Riemannian geometry
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1 First lecture

The following topics will be covered

1. Riemannian metrics
2. Connections
3. metric properties, geodesics
4. curvature
5. Jacobi fields and applications

1.1 History/motivation

Gauss studied the theory of curved surfaces in $\mathbb{R}^3$. A typical curve on the surface is $\gamma = r \circ \sigma$, $\gamma(t) = (u(t), v(t))$.

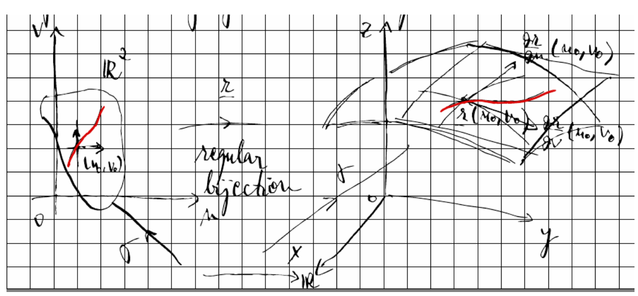

Figure 1: Figure by Botond Miklósi

The arc length is defined by $\int_a^b ||\gamma'(t)||dt$. $\gamma' = (r \circ \sigma)'(t) = \partial_u r u' + \partial_v r v'$, and the norm can be calculated by

$$\sqrt{< \partial_u r, \partial_u r > (u')^2 + 2 < \partial_u r, \partial_v r > u'v' + < \partial_v r, \partial_v r > (v')^2}$$

The three terms are denoted by E, F, G, and are called Gauss' first fundamental functions, together they constitute the first fundamental form with the matrix $\begin{matrix} E & F \\ F & G \end{matrix}$ being the Gram matrix of the standard dot product in the tangent space written out wrt the basis $\partial_u r, \partial_v r$. Whatver can be defined in terms of these functions is an "intrinsic" geometric property of the surface. The main discovery is that Gaussian curvature is intrinsic.

Notations we will commonly use: M will be a differentiable manifold. TM will denote the tangent bundle, T_pM, the tangent space of a point $p \in M$, vector fields are $X \in \mathfrak{X}(M)$, their lie bracket is denoted $[X, Y]$. Tensors of type (k, l) on an $n - dim$ real vector space V are denoted by $T_l^k(V)$. Tangent maps/derivatives are denoted df or $Tf : TM \to TN$. A tensor is covariant if $k = 0$. The pullback of a covariant tensor field $t \in T_l^0(N)$ by a diffeomorphism $f : M \to N$ is denoted $f^*t \in T_l^0(M)$, defined by the usual formula.

Definition 1.1. If M^n is a manifold, a Riemannian metric on M is a smooth assignment of a positive definite dot product to each tangent space T_pM.

In terms of local coordinates $x^1, ..., x^n$ $g_p(\partial_i, \partial_j) =< \partial_i, \partial_j >_p= (g_{ij})_p$, where the $g_{ij} : U \to \mathbb{R}$ are smooth functions, and $(g_{ij})_p$ is a positive definite symmetric matrix. From a tensorial viewpoint the matrix $g \in T_2^0(M)$ is a tensor field, locally written as $g = \sum g_{ij} dx^i \otimes dx^j$.[1]

Example 1.2. The easiest example is the standard Riemannian metric on $\mathbb{R}^n$. Here $g_{ij} = \delta_{ij}$.

A Riemannian manifold is a pair (M, g) where g is a Riemannian metric on M.

Definition 1.3. Two Riemannien manifolds $(M, g), (N, h)$ are called isometric, if there is a diffeomorphism $f : M \to N$ such that $f^*h = g$.

Remark 1.4. In Gauss' viewpoint the first fundamental form is just the pullback of the standard Riemannian metric to the parameter space.

Example 1.5. Submanifolds of $\mathbb{R}^n$ of any dimension are a rich source of examples. They all inherit[2] the Riemannian metric from the standard one on the ambient space.

We can be a bit more general even. If $f : M \to N$ is an immersion, and (N, h) is a R-manifold, then f^*h will again give an induced metric on M.

Definition 1.6. A map $f : M \to N$ between R-manifolds is a local isometry at $p \in M$, if there is an open neighbourhood $U \ni p$ in M such that $f|_U : U \to f(U)$ is an isometry.

Definition 1.7. Let (M_i, g_i) be two R-manifolds. Then $M_1 \times M_2$ endowed with the metric $\pi_1^* g_1 + \pi_2^* g_2$ is a R-manifold ($\pi_i : M_1 \times M_2 \to M_i$ are the standard projections).

2 Second lecture

More examples

Example 2.1. We saw the construction of the product manifold last lecture, for example $T^2 = S^1 \times S^1$, and the circle has a natural Riemannian metric. In the same way we get all the tori. Observe that the torus we get in this way is the "flat" torus, ergo locally isometric to the plane/higher Euclidean space. This happens because the Euclidean plane itself is a product. We can get general flat tori, by taking two independent vectors, and the group generated by their translations and quotienting out by the lattice they generate, $\mathbb{R}^2/\mathbb{Z}^2$ will give another torus. The lattices have to be congruent for two flat tori to be globally isometric.

[1] the tensor product symbol is usually omitted

[2] by restriction

These are called "space forms", spaces that are locally isometric to one of the standard classical geometries (Euclidean, spherical or hyperbolic). Complete R-mfds are all quotients of their respective geometries (if they are "space forms").

2.1 Riemannian coverings

Definition 2.2. A smooth map $f : N \to M$ between R-manifolds is a R-covering, if it is a covering map (in the sense of topology), and f is a local isometry.

Proposition 2.3. *If you have a smooth covering, and M has a R-metric, then there is a unique metric on N, which makes f into a R-covering.*

Proof. This is trivial, take f^*g as the metric on N. □

Proposition 2.4. *If N is an R-mfd, and Γ is a discrete group of isometries of N such that its action on N is a covering action*[1] *so that the factor N/Γ inherits a smooth manifold structure and the factor map is a covering map, then there is a unique R-mfd structure on $M = N/\Gamma$.*

Proof. To get the metric, pick a lift of a point, and take the metric on N, this will be well defined, since Γ acts by isometries, it doesn't matter which lift we pick. □

Remark 2.5. Notice that the torus was produced in just this way, we factor by translations (which are isometries of the plane).

Example 2.6 (More space forms). For spherical geometry the first thing that comes to mind is $P^n = S^n/\mathbb{Z}_2$, the action being the antipodal map. Sometimes this is called "elliptic space". For $n = 3$ we also have lens spaces. Take $p \geq 2$ a natural number, and think of $S^3 \subset \mathbb{C}^2$, and pick a primitive pth root of unity ξ, and q such that $(p, q) = 1$. Let the $\mathbb{Z}_p$ generated by ξ act as $(z, w) \mapsto (\xi z, \xi^q w)$, giving the lens space $L(p, q)$, giving more examples of elliptic[2] manifolds.

Remark 2.7. If n is even, then Z_2 is the only possible group which can act on the sphere by isometries.

Example 2.8. The most famous elliptic 3-mfd is the Poincaré homology sphere. It looks like S^3/I, where I is the binary icosahedral group. Take the orientation preserving symmetry group of the dodecahedron (which is abstractly an $A_5 \leq SO(3)$). Take the inverse image under the $2:1$ cover $S^3 \xrightarrow{q} SO(3)$, now $PHS^3 = S^3/q^{-1}(A_5)$.

Example 2.9 (Surfaces). S^2, RP^2 carry elliptic geometry.
T^2 and the Klein-bottle have flat R-metrics. For the latter take the translations by $(0, 1)$ and the translation by $(1, 0)$ composed by a reflection to ensure the right gluing of the fundamental domains.
All the rest have hyperbolic R-metrics.

2.2 Existence of Riemannian metrics

Theorem 2.10. *On any differentiable manifold there exists a Riemannian metric.*

[1] absolutely discontinous etc etc.
[2] positive constanst curvature

Proof. One possible approach is to embed it differentiably into some $\mathbb{R}^N$, and pull back the inherited metric. This works for compact manifolds easily, but the embedding theorem is true for non-closed mfds as well, but is a bit harder to see.
A different approach is to use partitions of unity. On a single chart it is not hard to define a metric, just pull back from $\mathbb{R}^n$. Now we can paste these together using a suitable partition of unity (in the noncompact case locally finite). We need the statement, that any convex combination of positive definite bilinear forms is again positive definite, which is more or less clear. □

Definition 2.11. A cover of open sets is locally compact, if every point has a neighbourhood intersecting only a finite number of elements in the cover.
A topological space is paracompact, if every open cover has a locally finite refinement

Proposition 2.12 (general topology fact)**.** *M_2 and T_2 implies paracompact.*

This condition is essential, the long line doesn't have a Riemannian metric.

3 Third lecture

What makes a Riemannian manifold a geometry? We would like to define distance, angle, volume and the like. Angle is clear, the angle between two curves is the inner product[1] of their unit tangent vectors in the tangent space of an intersection point. Distance is more intricate.

3.1 Riemannian distance

Arc length of smooth curves is well defined the same, as in the classical case: $L(\gamma) := \int_a^b ||\gamma'(t)||\ dt$, this is invariant under (regular) reparametrisation by the same argument as in the classical case. This can be extended to piecewise smooth curves easily.

Definition 3.1. Let (M, g) be a Riemannian manifold, $d_g(p, q) := \inf\{L(\gamma) : \text{piecewise smooth curve from } p \text{ to } q\}$ will be the Riemannian distance of p, q.

Remark 3.2. This is a finite number if p, q are in the same connected component of M. We could allow d_g to take the infimum over only smooth curves, and get the same distance, but then we would have to worry about technical dificulties with smoothing corners.

Theorem 3.3. *If (M, g) is a connected R-manifold, then d_g is indeed a metric on M, moreover the metric topology coincides with the manifold topology of M.*

Proof. $d_g \geq 0$ is obviously true, the infimum is taken over nonnegative numbers. Symmetry is also clear, we can parametrise the curve in the other direction, and the infimum is taken over the same set. The triangle inequality is also clear, take another point r, the curves $p \to r$ and $r \to q$ can be concatenated, so the infimum cantains all of these curves, maybe more.
$d_g(p, q) > 0$ if $p \neq q$ is the only nontrivial requirement for being a metric. Take p fixed, take a chart U, ϕ around p. There exists a ball such that $\overline{B}(\phi(p), r) \subset \phi(U)$ for some r. Take $K = \phi^{-1}(\overline{B}(\phi(p), r))$ is a compact set in U. Let S denote the unit tangent sphere bundle restricted to K, e.g. $\{v \in TM : \sqrt{g(v, v)} = ||v|| = 1, v \in$

[1]given by the metric

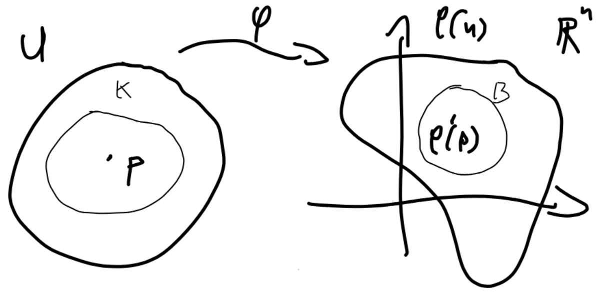

Figure 2: The setup for $d_g(p,q) > 0$

$T_pM, p \in K\}$. Take $v \mapsto ||(T\phi)(v)||, S \to \mathbb{R}^+$, strictily positive, since ϕ is a diffeomorphism. It is defined on a compact set, so it has a positive minimum and maximum (c_1, c_2 respectively). Now for all tangent vectors v at some point of K we see, that $c_1||v|| \leq ||T\phi v|| \leq c_2||v||$. This implies, that for any curve in K, we have the same estimate for its arclength, e.g. $c_1 L(\gamma) \leq$ Euclidean arclength$(\phi \circ \gamma) \leq c_2 L(\gamma)$ Pick a point $q \neq p$. If $q \in K$ and γ connects p with q in K, then $L(\gamma) \geq \frac{1}{c_1} d(\phi(p), \phi(q))$ for the euclidean distance of the images, which is nonzero, and this is independent of the curve γ, this means, that the infimum is bounded away from zero. Lastly, if $q \notin K$ we still get a piece of every connecting curve in K, and we can apply the previous case plus the triangle inequality.
We have actually seen that the metric topology is finer, than the original topology, repeating the same argument as before we find a metric ϵ-ball inside any neighbourhood of any point. Conversely, we can choose r arbitrarily small, and this will give open neighbourhoods of a point with arbitrarily small Riemannian distance. □

Now a little digression.

3.2 Tensors/tensor fields on Riemannian manifolds

Observe, that a nondegenerate inner product $<,>$ on a finite dimensional vector space V gives an explicit isomorphism between V and V^*[2]. This means that using this identification the whole tensor buisness gets simplified, since the difference between covariant and contravariant tensors was precisely that they had vectors/covectors as input. Applying this to tangent spaces of a R-manifolds g_p defines an isomorphism $T_pM \to T_p^*M$ simultaneously for all p, i.e. $\mathfrak{X}(M) \to \Omega^1(M)$ are isomorphic using the same formula $X \mapsto g(X, Y)$.

$$X = \sum x^i \partial_{x^i} \mapsto \omega = \sum \omega^i dx^i$$

where

$$\omega^i = \sum g_{ij} x^j.$$

Funny notation: $\omega = X^\flat$ and $X = \omega^\sharp$, and the above isomorphisms are called musical isomorphisms[3].

Example 3.4. $df^\sharp = \nabla f \in \mathfrak{X}$ is called the gradient of the smooth function f.

Remark 3.5. The inner product on T_pM can be copied to an inner product on T_p^*M. One can check, that $< \omega, \phi >$ can be expressed locally as $\sum g^{ij} \omega_i \phi_j$ with the inverse matrix of the metric.

[2] $v \mapsto < v, . >$
[3] lol

3.3 Volume

Let (M^n, g) be an oriented Riemannian manifold. For an n-form $\lambda \in \Omega^n(M)$ the following are equivalent:

1. for any local oriented orthonormal n-frame $E_1, .., E_n \in \mathfrak{X}(U)$ we have $\lambda(E_1, .., E_n) = 1$
2. for any local oriented orthonormal coframe $\epsilon^1, .., \epsilon^n \in \Omega^1(U),\ \lambda = \epsilon^1 \wedge ... \wedge \epsilon^n$.
3. for any oriented local coordinate chart $x^1, ..., x^n$ we have $\lambda = \sqrt{\det(g_{ij})}\ dx^1 \wedge ... \wedge dx^n$

Definition 3.6. This unique λ is called the volume form of (M, g), denoted by V_g.

This volume form generates a measure on the manifold, and we can talk about integrals of functions.

4 Fourth lecture

Back to volume, we have a unique volume form for any Riemannian manifold, so we can integrate smooth functions. This is a linear functional on $C^\infty(M)$, and by the Riesz representation theorem there exists a unique Borel measure on M, the volume measure. If M is not oriented we use the oriented double cover. We can equip this with the volume measure, lift a set and take half of its measure.
To do some elementary geometry on M, we need first of all straight lines. What makes a curve straight in the physical world? One possible definition would be the distance minimizing property of straight lines, but this is a bit hard to handle technically. We use the other definition motivated by physics, tha path of inert motion[1] will be straight. To phrase this condition precisely, we need the notion of second derivative of a curve, which we don't have unfortunately. We also would require derivatives of vector fields, so we need additional structure on our manifolds.

4.1 Covariant derivative (of vector fields)

The motivational example is the directional derivative of vector fields in $\mathbb{R}^n$. At a point $p \in \mathbb{R}^n, v \in T_p\mathbb{R}^n$ and $Y \in \mathfrak{X},\ Y = \sum Y^i \partial_i$, the derivative $\partial_v Y = \sum vY^i \partial_i \in T_p\mathbb{R}^n$. We list the properties:

- $\partial_v Y$ is $\mathbb{R}$-linear in both its variables.
- Leibniz rule for the second argument $\partial_v(fY) = (vf)Y + f\partial_v Y$.

The same thing could be stated globally. For $X, Y \in \mathfrak{X}$, we could define with the formula $(\partial_X Y)_p = \partial_{X_p} Y$. The previous properties extend as follows

- this map is $C^\infty(M)$ linear in its first argument, $\mathbb{R}$-linear in the second
- and it is a derivation in the second $\partial_X(fY) = (Xf)Y + f\partial_X Y$
- the commutator $\partial_X Y - \partial_Y X = [X, Y]$
- $X < Y, Z > = < \partial_X Y, Z > + < Y, \partial_X Z >$

[1] no force, i.e. no acceleration

4.2 Connection on a manifold

[2] Let M be a smooth manifold. A connection on M is a map $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ satisfying the first two of our previous properties:

1. this map is $C^\infty(M)$ linear in its first argument, $\mathbb{R}$-linear in the second
2. and it is a derivation in the second $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

We can give a local expression on a chart $x^1, ..., x^n$ in terms of the basis fields ∂_i. Everything can be expressed in terms of the base fields:

$$\nabla_{\partial_i}\partial_j = \sum \Gamma^k_{ij}\partial_k.$$

The functions Γ^k_{ij} are called the Christoffel symbols of the connection on this chart. On $\mathbb{R}^n$ the standard covariant derivative has $\Gamma^k_{ij} = 0$.

4.3 Riemannian (or Levi-Civita) connection

If (M, g) is a Riemannian manifold, then **additionally** we require the last two conditions:

3. the commutator $\nabla_X Y - \nabla_Y X = [X, Y]$
4. $X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z >$

Theorem 4.1 (Levi-Civita). *On any Riemannian manifold (M, g) there exists a unique Riemannian connection.*

Proof. We want to define a connection from the 4. axiom and the metric. The trick is to cyclically permute the inputs, and alternatingly add them together.

$$+ X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z >$$

$$+ Y < Z, X > = < \nabla_Y Z, X > + < Z, \nabla_Y X >$$

$$- Z < X, Y > = < \nabla_Z X, Y > + < X, \nabla_Y Z >$$

Collect like terms to get

$$X < Y, Z > + Y < Z, X > - Z < X, Y > = < [Y, Z], X > + < [X, Z], Y > + < [Y, X] + 2\nabla_X Y, Z > .$$

Now we have won, since this is a linear system of equations for the connection $\nabla_X Y$ by varying Z (here we use the nondegeneracy of the inner product). So we have uniqueness, and one can check that the formula we get actually defines a covariant derivation on the manifold. □

4.4 Covariant derivative along curves

It is enough to assume that M is a differentiable manifold equipped with an affine connection ∇. A vector field along a curve is a map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$. One such guy is the velocity vector field γ', or the restriction of any element of $\mathfrak{M}$. We define covariant differentiation along a curve $\frac{D}{dt} : V \mapsto \frac{DV}{dt} (= V')$ satisfying the following

[2] affine connections, no metric needed for now

1. $\mathbb{R}$ linear

2. Leibniz rule

3. if V is induced by restriction, then $\frac{DV}{dt} = \nabla_{\gamma'} X$

Proposition 4.2. *Such a D exists uniquely.*

Proof. straightforward :) Except its not, a curve can stop, and the vector field still change and such, but it still works. □

Knowing this D for every vector field along every curve determines ∇ completely.
In a Riemannian manifold we have the extra property that $\frac{d}{dt} < V, W > = < \frac{DV}{dt}, W > + < V, \frac{DW}{dt} >$.

4.5 Parallelism of vector fields

Definition 4.3. A vector field V along a curve $\gamma \subset M$ is called parallel if $\frac{DV}{dt} = 0$.

5 Fifth lecture

Recall $\mathfrak{X}(\gamma)$ denoted vector fields along the curve γ, and we defined the covariant derivative along curves $\frac{D}{dt}$
We can express this locally using the velocity vector field of the curve $\dot{\gamma} = \sum \dot{x}^j(\partial_j \circ \gamma)$, to get the expression

$$\dot{V} = \sum_i \dot{V}^i(\partial_i \circ \gamma) + V^i(\nabla_{\dot{\gamma}} \partial_i) = \sum_k (\dot{V}^k + \sum_{i,j} V^i \dot{x}^j (\Gamma_{ij}^k \circ \gamma))(\partial_k \circ \gamma).$$

Remark 5.1. $\Gamma_{ij}^k = \Gamma_{ji}^k$ holds for all i, j. This follows from the torsion free property, and the fact, that for base fields $[\partial_i, \partial_j] = 0$.

Definition 5.2. $V \in \mathfrak{X}(\gamma)$ is a parallel vector field, if $\dot{V} = 0$.

Locally all coefficient functions[1] must vanish, which is a first order ODE system for the V^i-s. From the Cauchy-Peano theorem we see, that for a given initial condition $V(0) = v \in T_{\gamma(0)}M$ there is a unique solution defined on the domain of γ.

Definition 5.3. Parallel transport of vectors along curves means precisely the solution of this equation system, a map $P_{\gamma(a)}^{\gamma(b)} : T_{\gamma(a)}M \to T_{\gamma(b)}M$.

This map is clearly linear, moreover it is orthogonal wrt the metric. This is true, because is V, W are parallel, then $< V, W >' = < V', W > + < V, W' > = 0 + 0 = 0$, thus an orthonormal basis in $T_\gamma(a)$ is taken to an orthonormal basis in $T_{\gamma(b)}M$. This construction clearly extend to piecewise smooth curves in a natural manner as well.

Remark 5.4. This parallel transport map depens on the curve, not only the endpoints.

This leads to the notion of holonomy at a point. Take all parallel transport maps generated by piecewise smooth curves beginning and ending at p, this obviously forms a subgroup in $O(T_pM)$, called the holonomy group of M at p.

[1] $\dot{V}^k + \sum_{i,j} V^i \dot{x}^j (\Gamma_{ij}^k \circ \gamma)$

Theorem 5.5 (de Rahm). *If the holonomy group is reducible (in the sense of group representations*[2]*) at* p*, then* M *is locally*[3] *a Riemannian product.*

5.1 Geodesics

Definition 5.6. $\gamma : I \to M$ is a geodesic, if $\gamma' = 0$.

An immediate consequence of this definition, is that $||\gamma'||$ is constant, so the parameter is proportional to arc length, we call a geodesic "normal", if its speed is 1.
By writing the equation for a vector field to be parallel for γ', we get the geodesic equation(s).

$$\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j, \ k = 1, \dots, n$$

where x^i are the local coordinate functions of our curve γ. We see, that this is a second order ODE, and from the general theory of ODEs, we get

Theorem 5.7. *For any* $p \in M$*, there exists an open neighborhood* U *of* p*, and* $\epsilon > 0$ *such that for all* $q \in U$ *and* $v \in T_pM$ *with* $||v|| < \epsilon$ *here exists a unique geodesic* $\gamma_v : (-1, 1) \to M$ *with* $\gamma_v(0) = p, \gamma'_v(0) = v$*. Furthermore, the map* $(v, t) \mapsto \gamma_v(t)$ *is smooth* $T^{<\epsilon}U \times (-1, 1) \to M$*.*[4]

Example 5.8.

- Straight lines in $\mathbb{R}^n$.
- Great circles in S^n.[5]
- Hyperbolic lines in H^n.
- For surfaces in $\mathbb{R}^3$ γ is a geodesic iff $\gamma'' \perp S$ at all points, or equivalently the principle normal of γ in $\mathbb{R}^3$ must be normal to S.
- Meridians on surfaces of revolution.

6 Sixth lecture

Remember, we've been doing geodesics, given a $p \in M$ *there is a unique geodesic for a given* $v \in T_pM$ *with this vector as its velocity vector. There is also a universal* ϵ *such that all of these geodesics are defined on the interval* $(-\epsilon, \epsilon)$ *or something bigger. Observe, that geodesics have a certain "homogeneity" to them. The geodesic for the vector* sv *for* $s \in \mathbb{R}$ *is* $\gamma_v(st)$*, as is easily seen by the chain rule+uniqueness of geodesics. With all this, we reach today's topic.*

[2] i.e. there are inariant subspaces of T_pM under the holonomy group

[3] we can't say anything globally, a flat torus has trivial holonomy

[4] $T^{<\epsilon}U := \{v \in TU : ||v|| < \epsilon\}$

[5] The "principle of symmetry" states, that if $f : M \circlearrowleft$ is an isometry with $N = fix(f)$ a submanifold, then any geodesic beginning from N, with initial velocity in TN will stay in N.

6.1 The exponential map

Definition 6.1. Consider the set $\Omega := \{v \in TM | \gamma_v \textit{is defined at } t = 1\}$, we define the exponential function $exp : \Omega \to M,\ v \mapsto \gamma_v(1)$.

The key observation is that this map is differentiable, and its derivative at the origin $T_0 exp_p : T_0\Omega_p \to T_pM$ is an endomorphism of T_pM.[1]

Proposition 6.2. *This endomorphism is the identity.*

Proof. $T_0 exp_p(v)$, take a representative curve for v, $t \mapsto tv$ will suffice. We compose this with the exponential map to get the value of the derivative.

$$\frac{d}{dt} exp_p(tv)|_0 = \frac{d}{dt}\gamma_{tv}(1)|_0 = \frac{d}{dt}\gamma_v(t)|_0 = v$$

□

Corrolary 6.3. *The exponential map takes a neighbourhood of* $0 \in T_pM$ *diffeomorphically to a neighbourhood of* $p \in M$. *Such a neighbourhood of* p *is called normal*[2]. *Normal coordinates around* p *are the image of an orthonormal basis of* T_pM *by the exponential map, so we get coordinates which are orthonormal around* p. *We can also talk about normal balls (or geodesic balls) about* p, *the image of the standard metric balls*[3] *in* T_pM *by the exponential map. Similarly we can define normal spheres. Finally we can talk about uniformly normal neighbourhoods. A neighbourhood is called uniformly normal, if it is contained in a normal neighbourhood of any of its points. These also always exist thanks to the theorem from last lecture.*

Lemma 6.4 (Gauss)**.** *Geodesics starting from* p *intersect normal spheres orthogonally.*

Proof. Let $B = B(0, \epsilon) \subset T_pM$ such that $exp_p : B \to M$ is a diffeomorphism, denote the image by D. Denote by $S \subset T_pM$ the unit sphere. We want to use polar coordinates to parametrise everything, $B \setminus \{0\} = (0, \epsilon] \times S$, and the exponential map takes this to $D \setminus \{p\}$ by $f : (r, v) \mapsto \gamma_v(r)$. Pick a vector field $\tilde{X} \in \mathfrak{X}(S)$ and extend it by pullback along the radial directions to $B \setminus \{0\}$. Also define X to be $r\tilde{X}$. We want to project this down to the manifold, take the pushforward of X to be $TfX = Y \in \mathfrak{X}(D \setminus \{p\})$. Now the claim of the theorem is that $< Y, \frac{\partial}{\partial r} >= 0$.[4]
First we will see, that $< Y, \frac{\partial}{\partial r} >$ is constant.

$$\frac{d}{dr} < Y \circ \gamma_v, \gamma_v' >=< \frac{D}{dr}(Y \circ \gamma_v), \gamma_v' > + < Y \circ \gamma_v, \gamma_v'' >=< \nabla_{\frac{\partial}{\partial r}} Y, \frac{\partial}{\partial r} > \circ \gamma_v =< \nabla_Y \frac{\partial}{\partial r} + [\frac{\partial}{\partial r}, Y], \frac{\partial}{\partial r} > \circ \gamma_v =$$

Where we used that γ_v is a geodesic, the vector fields come from vector fields on the manifold, and symmetry of the covariant derivative.

$$= \frac{1}{2} Y < \frac{\partial}{\partial r}, \frac{\partial}{\partial r} > \circ \gamma_v + < [\frac{\partial}{\partial r}, Y], \frac{\partial}{\partial r} > \circ \gamma_v =$$

The first term disappears since the inner product is constant (again, γ_v is a geodesic), secondly, the Lie-bracket is preserved by diffeomorphisms, so it is enough to evaluate it 'upstairs', in T_pM. There $\tilde{X}$, and $\frac{\partial}{\partial r}$

[1] since $T_0\Omega_p = T_0T_pM = T_pM$
[2] we also assume that it is a coordinate neighbourhood, which it always is but whatever
[3] if they are small enough
[4] along γ_v this vector $\frac{\partial}{\partial r}$ is just γ_v'

are the basis fields for the polar coordinates, meaning that they commute, we only have to compute $[r\tilde{X}, \frac{\partial}{\partial r}]$ now.[5] So in the end we are left with $\tilde{X}$. Continuing the calculation from before, we get

$$= \frac{1}{r} < Y \circ \gamma_v, \gamma_v' >$$

So denoting $< Y \circ \gamma_v, \gamma_v' >= \phi(r)$ we get the equation $\phi' = \frac{1}{r}\phi$. The solutions of which are functions of the form ar, we need $a = lim_0 \phi' = 0$ to conclude what we want.

$$\phi'(r) =< \frac{1}{r} Y_{\gamma(r)}, \gamma'(r) >=< f_* \tilde{X}, \gamma'(r) >$$

The lift of the last term is constant 0 in T_pM for $r > 0$, so the limit is zero as well. Finally $T_0 exp_p : T_0 T_p M \to T_p M$ is an orthogonal map, so an isometry, and thus preserves the inner product. □

7 Seventh lecture

We want to apply Gauss' lemma.

Theorem 7.1. *M is a Riemannian manifold, $p \in U$, where U is a normal neighborhood of p. Let $B \subset U$ be a closed proper normal ball about p. $\gamma : [0, 1] \to B$ is a geodesic with $\gamma(0) = p$, $\gamma(1) = q$. If δ is another curve from p to q, then $L(\delta) \geq L(\gamma)$, with equiality exactly when $\delta([0, 1]) = \gamma([0, 1])$ (and δ is a monotone reparametrisation of γ).*

Proof. We can assume, that the image of δ is also contained in B by the triangle inequality, and also that $\delta(t) \neq p$ for $t > 0$, and also, that $q \in \partial B$. We get

$$B \setminus \{p\} \xrightarrow{\text{radius}/r} (0, 1] \xrightarrow{\gamma} \text{one radius of } B$$

, call the composition σ. Gauss' lemma says that for any $p' \in B \setminus \{p\}, p' \in S(p, r')$ the tangent space splits as $T_{p'}M = \mathbb{R} \oplus T_{p'}S(p, r')$ ($\sim$ polar coordinates). Note, that $\sigma(p') = \gamma(\frac{r'}{r})$, and $T_{p'}\sigma : T_{p'}M \to T_{\gamma(\frac{r'}{r})}(\gamma([0, 1]))$ is just projection onto the $\mathbb{R}$ component described above. Now

$$L(\delta) = \int_0^1 ||\delta'(t)||dt \geq \int_0^1 ||\sigma \circ \gamma'(t)||dt \geq L(\sigma \circ \gamma) \geq L(\gamma).$$

□

Corrolary 7.2. *Geodesics locally minimize arc length.*[1]

Corrolary 7.3. *Any distance minimizing curve must be a geodesic.*

Remark 7.4. Geodesics don't necessarily minimize arclength globally.
Also in a uniformly normal ball, any two points are connected by a unique minimal geodesic, because exp_p is a local diffeomorphism.

The other noteworthy consequence will be the existance of convex neoghborhoods.

Theorem 7.5. *If r is small enough, then the minimizing geodesic segment between any two pont p, q in a normal ball $B(x, r)$ stays in $B(x, r)$. Moreover $B(x, r)$ is strictly convex, ergo $\gamma((0, 1)) \subset int(B(x, r))$ for all geodesics γ.*

[5] $[\partial_x, x\partial_y]f = \partial_x(x\partial_y f) - x\partial^2_{xy} f = \partial_y f$
[1] Choose a uniformly normal ball and apply the theorem

Proof. We need the following lemma, from which the theorem clearly follows.

Lemma 7.6. *If r is small enough and γ is a geodesic in $B(p,r)$, which is tangent to some geodesic sphere S about p of smaller radius at some point $q \in S$, then in some neighborhood of q γ stays outside S.*

Proof. Let W be the uniformly normal ball around p, and $S(TW)$ be the tangent unit sphere bundle. Denote the function $\gamma(t,v) \mapsto exp_q(tv)$ by $\Gamma : (-\epsilon, \epsilon) \times S(TW) \to M$, these are arclength parametrized geodesics. Now let $u(t,v) = exp_p^{-1}(\Gamma(t,v))$, so $u : (-\epsilon, \epsilon) \times S(TW) \to T_pM$ is a smooth map, also denote $F(t,v) := ||u(t,v)||^2 \in \mathbb{R}$. Note, that $F(t,v) = d_g(p, \Gamma(t,v))^2$. We want to show, that if $\Gamma'(t,v) \perp S$, then $F(t,v)$ has a strict local minimum at $t = 0$, this will prove the lemma. $\frac{\partial F}{\partial t} = 2 < \frac{\partial u}{\partial t}, u >$. If $\Gamma'(t,v) \perp S$, at $q = \Gamma(0,v)$, then Gauss' lemma implies that $< \frac{\partial u}{\partial t}, u >= 0$, but we need strict positivity of the second derivative, so we know the function is convex, and we have a strict local minimum.
It suffices to see $\frac{\partial^2 F}{\partial t^2}(0,v) > 0$ for $v \in T_pM$, i.e. the case $p = q$, then by continuity we get the same conclusion in a small neighborhood of $(0,v)$. In this case $u(t,v) = tv \in T_pM$, thus $\frac{\partial^2 F}{\partial t^2}(0,v) = 2||v||^2 > 0$. □

□

8 Eight lecture

Still working with geodesics, we have to discuss completeness of Riemannian manifolds. We call a geodesic complete, if its domain of definition can be extended to the whole of $\mathbb{R}$.

Definition 8.1. M a Riemannian manifold is called complete, if any geodesic curve can be extended to be defined on the whole real line.

A natural question would be, what conditions on M guarantee completeness. Compactness would be a good guess for example.

Theorem 8.2 (Characterisations of completeness)**.** *Let (M,g) be a connected Riemannian manifold. The following are equivalent:*

1. *M is complete*
2. *For all $p \in M$, $exp_p : T_pM \to M$ is defined on the whole tangent space.*
3. *There exists $p \in M$ such that exp_p is defined on the whole T_pM.*
4. *Closed and bounded[1] subsets are compact in M.*
5. *(M, d_g) is complete as a metric space.*

Proof. 1. $\to$ 2. $\to$ 3. are completely trivial.
3. $\to$ 4. is not hard either. If we have a bounded set in $H \subset M$, then it is contained in a certain metric ball around the point p, guaranteed by 3.. This means that it is in the image[2] of a closed ball in T_pM, a closed subset of a compact set is compact.

[1] w.r.t. the metric d_g on M
[2] and the image of a compact set is compact

4. $\to$ 5. is some general topology thing. Any chauchy sequence is bounded, its closure is compact, thus we get convergence.
5. $\to$ 1. is the observation that if a geodesic can only be defined on a proper subset of $\mathbb{R}$, we can pick parameter points, converging to one of the (finite) ends[3] of the interval. The image of this sequence in M is a Cauchy sequence, so it has a limit, so the geodesic is defined on some larger interval, a contradiction. $\square$

Theorem 8.3 (Hopf-Rinow theorem). *In a complete connected Riemannian manifold any two points can be joined by a minimal geodesic.*

Lemma 8.4 (completeness not needed here). *Given $p, q \in M$, choose a normal ball of radius $r << d_g(p,q)$ around p. Then there exists $p' \in S = \partial B(p,r)$ such that $r + d_g(p',q) = d(p,q)$.*

Proof. $d_g(q,S) = \inf\{d_g(q,s) : s \in S\}$ by definition, but since S is compact, this is a minimum, so we can pick a point, realizing this minimum, we claim this point will suffice for the requirements of p'.
If γ is any piecewise smooth curve from p to q, it will hit S at some point p''. Now $L(\gamma) \geq d_g(p,p'')+d_g(p'',q) \geq r + d(p',q)$, thus $d(p,q) \geq r + d(p',q)$, the other direction is given by the triangle inequality, and the lemma is proven. $\square$

Hopf-Rinow proof. Choose r and p' as in the lemma. By completeness $p' = exp_p(rv)$, where v is a unit tangent vector at p. Now take $\gamma(t) = exp_p(tv)$, which is defined on the whole real line, again by completeness.
We want to show, that γ hits q at some time T, and $\gamma|_{[0,T]}$ is minimal. Consider the set $J = \{t \in [0,+\infty] : t + d(\gamma(t),q) = d(p,q)\}$ of "good" parameter values. $0 \in J$ is clear, also $J \subseteq [0, d(p,q)]$, and it is an initial segment of this interval, if $t_2 \in J$, then $t_2 > t_1 \in J$, which is easy to see, for $|t_2 - t_1| << 1$, and then going by small steps.
Let $T = \max J$, and we claim, that it is actually $d(p,q)$. Assume otherwise, and take a small normal ball around $\gamma(T) \neq q$, as in the previous lemma. We get a point $p' \in S(\gamma(T), r)$, the lemma guarantees, that $r + d(p',q) = d(\gamma(T),q)$. $T \in J$ means that $T + d(\gamma(T),q) = d(p,q) \leq d(p,p') + d(p',q)$ adding this inequality to the previous equality we see that $r + T = d(p,p')$ and this is certanly less, than $d(p,\gamma(T)) + r = T + r$ by the triangle inequality. This means that we had equality everywhere, and $T + r \in J$, and we get a contradiction. $\square$

8.1 Curvature of Riemannian manifolds

Motivation: curvature should measure how much a manifold differs from a flat manifold/Euclidean space. We need to consider 2-d directions in the manifold, i.e. vector field pairs, and look at how they are related by the geometry of the ambient manifold. Look at basis fields first, do covariant differentiation in succesion: $\nabla_{\partial_i}\nabla_{\partial_j}Z$. For basis fields the order in which we differentiate does not matter, not so in the general case! For non-basis fields X, Y in the Euclidean case we get $\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = \nabla_{[X,Y]}Z$. For general Riemannian manifolds we can still form

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z : \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Definition 8.5. On a Riemannian manifold M, equipped with the Levi-Civita connection the map defined by the above formula is called the Riemannian curvature tensor field.

[3] the domain of definition is necessarily open, since locally we can always extend about a point by solving the equations

9 Ninth lecture

Last time, we should have proven the Hopf-Rinow theorem first, since for the $3 \to 4$ part of the characterisation theorem we need, that every point can be connected to the distinguished point by a minimal geodesic.

Now back to the curvature tensor. Since it is a tensor, we have a local expression for it. $R(\partial_i, \partial_j)\partial_k = \sum R^l_{ijk}\partial_j$, and we can evaluate the functions R^l_{ijk} explicitly in terms of the Christoffel symbols:[1]

$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \sum (\Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{jk}\Gamma^l_{ir})$$

Now we will use the musical isomorphism to turn this into a (0,4) tensor, instead of a (1,3) tensor. This version is sometimes called the Riemann-Christoffel tensor: $Rm = R^\flat$, and $Rm(X,Y,Z,W) =< R(X,Y)Z, W >$, we will write (X,Y,Z,W) for the same thing also.

There are certain symmetries of these functions, and at large, of the curvature tensor, which we will investigate next.

Theorem 9.1. *The following identities hold for the Riemann-Christoffel tensor:*

1. $(Y,X,Z,W) = -(X,Y,Z,W)$*, obviously*
2. $(X,Y,W,Z) = -(X,Y,Z,W)$
3. $(X,Y,Z,W) + (Y,Z,X,W) + (Z,X,Y,W) = 0$*, this is called the first/algebraic Biachi identity*
4. $(Z,W,X,Y) = (X,Y,Z,W)$

Proof. 2. If X, Y is fixed, $(X,Y,\cdot,\cdot) \to \mathbb{R}$ is a bilinear function, which we claim is antisymmetric, so we have to check that $(X,Y,Z,Z) = 0$.

$$< \nabla_X \nabla_Y Z, Z >= X < \nabla_Y Z, Z > - < \nabla_Y Z, \nabla_X Z >$$

$$< \nabla_Y \nabla_X Z, Z >= Y < \nabla_X Z, Z > - < \nabla_X Z, \nabla_Y Z >$$

$$< \nabla_{[X,Y]} Z, Z >= \frac{1}{2}[X,Y] < Z, Z >$$

By summing up these terms, we get the expression we are looking for being equal to

$$X < \nabla_Y Z, Z > -Y < \nabla_X Z, Z > -\frac{1}{2}[X,Y] < Z, Z >$$

Using the same trick of compatibility of the connection and the metric we can rewrite the first two terms as $\frac{1}{2}Y < Z, Z >$ and $\frac{1}{2}X < Z, Z >$ respectively, so the expression indeed vanishes.

3. We write with the original curvature tensor. Write out the expression, group terms by the outer covariant differentiation, and use torsion-freeness of the covariant differential to get terms like $\nabla_X [Y,Z]$, and group these with the corresponding $-\nabla_{[Y,Z]}X$ terms, and finally use the Jacobi identity for the Lie bracket.

4. Write out all 4 Bianchi identities and add them up. Notice that a lot of terms cancel, and we get twice what we wanted to prove (switch the first two and the last two arguments in two of the 4 remaining terms). □

[1] as seen above. We will never use this

Remark 9.2. 1., 2., 4. together means that this is a symmetric bilinear function on $\Lambda^2 TM$, if we interpret the first pair, and the second pair of inputs as bivectors. So it is a self-adjoint operator on this space, since it is equipped with an inner product, inherited from the metric on TM.[2] In our special case[3]

$$< x \wedge y, z \wedge w >= \det \begin{matrix} < x, z > & < x, w > \\ < y, z > & < y, w > \end{matrix}$$

If we interpret our tensor as this self-adjoint map $\Lambda^2 T_p M \to \Lambda^2 T_p M$ it is called the curvature operator. For example in dimension two, the second exterior product is of dimension one, so the curvature is encoded as a single scalar, the Gaussian curvature.

Now we turn our attention to sectional curvature. If V is a finite dimensional vector space, we denote by $G^k V$ the space of k-dimensional subspaces of V. For $p \in M$ we can form $G^2 T_p M$, which we will denote $G^2_p M$, the union of which will be the second Grassmann-bundle $G^2 M$.

Definition 9.3. For $p \in M$ and $\sigma \in G^2_p M$ choose a basis x, y of σ. The sectional curvature of M in σ is defined as

$$K(\sigma) := \frac{(x, y, y, x)}{< x \wedge y, x \wedge y >} = \frac{< R_p(X, Y)Y, X >}{< x, x >< y, y > - < x, y >^2}$$

where we use the inherited metric on the second exterior product discussed before.

This K gives a smooth map $G^2 \to \mathbb{R}$, once we argue that is well defined. But that is a simple check, scalar multiples obviously don't change anything, and for an elementary basis transformation the new terms drop out. Things are a bit clearer, if we restrict ourselves to orthonormal bases in σ, in this case $K(\sigma) = (x, y, y, x)_p$.

Theorem 9.4. *K determines R uniquely. More precisely, this is an algebraic statement: If V as any finite dimensional real vector space and Rm and Rm' are algebraic curvature tensors (meaning a $(0, 4)$ tensor on V satisfying the symmetries $1 - 4$ from before) for which K and K' coincide (as functions on $G^2 V$), then $Rm = Rm'$.*

Proof. We will use (x, y, z, w) and $(x, y, z, w)'$ for the two tensors. By assumption $(x, y, x, y) = (x, y, x, y)'$, we will manipulate this. $(x + z, y, x + z, y) = (x + z, y, x + z, y)'$ is also true for all $x, y, z \in V$, we expand by multilinearity. $(x, y, x, y) + (z, y, x, y) + (x, y, z, x) + (z, y, z, y)$, and this is equal with the same expression with $'$-s, the first and last terms are also on the right hand side, and thus vanish, we are left with

$$2(x, y, z, y) = 2(x, y, z, y)'.$$

We do the same trick for this identity again, substitute $y + w$ for y in both places and expand. $(x, y, z, y) + (x, w, z, y) + (x, y, z, w) + (x, w, z, w)$ and this is equal to the same thing with $'$-s, and by the original assumption the first and last terms are equal, and thus vanish again. What remains is $(z, y, x, w) + (x, y, z, w)$ being equal to its primed counterpart (after an application of the 4th symmetry). Subtracting we get

$$(x, y, z, w) - (x, y, z, w)' = (y, z, x, w) - (y, z, x, w)'$$

and notice, that this identity implies, that the expression is invariant under cyclic permutations of the first three variables. Now we apply the Binachi identity, to get that

$$3[(x, y, z, w) - (x, y, z, w)'] = 0$$

□

[2] basically fix an orthonormal ordered basis, and declare the basis they induce on some $\Lambda^k V$ to be orthonormal

[3] a special case of this formula is the Lagrange identity in $\mathbb{R}^3$ expressing $(a \times b) \cdot (c \times d)$ as a determinant of their inner products

Remark 9.5. K is the quadratic form of the induced symmetric map on $\Lambda^2 V$, and this determines the bilinear map completely, think this through!

10 Tenth lecture

Continuing curvature. Recall the formula for $R(X,Y)Z$, and the modified version $Rm(X,Y,Z,W)$. We also had the $R : \Lambda^2(TM) \to \Lambda^2(TM)$ curvature operator, a self adjoint linear map. If $\sigma = span(x,y) \in G^2(TM)$, then $K(\sigma) = \frac{Rm(x,y,y,x)}{||x||^2||y||^2 - <x,y>^2}$, and these sectional curvatures determine the curvature tensor completely.

Remark 10.1. A small digression into the world of Grassmannian "things". If we have an n-dimensional vector space, we can create the $G^k V$ manifolds, consisting of the k-dimensional subspaces of V, and also the Grassmann algebra $\Lambda V = \oplus \Lambda^k V$. There is a natural map $G^k V \to P(\Lambda^k V)$, which sends a basis of a subspace in $Gr^k V$ to its class in the target, which is well defined, any two bases of the same subspace differ only by a scalar. This map is not surjective by simple dimension count[1]. These dimensions do coincide just by accident for $k = 2, n = 2, 3$ for example.

Example 10.2 (algebraic curvature tensor). Suppose an inner product $<,>$ is given on V and define

$$Rm^0 = (x,y,z,w)_0 = < x, w >< y, z > - < x, z >< y, w > .$$

It is an immediate check, that this satisfies the symmetries we want from a curvature tensor. In an orthonormal basis $R_{ijkl} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$ will be the coefficient system. Substituting $z = y, w = x$ we see that the sectional curvatures are all equal to 1.

Corrolary 10.3. *Suppose that in a manifold M, $p \in M$ is such that K is constant on $G^2_p M$, then $Rm_p = K(p)Rm^0$.*

Theorem 10.4 (Schur)**.** *If $K(p)$ is constant at all $p \in M$, where M is connected and $\dim M > 2$, then this constant does not depend on p.*

Example 10.5 (Manifolds of constant curvature).

1. $\mathbb{R}^n$, everything is 0.
2. S^n, we notice that isometries preserve curvature, and they are transitive on $G^2 S^n$
3. $\mathbb{H}^n$ by the same reasoning[2]
4. also anything locally isometric to any of the above

Remark 10.6. If $q : N \to M$ is a Riemanian covering, all the sectional curvatures are preserved by q, and $R_N = q^* R_M$. In particular having constant curvature is preserved in either direction.

10.1 Curvature of submanifolds

The setup: $M \subset \tilde{M}$ Riemannian submanifolds, every symbol will be distinguished by the tilde. We have $\nabla, \tilde{\nabla}$, we get ∇ by orthogonal projection (a routine application of Koszul's formula). We know, that for vector fields $X, Y \in \mathfrak{X}(M)$ $\nabla_X Y = (\tilde{\nabla}_{\tilde{X}} \tilde{Y}$ projected orthogonally to $TM)$, where $\tilde{X}, \tilde{Y}$ are local extensions of X, Y to $\tilde{M}$, so moving forwards, we will denote this by just writing $\tilde{\nabla}_X Y$, since it does not depend on the extension.

[1] source is $k(n-k)$, traget is $\binom{n}{k} - 1$
[2] we don't know the exact value of the constants just yet

Definition 10.7. Second fundamental form of M will be defined as the normal component of $\tilde{\nabla}_X Y$, i.e. $b(X,Y) := \tilde{\nabla}_X Y - \nabla_X Y$ for $X, Y \in \mathfrak{X}(M)$.

Proposition 10.8. *This b is symmetric and tensorial in both variables.*

Proof. Symmetry can be seen by looking at $b(X,Y) - b(Y,X) = [\tilde{X}, \tilde{Y}] + [Y, X]$ by writing out the defining formula, and this vanishes, when we restrict to M.
Tensoriality comes from this for free, since the covariant derivative is tensorial in the first variable, by symmetry it is tensorial in the second as well. □

Remark 10.9. If M is a hypersurface, then locally b is a real valued function, in general b takes its values in the normal bundle of M. In this way we recover Gauss' classical second fundamental form from this construction. In hypersurface theory we identify $b(X,Y)$ with $b(X,Y)N$, where N is the unit normal vector field. If $X, Y \in \mathfrak{X}(M)$ then $< \tilde{Y}, \tilde{N} > |_M = 0$, so

$$0 = \tilde{X} < \tilde{Y}, \tilde{N} > |_M =< \tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{N} > |_M + < \tilde{Y}, \tilde{\nabla}_{\tilde{X}} \tilde{N} > |_M$$

here the first term is just $b(X,Y)$, the second is the Weingarten map.
So we have $\nabla_X Y = \tilde{\nabla}_X Y - b(X,Y)$, this is called the Gauss formula. We need the formula for curvature[3].

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z =$$

$$= \nabla_X (\tilde{\nabla}_Y Z - b(Y,Z)) - \nabla_Y (\tilde{\nabla}_X Y - b(X,Y)) - (\tilde{\nabla}_{[X,Y]} Z - b([X,Y],Z)) =$$

Next we expand the outer ∇s as well, collecting like terms, we get

$$R(X,Y)Z = \tilde{R}(X,Y)Z - \tilde{\nabla}_X b(Y,Z) + \tilde{\nabla}_Y b(X,Z) + b([X,Y],Z)$$

Similarly for the Riemann-Christoffel tensor

$$< R(X,Y)Z, W >=< \tilde{R}(X,Y)Z, W > + < b(Y,Z), b(X,W) > - < b(X,Z), b(Y,W) >$$

this is called the Gauss equation for the cirvature tensor. For sectional curvature $\sigma =< x, y >\in G_p^2 M$ we get

$$K(\sigma) = \tilde{K}(\sigma) + \frac{< b(x,x), b(y,y) > - < b(x,y), b(x,y) >}{< x,x >< y,y > - < x,y >^2}.$$

This is the Gauss equation for sectional curvature.
An important special case is when M is a hypersurface $K(\sigma) = \tilde{K}(\sigma) + \frac{\det(b|_\sigma)}{\det g|_\sigma}$. In this case sectional curvature and Gausian curvature coincide.

Remark 10.10. The Theorema Egregium follows as a consequence.

11 Eleventh lecture

Definition 11.1. $p \in M \subset \tilde{M}$, M is called a geodesic submanifold at p, if any geodesic in $\tilde{M}$ through p in any tangent direction from $T_p M$ stays in M for some time.

[3] :'(

Proposition 11.2. *If M is geodesic at p, then $b_p = 0$.*[1]

Proof. It suffices to show, that $b_p(v, v) = 0$ for all $v \in_T pM$. Let γ be a geodesic from p, with $\gamma'(0) = v$. Extend γ' to a vector field X and $\tilde{X}$.
$\nabla_{\gamma'}\gamma' = 0 = \tilde{\nabla}_{\gamma'}\gamma'$ since it is a geodesic. It follows, that $\nabla_v X = \tilde{\nabla}_v X = 0$, thus $b_p(v, v) = 0$. □

Corrolary 11.3. *If M is geodesic at p, then $K(\sigma) = \tilde{K}(\sigma)$ for all $\sigma \in G_p^2 M$.*

Definition 11.4. $M \subset \tilde{M}$ is a totally geodesic submanifold, if it is geodesic at all $p \in M$.

Theorem 11.5. *For $M \subset \tilde{M}$ the following are equivalent.*

1. *M is totally geodesic*
2. *$b = 0$ everywhere*
3. *$\tilde{\nabla}_X Y$ is tangential for all $X, Y \in \mathfrak{X}(M)$*

Proof. The equivalence of 2. and 3. is clear. 1. $\to$ 2. follows from the preceding proposition.
2. $\to$ 1. follows from 3., the geodesics are the same. □

Example 11.6.
1. Let $p \in \tilde{M}$ arbitrary, choose an arbitrary linear subspace $V \leq T_p\tilde{M}$ and consider $M = exp_p(\Omega_p \cap V)$, then M is a geodesic submanifold at p.
2. If $f : \tilde{M} \circlearrowleft$ is an isometry with $fix(f) \neq \emptyset$. Then connected components of $fix(f)$ are totally geodesic.
3. In $\mathbb{R}^n, S^n, H^n$ totally geodesic submanifolds are just tje geometric subspaces.

Corrolary 11.7. *Sectional curvature of these are $0, 1, -1$ respectively.*
More precisely $K(rS^n) = \frac{1}{r^2}$, and similarly $K(rH^n) = \frac{-1}{r^2}$, so all real constants occur as sectional curvature of some constant curvature space.

11.1 Jacobi fields

The motivation behind this notion, is the interest in the singular behaviour of the exponential map. Take a Riemannian manifold M, a point p, and $v \in T_pM$, we are interested in $T_v exp_p$, if we denote $epx_p(v) = \gamma(1) = q$, this map has image in T_qM. Also remember, that we can identify the tangent space of a vector space with itself, so $w \in T_vT_pM = T_pM$. One may take $w = v'(0)$, where v is a curve in T_pM, with $v(0) = v$, representing the tangent vector w. From this $(T_v exp_p)w = \frac{d}{ds} exp_p v(s)|_{s=0}$.
The idea is, that we may consider this vector along the whole of γ as a vector field: $(T_{tv} exp_p)tw = \frac{\partial}{\partial s} exp_p(tv(s))|_{s=0}$. This gives the idea of "variation vector fields" along γ.

Definition 11.8. Let $\gamma : I \to M$ be a smooth curve. A variation of γ is a smooth map $\alpha : (-\epsilon, \epsilon) \times I \to M$ such that $\alpha(0, t) = \gamma(t)$. This is called a geodesic variation, if $\alpha(s, .) : I \to M$ is a geodesic for all $s \in (-\epsilon, \epsilon)$. The variation vector field corresponding to α is $V_\alpha(t) = \frac{\partial}{\partial s}\alpha(s, t)|_{s=0}$, clearly $V_\alpha \in \mathfrak{X}(\gamma)$.

Definition 11.9. Suppose γ is a geodesic in M. A vector field $J \in \mathfrak{X}(\gamma)$ is called a Jacobi field along γ if $J = V_\alpha$ for some geodesic variation α of γ.

[1] the converse is false

Theorem 11.10. *Given a geodesic $\gamma : I \to M$ a vector field $J \in \mathfrak{X}(M)$ is a Jacobi field iff it satisfies the "Jacobi equation"*

$$J'' + R(J, \gamma')\gamma' = 0 \quad J'' = R(\gamma', J)\gamma'$$

where $J'' = \frac{D^2}{dt^2} J$ is the second covariant derivative.

Proof. Firstly let α be a geodesic variation such that $J = V_\alpha$. Now $\frac{\partial \alpha}{\partial t}$ are parallel fields along $\alpha(s, .)$ for all s by definition, so $\frac{D}{dt}\frac{\partial \alpha}{\partial t} = 0$ for all s.

$$0 = \frac{D}{ds}\frac{D}{dt}\frac{\partial \alpha}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial \alpha}{\partial t} + R(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t})\frac{\partial \alpha}{\partial t}$$

Evaluating at $s = 0$ we get the equation we wanted, use $\frac{D}{ds}\frac{\partial \alpha}{\partial t} = \frac{D}{dt}\frac{\partial \alpha}{\partial s}$, and that $\frac{\partial \alpha}{\partial s}|_{s=0} = J$, $\frac{\partial \alpha}{\partial t}|_{s=0} = \gamma'$. □

12 Twelfth lecture

Remember, we discussed Jacobi fields along a geodesic, and variation vector fields through geodesics. We have already seen, that Jacobi fields along γ satisfy the Jacobi equation $J'' = R(\gamma', J)\gamma'$. Note, that this is a second order ODE for J, given $\gamma, J(0), J'(0)$, there is a unique solution along γ. In particular, for a given γ, the dimension of the vector space of Jacobi fields along γ is $2n$.

Proof of the other direction of 11.10. Given γ, choose $t_0, t_1 \in I$ close enough so that $p = \gamma(t_0), q = \gamma(t_1)$ are contained in a uniformly normal ball in M. For any Jacobi field along γ, assign the pair $(u, v) \in T_pM \times T_qM$, where $u = J(t_0), v = J(t_1)$. Put $\alpha(., t_0)$ to be some curve with derivative u at $s = 0$, and similarly for $\alpha(., t_1)$. For any small s, connect $\alpha(s, t_0)$ to $\alpha(s, t_1)$ with a geodesic defined on $[t_0, t_1]$ (note this is unique in U). This α becomes now a geodesic variation of γ with $J = V_\alpha$. This map is onto, thus the dimension of Jacobi fields is at least $2n$, they all satisfy the Jacobi equation, so from the converse direction we are done. □

Example 12.1. The trivial examples of Jacobi fields we can get by reparametrisation of γ. Choose $\alpha(s, t) := \gamma(t + s)$, with this choice $J(t) = \gamma'(t)$.
Another important special case is when $J(0) = 0$

Corrolary 12.2. *If J is a Jacobi field along γ with $J(0) = 0, \gamma'(0) = v, J'(0) = w$, then a formula for the geodesic variation α such that $J = V_\alpha$ is $\alpha(s, t) = exp_p(t(v + sw))$.*

Proof. This is clearly a geodesic variation, we only need to check $V_\alpha(0) = 0$, and $V'_\alpha(0) = w$, and then apply the preceding proposition.

$$\frac{D}{dt}V_\alpha = \frac{D}{dt}(\frac{\partial}{\partial s}\alpha(s, t))|_{s=0} = \frac{D}{dt}((T_{tv}exp_p)tw) = \frac{D}{dt}(t(T_{tv}exp_p)w) = (T_{tv}exp_p)w + t\frac{D}{dt}(T_{tv}exp_p)w$$

Evaluating at $t = 0$, this shows $V'_\alpha(0) = (T_0 exp_p)w = w$. □

Definition 12.3. Given $p \in M$, a point $q \in M$ is called conjugate to p, if q is a singular value of exp_p, thus q is conjugate to p along the geodesic γ if $\gamma = \gamma_v$ (here $v \in T_pM$) and $q = \gamma_v(1)$ and $T_v exp_p$ is non-invertible. $\dim ker T_v exp_p$ is called the order of conjugacy.

Proposition 12.4. *Suppose $\gamma : [0, 1] \to M$ is a geodesic with $\gamma(0) = p, \gamma(1) = q$. q is conjugate to p iff there exists a nonzero Jacobi field along γ which vanishes at p, q.*

Corrolary 12.5. *Conjugacy of points is a symmetric relation.*

Proof. '→' suppose q is conjugate to p. Let $0 \neq w \in kerT_v exp_p$, also denote $v = \gamma'(0)$. $J(t) = (T_{tv}exp_p)tw = \frac{\partial}{\partial s}exp_p(t(v+sw))|_{s=0}$ is a Jacobi fields, which vanishes at 0, and also $J(1) = (T_v exp_p)w = 0$, since its in the kernel.
'←' If J is as stated, then $\alpha(s,t) = exp_p(t\gamma'(0) + sJ'(0))$ is a geodesic variation such that $V_\alpha = J$. Now $0 \neq (T_v exp_p)J'(0) = J(1) = 0$ by assumption. □

Consider a Jacobi field J along a geodesic γ. What do we know about the angle between $J(t)$ and $\gamma'(t)$?

$$\frac{d}{dt} < J, \gamma' > = < \frac{D}{dt}J, \gamma' > + < J, \frac{D}{dt}\gamma' >$$

Here the first term the derivative is what we called J', and the second vanishes since γ is a geodetic.

$$\frac{d}{dt} < J', \gamma' > = < \frac{D}{dt}, \gamma' > + < J', \frac{D}{dt}\gamma' > = < R(\gamma', J)\gamma', \gamma' > = 0$$

Here we used the Jacobi equation.
From all this, we conclude that $< J', \gamma' >$ is constant. If $J(0) = J(1) = 0$, then somwhere in between $< J', \gamma' >$ must vanish, so it is constant 0, and from this $< J, \gamma' >$ is constant as well, i.e. it is zero everywhere. All this means, that if a Jacobi field shows two points' conjugacy along γ, we see that $J \perp \gamma'$ all the way between the two points.

Example 12.6.

1. In $\mathbb{R}^n$ $R = 0$, so $J(t) = J(0) + tJ'(0)$
2. In S^n geodesics are great circles, i.e. $\gamma(t) = cos(t)x + sin(t)v$, where $v \in T_x S^n, ||v|| = 1$. Choose $w \in T_x S^n, w \perp v, ||w|| = 1$, we get $\alpha(s,t) = cos(t)x + sin(t)(cos(s)v + sin(s)w)$ as a geodesic variation. Now $J(t) = sin(t)w$, the equation becomes $J''(t) = -J(t)$. From this we can calculate, that the unit sphere has curvature 1 as well.
3. A very similar calculation works for H^n as well, exchange sin, cos by their hyperbolis counterparts. $J(t) = sinh(t)w$ and $J'' = J$, and from this $K \equiv -1$.

13 Thirteenth lecture

13.1 Some applications of Jacobi fields

Recall $J'' = R(\gamma', J)\gamma'$, the Jacobi equation. Firstly, if $\dim M = 2$, we can express curvature in terms of "geodesic polar coordinates". Now we think of manifolds abstractly, since embedded surfaces in $\mathbb{R}^3$ have their own well-developed theory due to Gauss. Pick $p \in M$, and exp_p^{-1} as a chart around it. Pick an orthonormal basis in T_pM, and take r, θ polar coordinates correspondingly. Gauss' lemma from earlier states, that locally $exp_p^* = dr^2 + f(r,\theta)^2 d\theta^2$, we get no mixed terms![1]

Remark 13.1. All three classical geometries have metrics in this form, with $f = r$, or $sin(r)$, or $sinh(r)$

We want a formula for the curvature in terms of this function f. Geodesics have the form $\gamma(r) = exp_p(ru)$, and we define $V \in \mathfrak{X}(\gamma)$ by parallel transport of v along γ. We can turn this geodesic a little bit to get

[1] notice $f(0,\theta) = 0$ and $\partial_r f(0,\theta) = 1$

a variation $\alpha(\theta, r) = exp_p(r(cos\theta u + sin\theta v))$. We can calculate, that the corresponding Jacobi field will be $f(r, 0)V(r)$. Now $J''(r) = \partial^2_{rr} f V(r)$, since v is parallel. The Jacobi equation can be written now

$$R(\gamma'(r), J(r))\gamma'(r) = J''$$

From the sectional curvature formula we see $K(\gamma(r)) = -\frac{\partial^2_{rr}(r,0)}{f(r,0)}$, where we use that γ', V is an orthonormal basis in the tangent space, the previous calculations were also used, and symmetry properties of the curvature tensor, finally $||V|| = 1$.

This also implies the previous calculation on the curvature of $\mathbb{R}^2, S^2, H^2$.

Remark 13.2. This formula is not the most practical in practice, abstract surfaces aren't usually given by geodesic polar coordinate patches.

Secondly, we can finally classify manifolds of constant curvature. Note, that if the curvature is constant, we can scale by a positive factor, so only the cases $0, -1, 1$ need to be considered.

We also saw previously, that the curvature tensor looks like the simplest algebraic one, times a scalar $K < xz >< y, w > - < x, w >< y, z >$. Choose $p \in M$, $v \in T_pM$ of norm one, $\gamma(t) = exp_p(tv)$, and pick $u \in_T pM$ orthogonal to v. Let u be the parallel vector field along γ with $u(0) = u$. We get a Jacobi field satisfying $J(0) = 0, J'(0) = u$ as before.

First case is $K = 0$, so the Jacobi equation says $J'' = 0.$, a solution looks like $tu(t)$. Now we use Gauss' lemma once again, which says, that $T_{tv}exp_p : T_{tv}T_pM = T_pM \to T_{\gamma(t)}M$ is an orthogonal linear map, so the exponencial map is not only a local diffeomorphism, but a local isometry! If M is complete, then this is a Riemannian covering[2]. So if we assume M complete, we get that if M is simply connected, complete of constant 0 curvature, then M is isometric to Euclidean space.

If $K = -1$, the equation becomes

$$J'' = R(\gamma', J)\gamma' = -(< J, \gamma' > \gamma' - ||\gamma'||^2 J) = J.$$

We get the solution as $sinh(t)u(t)$. Choose now a reference point $a \in H^n$ and an orthogonal map $\phi : T_pM \to T_aH^n$. Do the same construction, which we did earlier in H^n, with $\tilde{u} = \phi(u)$ to get $\tilde{\gamma}, \tilde{u}, \tilde{J}$, the solution will be the same to the Jacobi equation $\tilde{J}'' = sinh(t)\tilde{u}(t)$.[3] Our map will be $f : exp_p \circ \phi^{-1} \circ \tilde{exp_a}^{-1}$, using the fact, that the exponential map is a local diffeomorphism. Now again Gauss' lemma implies that this f is a local isometry, thus if M is complete, then f is a Riemannian covering map.[4] As a corollary we get, that any simply connected complete Riemannian manifold with constant -1 curvature is isometric to hyperbolic space.

Some changes need to be made for the $K = 1$ case, since it is not diffeomorphic to its own tangent space. The exponential map itself is undefined outside the bal of radius π, so we have to delete the antipodal point of our chosen reference point, so we have to do the construction twice, with two different reference points, and we get the same corollary.

In all cases above, if we replace simply connected, with just connected, we get space forms.

One final application on manifolds of nonpositive curvature.

Theorem 13.3 (Hadamard (n=2), Cartan)**.** *Let M be a connected, complete Riemannian manifold with $K \leq 0$ everywhere. Then for any $p \in M$ $exp_p : T_pM \to M$ is a covering map.*

[2] by a statement from the problem sessions

[3] We know, that H^n has curvature $\equiv -1$.

[4] We need to do the construction in all directions, to see that it is norm preserving in the complementary directions

Proof. Choose an arbitrary geodesic γ in M, and a Jacobi field J along it, which is normal[5]. Look at $(||J||^2)' = 2 < J, J' >$. Second derivative will be $(||J||^2)" = 2 < J', J' > +2 < J, J'' >$, this second term can be expanded using the Jacobi equation. $< J, R(\gamma', J)\gamma' >= - < R(\gamma', J)J, \gamma' >=$[6]$-K(span(J, \gamma')) \geq 0$. Here we used, that $J \perp \gamma'$, and the supposition on the sectional curvatures. This means, that $||J||^2$ is convex function, so if $J(0) = J(1) = 0$, then $J = 0$ everywhere, there are no conjugate points along γ in M. This means exp_p is regular everywhere, so $exp_p : T_pM \to M$ is a local diffeomorphism. Pull back the metric of M using this local diffeomorphism, to get another metric on T_pM, wrt this metric, our map is now a local isometry. The main thing is that our tangent space will be complete with this new metric, since all geodesics through p are complete. □

Corrolary 13.4.

1. *If M is also simply connected, then it is diffeomorphic to $\mathbb{R}^n$*
2. *In general, the universal covering of M is diffeomorphic to $\mathbb{R}^n$, so M is aspherical.*

[5] orthogonal to γ'

[6] this isn't completely true, since we don't know if J, γ' are orthonormal, but only a positive constant comes if if they are not.