$

Consider the subcomplex

$C' = \mathrm{Cone}([[\text{(circle)}]]_{V_+} \xrightarrow{m} [[\Omega]]) \le C$

subspace generated by the +1 graded generator of that circle.

we need two things: $H(C') = 0$; $C/C' = [[\sim]]$.

Claim: $m : [[\text{(circle)}]]_{V_+} \to [[\Omega]]$ is an isomorphism of chain complexes.

$m|_{V_+ \times \mathbb{F}_2}$ is invertible, thus it's an isomorphism.

Now consider $\mathrm{Cone}([[\text{(circle)}]]/V_+ \to 0)$ and profit???

HW R_1 proof with gradings!

Now onto R_2! Yaay

$[[\text{(crossings 1, 2)}]] \longleftrightarrow [[\approx]]$

we have 4 resolutions

$[[\supset\circ\subset]] \xrightarrow{m} [[\text{(smoothing)}]]$

$\Delta\uparrow$ $\quad C \quad$ Id?

$[[\supset\approx]] \xrightarrow{d?} [[\approx]]$

Take a subcomplex

$[[\supset\circ\subset]]_{V_+} \xrightarrow{m} [[\supset\subset]]$ acyclic, since m is invertible (H=0)

$\uparrow \quad C' \quad \uparrow$

$0 \longrightarrow 0$

so this has the same homology as:

$[[\supset\circ\subset]]/V_+ \to 0$

$\Delta\uparrow \quad C/C' \quad \uparrow \quad \cong \quad 0 \to 0$; $\uparrow C'' \uparrow$

$[[\supset\approx]] \to [[\approx]]$; $0 \to [[\approx]]$

subcomplex of C/C'

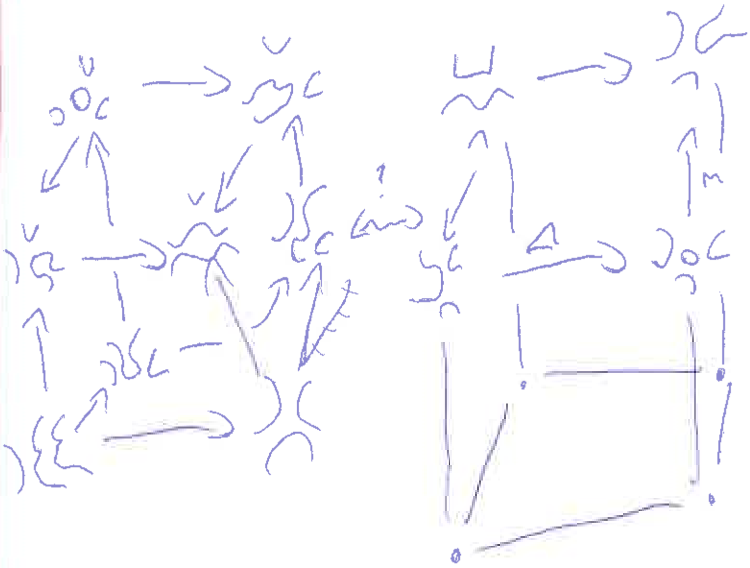
$(C/C')/C'' = [[\supset\circ\subset]]/V_+ \to 0$, and Δ is an isomorphism on this subcomplex, $\Rightarrow$

$\Delta\uparrow \qquad \uparrow$

$[[\supset\subset]] \to 0$

$\Rightarrow H = 0$

R_3:



bottom face the same, top face changed by a reflection.

back to R_2:

$\supset\circ\subset|_{V_+=0} \longrightarrow 0 \Rightarrow C''' = \{(\alpha, \beta, \tau\beta) \mid \alpha \in \supset\approx,\ \beta \in \supset\circ\subset|_{V_+=0}\}$

$\Delta\uparrow \quad C/C' \quad \tau = d\Delta^{-1}$

$\supset\approx \xrightarrow{d} \approx$

- C''' subcomplex
- C''' acyclic

$(C/C')/C''' = \begin{matrix} 0 \to 0 \\ \uparrow \quad \uparrow \\ 0 \to \approx \end{matrix}$

Remarks: Kohanov hom. detects the unknot, left and right trefoil, figure 8 and even $T_{2,5}$!

$g_4(T_{p,q}) = \frac{1}{2}(p-1)(q-1)$!

next time grid homology :>

---

open book on $\mathbb{R}^3$, take cylindrical coordinates (z, r, φ), $H_t = \{\varphi = t\}$ inverse images are half-planes. A link is in braid position if it's tangent to every H_t, and φ is monotone.

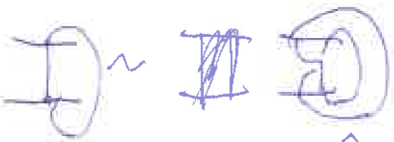
Close it by connecting the corresponding endpoints.

$\beta \mapsto \hat{\beta}$ (braid ↦ link)

$B_n = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$

HW: prove, that those are relations in B_n.

The braid representation is not unique!

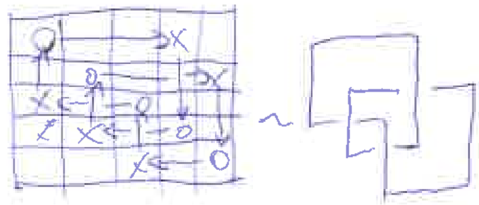


$\sim O$,

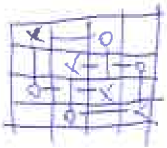
$B_n \hookrightarrow B_{n+1}$ $\quad \widehat{\beta\sigma_n} \sim \hat{\beta}$

$\beta \mapsto \beta\sigma_n$

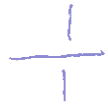
Now consider an $n \times n$ grid of squares. Put an X & O into it s.t. X, O is in every row and col. (rooks on a chessboard).

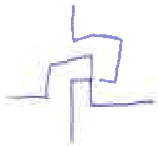


Connect in every column X → O, and after in every row O → X, if there is a conflict, go under.

Ex.:  ~ Hopf

Fact: every link can be represented by such a grid.

We have to get rid of  crossings

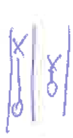
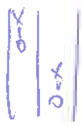


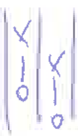
(swastika lol)

We want to imagine that we are in the torus, so glue the sides of the square grid.

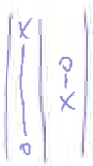
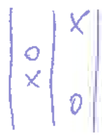
HW: Consider a link given by a grid, show that it can be isotoped to a braid.

2 grid moves:

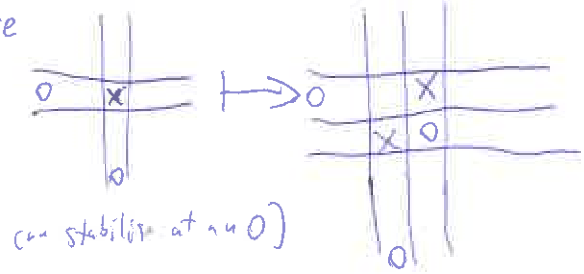
 nested,  disjoint,

 linked

Commutation move:

 ↦ , this is an R_2

Similarly for rows, and for disjoint cols/rows the second one is the stabilisation move



(also can stabilise at an O)

we make a choice in the middle square, we need to chose which of the 4 squares to leave ~~open~~ empty. Note, the rows and cols. of a stabilisation can be commuted! So we only need one to generate all the others.

Theorem: G_1, G_2 two grids represent isotopic knots $\Longleftrightarrow$ they can be moved into each other by commutation and stabilisation. (Reidemeister theorem for grids.)

we color squares according to winding number, take a matrix with entries $t^{\text{winding \#}}$, the determinant will be ~~constant~~ unnormalized Alexander poly,

Call a bijection between the vertical and horizontal lines of a grid diagram a grid state. These correspond to terms in the determinant above. We want another homology theory. Number the O-s 1..n. Consider $\mathbb{F}[V_1, \ldots, V_n]$ (= R). CG = free R module generated by the grid states

Let $P, Q \subset \mathbb{R}^2$ be finite sub sets. Let $I(P,Q) = \#\{(p_1,p_2),(q_1,q_2) \mid p_1 < q_1, p_2 < q_2\}$. Now $J(P,Q) = \frac{1}{2}(I(P,Q)+I(Q,P))$. The Maslov-grading is defined for a grid state $\underline{X}$ is

$$M(\underline{X}) = J(\underline{X},\underline{X}) - 2J(\underline{X},\mathbb{O}) + J(\mathbb{O},\mathbb{O}) + 1$$

(set of O-s in a fundamental domain.)

[Hw^3] Show, that $M(\underline{X}), A(\underline{X})$ are well defined on the torus. (independent of the cuts).

The Alexander grading:

$$A(\underline{X}) = J(\underline{X},\mathbb{X}) - J(\underline{X},\mathbb{O}) + \frac{1}{2}(J(\mathbb{O},\mathbb{O}) - J(\mathbb{X},\mathbb{X}) - n + 1)$$

(set of x-es)

Def. $M(V_i \cdot \underline{X}) = M(\underline{X}) - 2$, moreover $V_1^{k_1} \cdots V_n^{k_n} \cdot \underline{X}$ -s M is $M(\underline{X}) - 2\sum k_i$, A is $A(\underline{X}) - \sum k_i$.

Now the boundary, $\partial: CG \to CG$. ∂ should be $\in End_R(CG)$, we have to specify over the generators (grid states) since we took free modules.

$$\partial \underline{x} = \sum_{\text{grid states}} p_{\underline{y}}(V_1, \ldots, V_n) \cdot \underline{y}$$

p will be 0 unless x, y differ by a transposition (so only on 2 lines). These 4 intersection points where they differ give 4 squares on the Torus. each square is oriented based on weather the horizontal edges are $X \to Y$ (+) or $Y \to X$ (−).

~~Count the empty squares inside the~~

~~...,~~ ~~the corresponding R will be~~

the rectangle shouldn't contain x-es, and other points of $\underline{x}$ (equivalently $\underline{y}$). If this is satisfied $p_r(v_1,\dots,v_n) = v_1^{O_1(r)} \cdots v_n^{O_n(r)}$

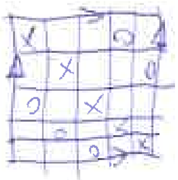
$O_i(r) = 1$ if the i-th o is in r, 0 otherwise.

Theorem (Milnor's conjecture): $u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

remark: $g_4(T_{p,q})$ and $g_3(T_{p,q})$ exist, $g_4 \leq u$

HW: Show that $u(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$

Fact: $\frac{(p-1)(q-1)}{2} \leq g_4(T_{p,q}) \overset{\text{triv}}{\leq} g_3(T_{p,q}) \overset{\text{easy}}{\leq} \frac{(p-1)(q-1)}{2}$



knot ⇝ G grid ⇝ $S(G)$ grid states

gradings: $M(\underline{x}) = M_O(\underline{x})$ Maslov

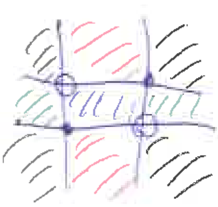
$A(\underline{x}) = \frac{1}{2}(M_O(\underline{x}) - M_X(\underline{x})) - \frac{n-1}{2}$

Alexander.

Fact: $M(\underline{x}) - M(\underline{y}) = 1 - 2\#(r \cap O) + 2\#(\underline{x} \cap \text{int}\, r)$

$A(\underline{x}) - A(\underline{y}) = \#(r \cap X) - \#(r \cap O)$

where r is a rectangle between $\underline{x}$ and $\underline{y}$.



4 rectangles, 2 of them go from $\underline{x}$ to $\underline{y}$ because of orientation, either of the two will suffice.

We pick $M(x^{NWO}) = 0$.

$$CG^-(G) = \bigoplus_{\underline{x} \in S(G)} \overbrace{\mathbb{F}[V_1,\dots,V_n]}^{R} \langle \underline{x} \rangle$$

$rk_R^{CG} = n!$

we want $\partial \in End_R(CG^-(G))$

$$\partial^- \underline{x} = \sum_{\underline{y}} \sum_{\substack{r \in Rect(x,y) \\ r \cap \mathbb{X} = \emptyset}} V_1^{O_1(r)} \cdot V_2^{O_2(r)} \cdots V_n^{O_n(r)} \cdot \underline{y}$$

($O_1(r) \in$? ; $O_2(r) \in \{0,1\}$)

$Rect^o = \{ r \mid \text{rectangle from } x \text{ to } y, \text{ with int } r \cap \underline{x} = \emptyset \}$

$M(V_1^{k_1} \cdots V_n^{k_n} \underline{x}) = M(\underline{x}) - 2\sum k_i$

$A(\quad - \quad) = A(\underline{x}) - \sum k_i$

homogeneous elements are not a submodule ↑.
We can grade R to fix this, but actually we don't care.

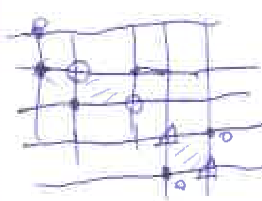
Prop: ∂^- is of bigrading $(-1, 0)$ (M, A)

$\cdot (\partial^-)^2 = 0$

Proof: $\partial\partial \underline{x}$

$\underline{x} \xrightarrow{\partial} \underline{y} \xrightarrow{\partial} \underline{z}$ ↑ is what we want: w/ come in pairs, which cancel, or component

4 vertical/horizontal coord.s move

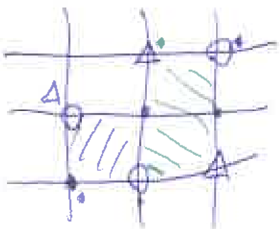


(2)

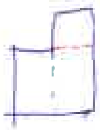
We can't have both ~~sets~~ rectangles

4 $X \to Y \to Z$ is two disjoint transpositions, we switch these to get w_1 and $X \to w \to Z$ will give the same term, so they cancel.

3



L shape



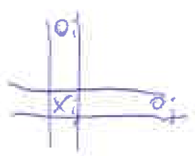
with w_1 we decompose it in the other way, giving w_1. □

Prop: V_i & V_j are chain homotopic maps on $GC^-(G)$.

corollary: on $H(GC^-(G))$ they induce the same map $GH^-(G)$.

proof: we need a map $H: CG^-(G) \to CG^-(G)$ s.t. $V_i - V_j = \partial H + H\partial$

call O_i, O_j neighbouring, if they are like this:



$$H_{ij} = \sum_Y \sum_{\substack{r \in Rect^o(x,y) \\ r \cap X = \{x_i\}}} V_1^{O_1(r)} V_2^{O_2(r)} \cdots V_n^{O_n(r)} Y$$

again 3 cases, the 4,) moving lines cancel as before, but in the 2 case we can have width 1 rectangles

through x_i. This gives an $X \to V_i X$, or $X \to V_j X$, and this is what we wanted. □

remark: thus the homology is a module over $\mathbb{F}[u_1, \dots, u_\ell]$ where the link has ℓ components

Def: $H_*(CG^-(G), \partial) = GH^-(G)$, the unblocked grid homology of G. This is a bigraded $\mathbb{F}[u]$-module, if we have a knot.

- $H_*\left(CG^-(G) / V_1(CG^-(G))\right) = \widehat{GH}(G)$

simply blocked grid homology. This is a bigraded finite dimensional vector space over $\mathbb{F}$.

- $H_*\left(CG^-(G) / (V_1 = 0, V_2 = 0, \dots, V_n = 0)\right) = \widetilde{GH}(G)$ is a bigraded finite dim vector space, fully blocked grid homology.

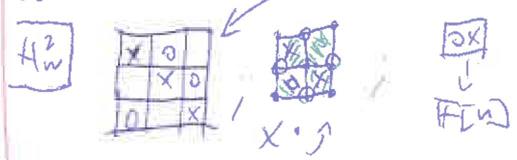
Theorem: $GH^-(G)$ is a finitely generated rank-1 $\mathbb{F}[u]$ module: $\mathbb{F}[u] \oplus \bigoplus_1^N \left(\mathbb{F}[u] / u^{n_i}\right) = GH^-(G)$.

Theorem: $GH^-(G)$ is a knot invariant

$\widehat{GH}(G)$ is a knot invariant.

$\widetilde{GH}(G) \cong \widehat{GH}(G) \otimes (\mathbb{F} \oplus \mathbb{F})^{\otimes n-1}$ ← grid number.

we proceed to invariance under commutation and stabilisation. compute $\widehat{GH}$ 6 generators 3 boundary maps.



$\partial_X = (V_1 + V_2) Y \Rightarrow \ker \partial = \mathbb{F}[V_1, V_2] Y \Rightarrow H = \mathbb{F}[u] \langle Y \rangle$
$\operatorname{Im} \partial = \mathbb{F}[V_1, V_2] \cdot (V_1 + V_2) Y$

$\partial Y = 0$ (rectangles forbidden)

$K \rightsquigarrow \mathbb{G} \rightsquigarrow$ grid states, bigrading (M, A) $\rightsquigarrow CG^-(\mathbb{G})$, a f.g. module over $\mathbb{F}[V_1, \dots, V_n]$ bigraded, $\partial: CG^- \to CG^-$

$\deg(V_i) = (-2, -1)$

of bidegree $(-1, 0)$.

$H_*(CG^-) =: GH^-$ homology, bigraded and a module over $\mathbb{F}[U]$. (also $\widehat{GH}$ and $\widetilde{GH} = \widehat{GH} \otimes (\mathbb{F}^2)^{n-1}$)

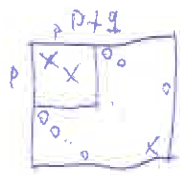
<u>Theorem</u>: GH^-, $\widehat{GH}$ are knot invariants.

• rk $GH^- = 1 \iff GH^- = \mathbb{F}[U] \oplus V$ torsion
$(-2\tau, -\tau)$

• $|\tau(K)| \leq u(K)$, • $|\tau(T_{p,q})| = \frac{1}{2}(p-1)(q-1)$

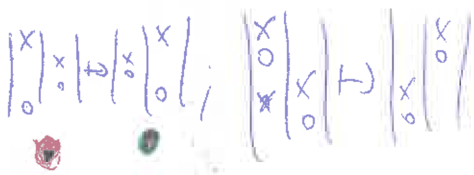
$(-\tau = \max\{A(x) \mid x \in GH^-, U^n x \neq 0 \ \forall n\})$

H^1_W $(p+q) \times (p+q)$ grid:

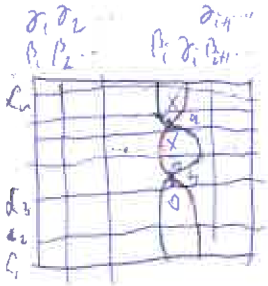


, this is $\pm T_{p,q}$

<u>Proof</u> (commutation):



but those are the same.



$CG(\mathbb{G}) \xrightarrow{P} CG(\mathbb{G}')$ we need a $P(\sim)$ chain homotopy equivalence

$P'(\leftarrow)$

$PP' \sim id_{G'} \iff PP' - id_{G'} = H'\partial + \partial H'$
$P'P \sim id_G$ $\qquad H': CG^-(\mathbb{G}')$

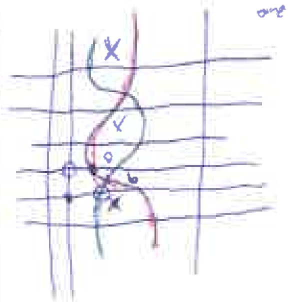
$P \circ \partial_G = \partial_{G'} \circ P$

$P(x) = \sum_{y \in S(\mathbb{G})} \sum_{\text{Pentagon } x \to y} V_1^{O_1(P)} \cdots V_n^{O_n(P)} y$

grid state $= S(\mathbb{G})$

$\text{int } P \cap \mathbb{X} = \emptyset$, $P \cap \mathbb{X} = \emptyset$

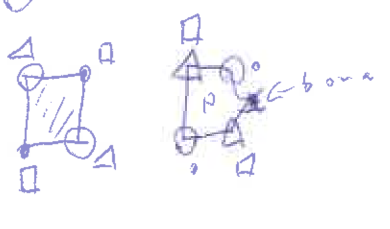
grid state in $\mathbb{G}$, and another in $\mathbb{G}'$ where the switch happens of the two one happens from • to •



H^*_W P preserves (M, A)

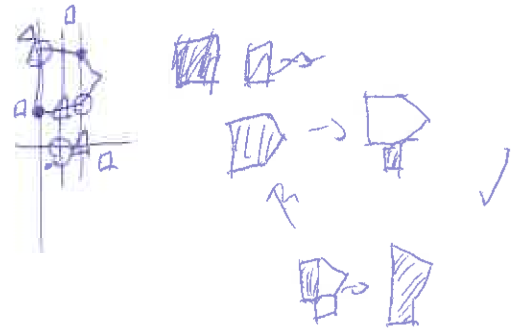
chain map?

④ 3, 2 moving parts can be

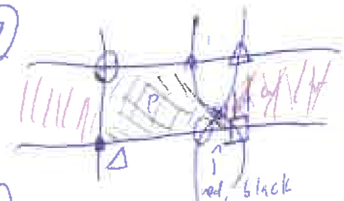


cancel in pairs

③

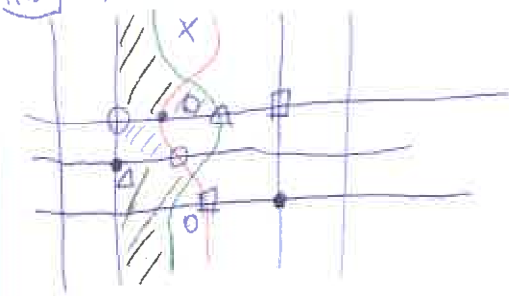


②



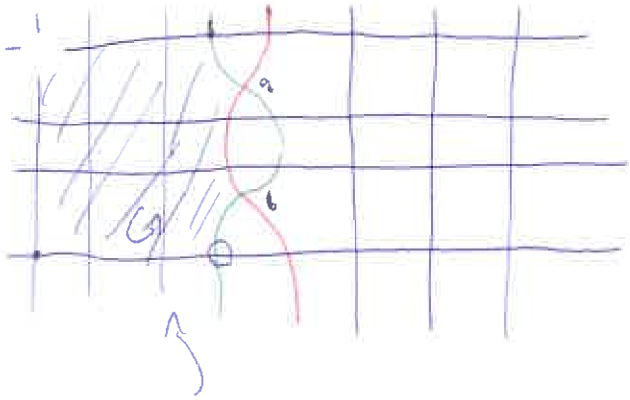
no lie inside, it would have a point.

H^2_W reproduce the cancellation in this case.



the next line after the twisted line will be it.

Compare $P' \circ P$ id.



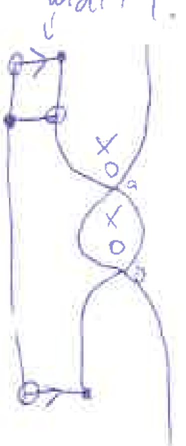
• O b a hexagon

$$H(x) = \sum_{y \in G} \sum_{\substack{u \times \to y \\ \mathrm{int}\, u \cap x = \phi \\ u \cap \mathbb{X} = \phi}} V_1^{O_1(u)} \cdots V_n^{O_n(u)} \, y.$$

we want $H\partial + \partial H + PP' = id$.

4, 3 cancel. 2?

width 1.



this stays, no X and O inside, so it's really id.

∎

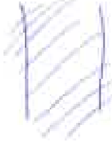
Stabilization:

⊠ ↦ $\frac{X_1 \mid O_1}{c \mid X_2}$ no subcomplex

$\widetilde{GC}(G) = I \oplus N$ ← subcomplex; N: no c; I: grid states which have c.

$\overset{3}{\mathrm{Hw}}$ (6)

Claim: N is a subcomplex



Summary: $\exists$ map: $I \to N$, denoted ∂_N^I, the total differential is $\begin{pmatrix} \partial^I & 0 \\ \partial_N^I & \partial^N \end{pmatrix} \leftarrow \mathrm{cone}(\partial_N^I : I \to N)$

2) $H_*(N, \partial^N) \cong \widetilde{GH}(G)$, 4) $H_*(I, \partial^I) \cong H_*(N, \partial^N)$

3) $(\partial_N^I)_* = 0$.

we want $(I, \hat{\partial}_I) \cong (GC(G), \hat{\partial})$, also

$x \cup \{c\} \leftarrow\!\!| \; x$

$H_*(N, \tilde{\partial}_N) \cong H_*(I, \tilde{\partial}_I)$

$\mathcal{H}_{X_2} : N \to I$, $\mathcal{H}_{O_1} : I \to N$

$$\mathcal{H}_{X_2}(x) = \sum_{y \in I} \sum_{\substack{r \in \mathrm{Rect}^\circ(x,y) \\ r \cap \mathbb{X} = \phi \\ r \cap \mathbb{O} = \phi \\ r \cap \mathbb{X} = \{x_2\}}} ,$$ the other one similarly.

$\mathcal{H}_{X_2} \mathcal{H}_{O_1} = id_I$, the other direction, we need a homotopy: $\mathcal{H}_{X_2, O_2}$ will do the trick.

$$(N, \tilde{\partial}_N) \to (I, \tilde{\partial}_I) \to (I, \hat{\partial}_I) \to 0$$

$0 \to (N, \tilde{\partial}_N)$; $(I, \tilde{\partial}_I) \to (N, \partial_N)$

cone S.E.S. ⟹ Homology LES

$$\widetilde{GH}(G) \to \widetilde{GH}(G') \to \widetilde{GH}(G)$$ with connecting map $(\tilde{\partial}_N^I)_*$

We want $(\tilde{\partial}_N^I)_*$ to be 0. And it is.

Claim: $\mathcal{H}_{x_2} \circ \tilde{\partial}^I_N = 0$, because again we can pair the rectangles,

$$\Rightarrow (\mathcal{H}_{x_2} \circ \tilde{\partial}^I_N)_* = (\mathcal{H}_{x_2})_* \circ (\partial^I_N)_* = 0$$

isomorphism

$$\Downarrow$$

$$(\tilde{\partial}^I_N)_* = 0 \text{ as we wanted.}$$

This proves that after a stabilisation

$$\widetilde{GH}(G') \simeq \widetilde{GH}(G) \oplus \widetilde{GH}(G)[1,1],$$

$GC^-(G')$ $\quad$ $GC^-(G)$

Theorem: $GH^-(G') \simeq GH^-(G)$ as bigraded $\mathbb{F}[u]$ modules.

Consider the map $GC^-(G)[V_2] \xrightarrow{V_1 - V_2} GC^-(G)[V_1]$, and the cone over this.

Fact: $H_*(\mathrm{Cone}(G)) \simeq H_*(GC^-(G)) \simeq GH^-(G)$

Also $GC^-(G') = I \oplus N$ as submodules, N is a sub complex. This is a cone, as before, also $\overset{G}{X} \overset{e}{\mapsto} \overset{G}{X} \cup \{c\}$, so $GH^-(G) = H_*(C_I, \partial_I)$,

$\mathcal{H}_{x_2}: N \to I$ again, rectangles with x_2 in them

$\mathcal{H}_{o_1}: I \to N$ w.r.t. o_1 is in them (now others can be as well).

$\mathcal{H}_{x_2,o_1}$ again.

$\mathcal{H}_{x_2}\mathcal{H}_{o_1} = id_I$, the other direction is chain homotopic to id_N.

the cone $I \xrightarrow{\partial^I_N} N$ gives $GH^-(G')$

$$\begin{array}{ccc} I & \xrightarrow{\partial^I_N} & N \\ e^{-1}\downarrow & \circlearrowleft & \downarrow e^{-1}\circ\mathcal{H}_{x_2} \\ GC^-(G)[V_2] & \xrightarrow{V_1 - V_2} & GC^-(G)[V_1] \end{array}$$

Square commutes, e^{-1} gives a map between cones, thus the homologies are isomorphic.

So $GH^-(K)$ is a bigraded $\mathbb{F}[u]$ module.

Theorem: K^+, K^- differ in a single crossing.

$\exists C_-: GH^-(K^+) \to GH^-(K^-)$ $(0,0)$ and

$C_+: GH^-(K^-) \to GH^-(K^+)$ $(-2,-1)$

and $C_- \circ C_+ = u \cdot \underline{v} \in GH^-(K^-)$

$C_+ \circ C_- = u \cdot \underline{v} \in GH^-(K^+)$

Cor: $GH^-(K) = \mathbb{F}[u] \oplus \overset{u}{\text{torsion}}$ ✓
$(-2\tau, -\tau)$ is the grading of the generator.

Cor: $\tau(K)$ is a knot invariant.

Cor: under a crossing change τ changes by ≤ 1. $(0 \leq \tau(K_+) - \tau(K_-) \leq 1)^*$

Cor: $|\tau(K)| \leq u(K)$

Conjecture (Milnor) $u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.

Proven by Kronheimer–Mrowka '94 by Gauge theory

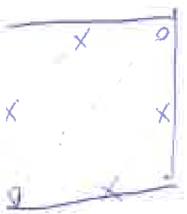
Rasmussen '10 using Khovanov

$|\tau(T_{p,q})| \leq u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$, we want $\frac{(p-1)(q-1)}{2} \leq \tau(T_{p,q})$

$-\tau = \max\{A(x) \mid x \in GH^-(\emptyset)$ s.t. $u^n x \neq 0\}$

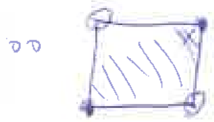
we need an element x s.t its $A(x) = \frac{(p-1)(q-1)}{2}$, its a cycle in $GH^-(K)$, and its not a torsion element.

Proof. $\xi \in GH^-(K_+)$ generates $\mathbb{F}[u]$, $A(\xi) = -\tau(K_+)$, $C_-(\xi) \in GH^-(K_-)$, and its not torsion since its image $C_+ C_- \xi = u \cdot \xi \Rightarrow -\tau(K_+) \leq -\tau(K_-)$, also $\eta \in GH^-(K_-)$ generating the free part, $A(\eta) = -\tau(K_-)$ $C_+\eta$ has A grading $-\tau(K_-) - 1$ ... ✓



, this is $T_{p,q}$. take a state: $\boxed{X}$, this is X. This will be all right.

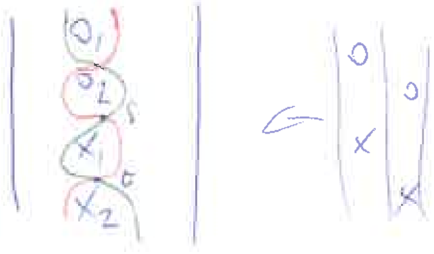
... grading $\frac{(p-1)(q-1)}{2}$, $\partial X = 0$ is clear



✓ $u^n[X] \neq 0$? we need a trick:

$C^-(G) \xrightarrow{\varphi} GC^-(G) / _{V_1 = V_2 = \dots = V_n = 1}$, here the u-action is trivial ($ux = x$)

$\varphi_*(u^n[X]) = \varphi_*([X]) \neq 0$



commutation is a crossing change

$C_-(x_+) = \sum_{y_-} \sum_{p \in Pent^o_s(x_+, y_-),\ p \cap \mathbb{X} = \emptyset} \prod V_i^{O_i(p)}\, y_-$

$C_-: GC^-(G^+) \to GC^-(G^-)$, C_+ similar.

chain homotopy again hexagons using only s, t, we get an O_1 or O_2 with the hexagons, thus $C_+C_- \sim V_1$, $C_-C_+ \sim V_2$.