The Geometry of Rings and Schemes

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2024/2025/1

1 First lecture

1.1 The basic idea

Rings are commutative with identity, don't let anyone convince you otherwise.

Proposition 1.1. *X, Y top spaces, U_α an open cover of X, we define $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Given $\phi_\alpha : U_\alpha \to Y$ continous maps such that $\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$. Then there is a unique $\phi : X \to Y$ such that it restricts to ϕ_α on U_α.*

Proposition 1.2. *R, S rings, $f_\alpha \triangleleft R$ are chosen such that $(f_\alpha | \alpha \in A) = (1)$. Note, that localisation at f_α and after at f_β is the same, as localisation in the other order. If $\phi_\alpha : S \to R_{f_\alpha}$ are given ringmaps, such that $S \to R_{f_\alpha} \to R_{f_\alpha f_\beta}$ is the same as $S \to R_{f_\beta} \to R_{f_\alpha f_\beta}$ agree. Then there is a unique ringmap $\phi : S \to R$ such that $S \to R \to R_{f_\alpha}$ is f_α.*

Remark 1.3. AG concepts usually have an algebraic and a geometric interpretation. The algebraic side is coming from commutative algebra and/or number theory, which we can then reinterpret in geometric terms, as seen above. This can go in the other direction as well, starting froma geometric/topological concept we can rephrase them in categorical/algebraic terms.

1.2 More geometry of rings

What ring should be analogous to the empty space?

Proposition 1.4. *The empty space is the initial object in TOP.*

So we would like to find the final object of $CRing$, since all of our arrows get reversed in this correspondance between spaces and rings. This is clearly the trivial ring 0, note that we consider this a unital ring.

Remark 1.5. The map $R \to 0$ is a localisation.

If R is a ring, $f \in R$. We can localise $R \to R_f$, another ring we can consider is $R \to R/(f)$.

Remark 1.6. $R_f/(f) = R/(f)_f = 0$, and in general localisation and taking quotients commute.

This gives us the idea, that just as localisation corresponds to open subspaces, quotients should correspond to inclusions of closed subspaces.
Next question is what should be the points of a ring?

Definition 1.7. If X is a topological space, then a point of X is a map $P \to X$ where $P = \{*\}$.

Now we want to know what the analogue of P is, and the answer to this is that it is the unique non-empty space with no nontrivial subspaces. Note, that P is the final object in TOP, but this way ruins the analogy, for example if one would use this definition, many rings would have no points (morphisms to $\mathbb{Z}$) at all. So we turn to the question of what nontrivial rings have no nontrivial quotients or loalisations. These are exactly fields.

Definition 1.8 (questionable)**.** If R isa ring, a point of R is a map $R \to k$, where k is a field.

The problem is, that if we have a field extension $l|k$, this gives a different map, and there are *very* many extensions of any given field.

Definition 1.9 (better)**.** Let R be a ring. A point of R is the natural map $R \to R_p/pR_p$ for p a prime ideal.

Homework 1.10. *If k is a field, R is a ring and $R \to k$ is a ringmap, show that there is a unique prime ideal $p \triangleleft R$ such that the map $R \to k$ factorises through $R \to R_p/pR_p$. (verify that R_p/pR_p is a field)*

Remark 1.11. Given R a ring, $f \in R$ $I \triangleleft R$ then a point $R_f \to k$ can be considered a point of $R \to R_f \to k$. Similarly a point $R/I \to k$ is a point of R by the same remark $R \to R/I \to k$.

Homework 1.12. *The points of R which factor through R/I like this correspond to prime ideals p such that $I \subset p$. The points of R_f correspond to the prime ideals p such that $f \notin p$.*

Definition 1.13. Let R be a ring. The *spectrum* of R is denoted *spec* R is a topological space, whose underlying set is the set is the set of prime ideals of R and whose topology is given by either of the following

1. the closed subsets correspond to ideals $I \triangleleft R$: $\{\phi : R \to k | \phi$ factors through $R/I\}$
2. A base for the topology is for any ideal $I \triangleleft R : \{\phi : R \to k | \phi$ factors through $R \to R_f\}$ for some $f \in R$

Homework 1.14. *Verify that the above two definitions give you a topology, and that they agree.*

Homework 1.15. *Find the spectrum of the following rings:*

1. $\mathbb{Q}$
2. $\mathbb{C}$
3. $\mathbb{C}[x]$
4. $\mathbb{C}[x]_x$
5. $\mathbb{C}[x]_{(x)}$
6. $\mathbb{Z}$
7. $\mathbb{C}[x, y]$

Remark 1.16. Note, for R_f we localise at the multiplicative set $1, f, f^2, ...,$ for a prime ideal p we localise at the multiplicative set $R \setminus p$.

Homework 1.17. *Show that Spec is functorial, i.e. for $\phi : R \to S$ there is an induced map spec ϕ : spec $S \to$ spec R which takes the identity map to the identity map, and respects composition.*

Homework 1.18. *Find two non-isomorphic quotients of $\mathbb{C}[x]$ whose spectra correspond to the same closed subset. I.e. find two ideals where the points which factor through are the same.*

Homework 1.19. *Recall the notion of the characteristic of a field. Give a geometric interpretation. (hint:do this afer computing spec $\mathbb{Z}$)*

Homework 1.20. *spec $R = \emptyset$ iff $R = 0$.*

Homework 1.21. *Let R be a ring, show that spec R is compact.*

2 Second Lecture

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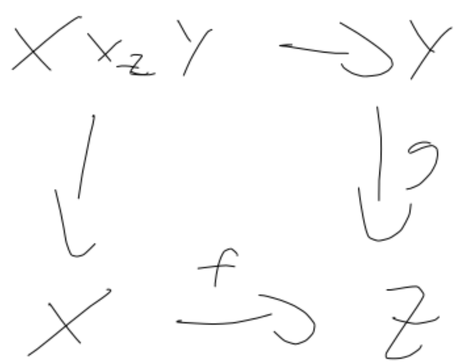
3 Fiber products and such

3.1 Fiber produxts of topoloical spaces

Definition 3.1. Given X, Y, Z topological spaces and maps $f : X \to Z$ and $g : y \to Z$ then the fiber product of X and Y over Z is denoted $X \times_Z Y := \{(x, y) : f(x) = g(y)\}$. There are natural projection maps to X and Y, so that the square commutes.



Example 3.2.

- Given $Z = \{*\}$ then $X \times_Z Y = X \times Y$.
- If $g : Y \hookrightarrow Z$ is an inclusion, then $X \times_Z Y = f^{-1}(Y)$. If $f : X \hookrightarrow Z$ is also an inclusion, then $X \times_Z Y = X \cap Y$.
- If $g : Y \to Z$ is the projection of a vector bundle, then $X \times_Z Y \to X$ is the projection of the pullback bundle f^*Y.

Homework 3.3. *Verify the examples.*

Definition 3.4. We call the diagram of Figure 3.1 a *pullback square* (or a fiber square). The arrow parallel to f is denoted f' and is called the *pullback of f along g*, and symmetrically the arrow parallel to g is the pullback of g along f.

Example 3.5. Being an open (or closed) inclusion is preserved along pullback. The same is true for vector bundle maps.

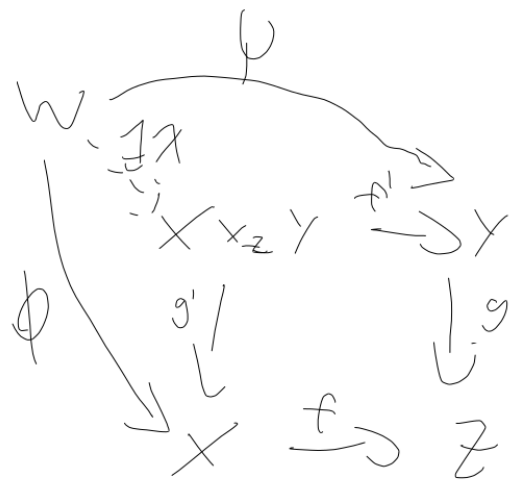


Figure 1:

Proposition 3.6 (Universal property of the fiber product)**.** *The fiber product if the unique space where for each W, ψ, ϕ there is a unique χ such that the following diagram commutes.*

Homework 3.7. *Given a pullback diagram and a map $\tilde{X} \to X$ show that $(X \times_Z Y) \times_X \tilde{X} = \tilde{X}_Z Y$. Also show that given a map $\tilde{Z} \to Z$, we have $(X \times_Z \tilde{Z}) \times_{\tilde{Z}} (Y \times_Z \tilde{Z}) = (X \times_Z Y) \times_Z \tilde{Z}$.*

3.2 Fiber products of schemes

Given R, S, T rings and two maps $r : T \to R$ and $s : T \to S$ we want to produce the pushout P, which is the opposite of the pullback square considered before, with the dual universal property. This exists, and is called the *tensor product* $R \otimes_T S$!

Definition 3.8. Given maps of affine schemes, we can use this to define the fiber product of affine schemes as the spectrum of the tensor product of the underlying maps of rings. For general schemes we define the fiber product on the affine open subsets.

Homework 3.9. *Verify the existence and uniqueness of this construction, and the unviersal property.*

Remember the first lecture,where we localised at two different elements of R, f and g, and we saw that we get a pushout diagram to R_{fg}, from the preceeding discussion we see that this is the same sa $R_f \otimes_R R_g$.>:(A similar procedure happens when we do a localisation at f and factorisation by an ideal I, to get $R_f/IR_f = R_f \otimes_R R/I$, similarly $R/(I + J) = R/I \otimes_R I/J$.

Proposition 3.10. *$\phi : X \to Y$ is a map of topological spaces, U_α is an open cover of Y. ϕ defines an open (closed) inclusion iff $X \times_Y U_\alpha \to U_\alpha$ is an open (closed) inclusion $\forall\alpha$.*

Definition 3.11. For schemes an open inclusion is an open subset, which is a sub-ringed space, which is automatically a subscheme.

Proposition 3.12. *Being an open or closed inclusion is preserved under pullback.*

What should be the product of two schemes? One candidate would be the fiber produt over $Spec\mathbb{Z}$, since every scheme admits a unique morphism to it, but as seen previously this is not the best way to do things.

3.3 Schemes over a scheme

Definition 3.13. Given a scheme S, a *scheme over* S is a map of schemes $X \to S$ (also called an S*-scheme*). This map is called the *structure map* of X If X, Y are schemes over S, then a morphism of S-schemes is defined as a map of schemes from $X \to Y$, commuting with the given maps from $X, Y \to S$.
If $S = spec\ k$ with k a field, then we will just call this k-scheme.

If $T \to S$ is map of schemes and X is an S-scheme, then we can produce $X \to X_s T$ which will be a T-scheme. This construction is called *base-change*. If X, Y are S-schemes we define their product $X \times Y := X \times_S Y$ as an S-scheme.

Homework 3.14. *Let k be a field, show that $\mathbb{A}^2_k = \mathbb{A}^1_k \times_{spec\ k} \mathbb{A}^1_k$. (More generally $\mathbb{A}^n_k = \bigtimes_{spec\ k} \mathbb{A}^1_k$ n times.)*

Homework 3.15. *$X \to spec\ \mathbb{C} \to spec\ \mathbb{R}$. Give an example of X, Y $\mathbb{C}$-schemes and a map of $\mathbb{R}$ schemes $X \to Y$ which is not a map of $\mathbb{C}$ schemes. (hint: don't get too fancy, an affine scheme is enough).*

Homework 3.16. *k a field, $n \geq 0$ we can define $\mathbb{A}^n_\mathbb{Z} := spec\ \mathbb{Z}[x_1, ..., x_n]$. Show that $A^n_k = A^n_\mathbb{Z} \times_{spec\ \mathbb{Z}} spec\ k$.*

Definition 3.17. Let S be a scheme, $n \geq 0$ we can define $\mathbb{A}^n_S := \mathbb{A}^n_\mathbb{Z} \times_{spec\ \mathbb{Z}} S$.

Homework 3.18. *Let S be a scheme, $spec\ k \hookrightarrow S$ a point, show that the fiber of $\mathbb{A}^n_S$ over this point is $\mathbb{A}^n_k$, i.e. $\mathbb{A}^n_S \times_S spec\ k = \mathbb{A}^n_k$.*

Definition 3.19. $\phi : X \to Y$ a map of schemes is called *finite type* if there is an open cover V_α of Y such that each $\phi^{-1}(V_\alpha)$ admits a finite open cover by some U such that the restricted map $U \to V\alpha$ factors as a closed inclusion followed by some map from affine space over the given cover element $U \hookrightarrow \mathbb{A}^{n_{\alpha_i}}_{V_\alpha} \to V_\alpha$.
We call an S-scheme finite type, if its structure map is finite type.

4 Some properties of Schemes

4.1 Local rings

Let R be a ring. $R \to R_p/pRp$ for a prime p factorises as $R \to R_p \to R_p/R_p/pRp$. What does this mean topologically? Let $\phi : R \to S$ be a ring map. Let ϕ factor through $R \to R_p$, equivalently $\phi(f)$ is a unit for every element not in p. This happens iff ϕ factors through every $R \to R_f$ for all $f \notin p$. This means, that *spec* R_p is the intersection of every open subscheme containing the point p. Thus *spec* R_p contains all points corresponding to primes $q \subset p$.

To show some statement for a neighborhood of p, we first see the statement in *spec* R_p, and realise that you only used finitely many inverses. This allows us to lift our argument to the corresponding affine open.

Proposition 4.1. *Let X be a scheme and a point $x \in X$ and $x \in spec\ R \subset X$ is an affine chart around x. Then the inclusion $x = spec\ R_p/pR_p \to spec\ R_p \to X$ is independent of spec R. We call $\kappa(x) = R_p/pR_p$ the residue field of x and $\mathcal{O}_{X,x} = R_p$ is called the local ring at x.*

4.2 Nilpotents and Reducedness

Example 4.2. Compare *spec* $\mathbb{C}[x]/(x)$ with *spec* $\mathbb{C}[x]/(x^2)$. The first one is just a closed point of the affine line. The second ring is the same point of the affine line, but it contains *something more*.
If there is a polynomial f, then restricting onto these subschemes we either keep the constant term, or the first two terms, so we are seeing not just the value at the origin, but also the first derivative!

Example 4.3. *spec* $\mathbb{C}[x, y]/(xy, y^2)$. The first generator of the ideal is the union of two lines, the second one is the x axis with first order derivative information. Doing both leaves us with the x-axis with derivative information, plus some extra "data" at the origin.

We wish to understand for a scheme X, whene is there a proper closed subscheme Y such that its underlying topological space $sp(Y) = sp(X)$. Easiest to start with affine schemes. This translates to the question that when is there a nonzero ideal, such that R and R/I have the same set of prime ideals? The prime ideals of a quotient are (canonically identified with) the prime ideals of R containing I, so when is there an ideal which is $\subset \bigcap p$ for all primes? This means, in the sense discussed on the second week, that this means, that these are functions which vanish at every point, but are nonzero.

Proposition 4.4. *Let R be a ring, $f \in R$. $f \in p$ for all prime p is equivalent to f being nilpotent (i.e. there is an n such that $f^n = 0$). We define $nil(R) = \bigcap p = \sqrt{0}$ and call it the nilradical of R. If $nil(R) = 0$ we call R reduced.*

Homework 4.5. *Prove this!*

Definition 4.6. $R_{red} := R/nil(R)$ is called the *reduction* of R. *spec* R_{red} is the smallest subscheme of R which contains all the same points.

Proposition 4.7. *For a scheme X, we can define X_{red} by gluing up the reductions of an affine cover of X. X is said to be reduced if any of the following equivalent properties hold:*

1. *the only closed subscheme of X with all of the same points is X*
2. $X = X_{red}$
3. $\forall\ U \subset X$ *open* $\mathcal{O}_X(U)$ *has no nilpotent elements*
4. $\forall x \in X : \mathcal{O}_{X,x}$ *has no nilpotent elements*

4.3 Connectedness, Irreducibility, Integrality

Definition 4.8. A scheme X is *connected* iff $sp(X)$ is connected.

Definition 4.9. A topological space X is *irreducible* if whenever $X = X_1 \cup X_2$ where X_i are closed at least one of the $X_i = X$ (equivalently every nonempty open subset is dense). A scheme X is irreducible iff $sp(X)$ is.

Example 4.10. *spec* $\mathbb{C}[x,y]/(y^2 - x^2(x+1))$ and *spec* $\mathbb{C}[x,y]/(y^2 - x^2)$ a nodal cubic and the union of two lines.

Proposition 4.11. *Let R be a ring, then spec R is irreducible iff $nil(R)$ is prime.*

Proof. If $nil(R)$ is prime, then R_{red} is a domain, so *spec* R_{red} has a generic point, which implies that every open subset is dense, since they all contain the generic point.
If $nil(R)$ is not prime, then there are $f, g \in R \backslash nil(R)$ such that $fg \in nil(R)$. This means, that $sp(spec\ R/(fg)) = sp(spec\ R)$. On the other hand *spec* $R/(f)$ has fewer points than *spec* R, and similarly for g. □

Corollary 4.12. *A scheme X is irreducible iff there is an $x \in X$ such that $sp(\overline{x}) = sp(X)$.*

Proposition 4.13. *A non-empty scheme X is called integral if the following equivalent properties hold:*

1. *X is reduced and irreducible*
2. *$\forall\ U \subset X$ open $\mathcal{X}(U)$ is a domain*
3. *X has a point whose scheme-theoretic closure is X*

In particular spec R is integral iff R is an integral domain.

4.4 Noetherianity

Definition 4.14. A ring R is called Noetherian if every ascending chain of ideals is eventually constant. Geometrically this means that every descending chain of closed subschemes in *spec* R is eventually constant.

Definition 4.15. A scheme X is called locally Noetherian if it has an affine open cover with Noetherian charts (equivalently every affine open is Noetherian).

Proposition 4.16. *If X is a scheme, we call X Noetherian if it satisfies the following equivalent properties.*

1. *every descending chain of closed subschemes stabilises after finitely many steps*
2. *X is locally noetherian and quasi-compact**
3. *X admits a finite affine Noetherian open cover*

Proposition 4.17. *Any locally† closed subscheme a (locally) Noetherian scheme is (locally) Noetherian*

Proposition 4.18. *Any (locally) finite-type scheme over a (locally) Noetherian scheme is (locally) Noetherian.*

Definition 4.19. A topological space is called Noetherian iff every descending chain of closed subsets stabilises after a finite amount of steps.

*=compact -.-'
†open∩closed

Proposition 4.20. *If X is a Noetherian scheme, then $sp(X)$ is Noetherian.*

Example 4.21. $k[x_1, x_2, ...]/(x_1^2, ...)$ is a Noetherian topological space, but not a Noetherian scheme.

Homework 4.22. *Counterexample with reduced structure?*

Proposition 4.23. *Let X be a noetherian topological space. We can write $X = X_1 \cup ... \cup X_n$ as the union of irreducible closed subspaces which do not contain each other, and this decomposition is unique up to order. The X_i are called the irreducible components of X*

Proposition 4.24. *Let R be a ring the irreducible components of spec R are $sp(spec\ R/p)$ given by minimal primes p.*

5 Dimension theory and singular points

5.1 Krull dimension

If k is a field, then the "dimension" of *spec* k we expect to be 0 intuitively. A bit more generally we should want $dim \mathbb{A}_k^n = n$.
Question: Let X be a scheme, Y a proper closed subscheme. "$dimY \leq dimX$", when should/shouldn't there be equality in this formula?
We want equality for example when Y has all of the same points as X, i.e. $Y = X_{red}$. There could also be some irreducible components, whose dimension is the same as the whole space.
Finally if we restrict ourselves to the case when X is integral. This means in particular that $X \setminus Y$ is dense, so we should expect that $dim\ Y < dim\ X$. In particular if $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_k$ irreducible closed subschemes of X, we expect $dim\ X \geq \dim Y_k \geq dimY_{k-1} + 1 \geq ... \geq dimY_0 + k \geq k$.

Definition 5.1. Let X be a nonempty scheme, $dimX := max\{k \geq 0 : \exists Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_k$ integral closed subschemes$\}$.

Proposition 5.2. *Let X be a nonempty scheme, $dim\ X = max\{k \geq 0 : \exists p_0, ..., p_k \in X$ distinct points such that $p_i \in \overline{p_{i-1}}\ \forall 1 \leq i \leq k\}$*

Corollary 5.3. *$dim\ spec\ R = max\{k \geq 0 : \exists p_0 \subsetneq ... \subsetneq p_k$ prime ideals of $R\}$.*

Proposition 5.4. *X nonempty Noetherian, then $dim\ X = max\{dim\ Y : Y \subset X$ irreducible component$\}$.*

Definition 5.5. Let X be a scheme, $x \in X$. The local dimension of X at x is defined as $dim_x X := dim \mathcal{O}_{X,x}$.

Proposition 5.6. *Let X be a nonempty scheme, $dim\ X = max\{dim_x X | x \in X\}$ and if X is Neotherian or finite dimensional one can restrict this set to the closed points of X.*

Remark 5.7. There are schemes, which don't have closed points at all!

Homework 5.8. *Let $R = \mathbb{C}[x, x^{1/2}, x^{1/3}, ...]_{(x, x^{1/2}, x^{1/3}, ...)}$. Show that R is not Noetherian, and $dim\ R = 1$.*

Example 5.9 (Nagata, Vakil excercise 12.1.M). There exists an infinite dimensional Noetherian ring!

5.2 Codimension and Krull's height theorem

Consider a plane and a transverse line through it, and a line inside it. We wish to compare these two.

Definition 5.10. Let X be a scheme and Y an integral closed subscheme. We define $codim_X Y := max\{k \geq 0 : Y = Y_0 \subset Y_1 \subset ... \subset Y_k$ integral closed subschemes in $X\}$.
For an arbitrary Y $codim_X Y := min_{Z \subset Y \text{ integral}} codim_X Z$.
If R is a ring, p a prime $codim_{spec\ R}(specR/f) = max\{k \geq 0 : p_0 \subset ... \subset f_k = p \text{ primes}\}$ is called the height of the prime p.
In general for a scheme X and Y an integral closed subscheme, $\eta \in Y$ its generic point $codim_X Y = dim_\eta X = \mathcal{O}_{X,\eta}$.

Remark 5.11. $codim_X Y \neq dimX - dimY$

Example 5.12. A plane and a transverse line is X. The codimension of the plane is 0! There is no larger integral subscheme. The codimension of the line is 0 for the same reason. The codimension of the points of the line is 1, of the plane is 2. The codimension of lines inside the plane is 1.

Theorem 5.13 (Krull's height theorem or Hauptidealsatz)**.** *R Noetherian ring and $f_1, ..., f_c \in R$. Let Z be an irreducible component of $specR/(f_1, ..., f_c)$. Then $codim_{spec\ R} Z \leq c$. (The $c = 1$ case is called Krull's principal ideal theorem)*

Homework 5.14. *Use this theorem to show that if (R, m) is local Noetherian and $m = (f_1, ..., f_c)$, then prove that $dimR \leq c$.*

Homework 5.15. *Use the previous excercise to show that if k is a field, $n \geq 0$ then $\mathbb{A}^n_k = n$.*

5.3 Regularity

Definition 5.16. If X is a scheme, $x \in X$ a point and $k \geq 0$ an integer denote $R = \mathcal{O}_{X,x}$, and m is the maximal ideal of R. $SpecR/m^{k+1}$ is called the *k'th order infinitesmial neighborhood* of x in X.

m^{k+1} represents function germs vanishing to order at least $k+1$ at x. Restriction preserves from a "function" f its derivatives of order $\leq k$. The underlying topological space of this infinitesimal neighborhood is just x, we are just considering some extra nilpotents.
For $k = 1$ we record the value, and the first order derivatives, the maximal ideal is now m/m^2.

Definition 5.17. If X is a scheme, x is a point of X, m_x is the corresponding maximal ideal of its local ring, m_x/m_x^2 is called the Zariski cotangent module of X at x. Its elements are called cotangent vectors.

Clearly it is a module over $\mathcal{O}_{X,x}$, but this action factors through the residue field $\kappa(x)$, so we indeed get a vector space over the residue field at x.
Why is this the *co*tangent space? $m_x \to m_x/m_x^2$ sends a "function f" and produce "df".

Proposition 5.18. *$\phi : X \to Y$ a map of schemes, $x \in X$ a point the pullback map $\phi^\sharp : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ induces a map on the local rings, and this in turn induces a pullback of cotangent spaces $m_{\phi(x)}/m^2_{\phi(x)} \to m_x/m_x^2$.*

Homework 5.19. *Let (R, m) be a Noetherian local ring, κ be the residue field at m. Given $r_1, ..., f_r \in m$. Show that their images span m/m^2 iff they generate m and conclude that $dimR \leq dim_\kappa m/m^2$.*

Definition 5.20. If we have equality in the previous the ring is called regular.
A Noetherian ring is called regular if all of its local rings are. A locally Noetherian scheme is regular at a point, if the local ring is regular. A locally Noetherian scheme is regular if all of its points are.

Proposition 5.21. *k a field, $n \geq 0$ then $\mathbb{A}^n_k$ is regular.*

Homework 5.22. *Identify (with proof!) the (non-)singular points of the following schemes:*

- *spec*$\mathbb{C}[x, y]/(y^2 - x^2)$
- *spec*$\mathbb{C}[x, y]/(y^2 - x^2(x + 1))$
- *spec*$\mathbb{C}[x, y]/(y^2 - x^3)$
- *spec*$\mathbb{C}[x, y, z]/(xz, yz)$
- *spec*$\mathbb{Z}$

Homework 5.23. *Show that every point of spec$\mathbb{C}[x, y]/(y^2)$ is singular!*

Homework 5.24. *Show that any non-singular scheme is reduced.*

Homework 5.25. *Compute the Zariski cotangent space of $\mathbb{C}[x, x^{1/2}, x^{1/3}, ...]_{(x,x^{1/2},x^{1/3},...)}$.*

6 Geometry of Modules

6.1 Setup

Definition 6.1. R, S rings, and $\phi : R \to S$ a ring map, if $M \in R - Mod$, then $S \otimes_R M$ is the *pullback* of M.

Proposition 6.2. *If R is a ring, $M, N \in R - Mod$, $\{f_\alpha \in R\}$ is a generating set, we denote $M_\alpha = R_{f_\alpha} \otimes_R M$, and similarly for N, double indicies denote intersections as usual. Suppose we have module maps $\phi_\alpha : M_\alpha \to N_\alpha$ and suppose that the maps induced on $M_{\alpha\beta} \to N_{\alpha\beta}$ by ϕ_α and ϕ_β agree. Then there is a unique $\phi : M \to R$, which restricts to ϕ_α for any given α under $\otimes_R R_{f_\alpha}$.*

Corollary 6.3. *Modules, elements of modules, exactness, etc. are all affine locally determined.*

Proposition 6.4. *If R is a ring, $M \in R - Mod$ and uhh $R \to \kappa(p)$ is a point of spec R then $\kappa(p) \otimes_R M$ is a $\kappa(p)$ vector space.*

Example 6.5. $R = \mathbb{C}[x, y]$ and $M = R \oplus R, N = (x, y) = \frac{Re_1 \oplus Re_2}{(ye_1 - xe_2)}$. For $p \subset R$ prime, $\kappa(p) \otimes_R M = \kappa(p)^2$, on the other hand $\kappa(p) \otimes_R N$ is $\kappa(p)^2$ if $p = (x, y)$ and $\kappa(p)$ otherwise.

So we want to realize M as a scheme over M with fibers, which are affine spaces, and also to figure out what happens over non-affine schemes. This leads us to something called linear fiber spaces, and over arbitrary spaces to the notion of quasicoherent sheaves.

6.2 Linear fiber spaces in topology

If T is a topological space, a rank n vector bundle is a motivating example, but we also want to be able to have fibers of different rank. $\mathbb{R} \times T \to T$, the trivial vector bundle of rank 1. It has a zero section $T \to \mathbb{R} \times T$, and also a well-defined "one section" and a fiberwise addition map $\mathbb{R} \times \mathbb{R} \times T = (\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \to \mathbb{R} \times T$, all of which respect the projection to T. Similarly there is a fiberwise multiplication and all of these maps satisfy the axioms for a ring fiberwise. Now we define modules over this ring.

Definition 6.6. If T is a topological space, a *linear fiber space* over T is a topological space X and a map $\pi : X \to T$, so that

- a map $z : T \to X$ is given, which represents the zero section
- there is a map $+ : X \times_T X \to X$ a fiberwise addition map
- a map $\cdot : \mathbb{R} \times T \times_T \to T$ a fiberwise map representing scalar multiplication

satisfying the module axioms over $\mathbb{R} \times T$, and all of the above commutes with the natural projections to T. A map of linear fiber spaces over T is a continous map $X \to Y$, commuting with the projections and the operations defined above.

6.3 Linear fiber spaces over schemes

If S is a scheme, $\mathbb{A}^1_S = S \times_{spec\mathbb{Z}} (Spec\mathbb{Z}[t])$. Over any affine open $specR \subset S$ this is just $specR[t]$, and the fiber over a point is $spec\kappa(p) = \mathbb{A}^1_k$.

Proposition 6.7. *$\mathbb{A}^1_S$ is a fiberwise ring over S, i.e.*

- *there is a zero section $z : S \to \mathbb{A}^1_S$, locally this corresponds to a map $R[t] \to R$, where $t \mapsto 0$*
- *$u : S \to \mathbb{A}^1_S$ the one section, locally $t \mapsto 1$*
- *$+ : \mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S$, as before the domain is just $\mathbb{A}^2_S$, the corresponding local map should be $t \mapsto t_1 + t_2$*
- *$\cdot : \mathbb{A}^2_S \to \mathbb{A}^1_S$, locally $t \mapsto t_1 t_2$*

satisfying the fiberwise ring axioms.

Homework 6.8. *Write out the whole definition, with all of the diagrams.*

Definition 6.9. Let S be a scheme, a linear fiber space over S is an S-scheme X together with maps of S-schemes:

- $z : S \to X$ zero section
- $+ : X \times_S X \to X$ addition
- $\mathbb{A}^1_S \times_S X \to X$ scalar multiplication

satisfying the axioms of a fiberwise module.
A map of linear fiber spaces over S is a map of schemes over S, wich commutes with all of the operations.

Homework 6.10. *Let k be a field, $n \geq 0$. Show that $\mathbb{A}^n_S$ is a linear fiber space over spec k.*

- $[k[t_1, ..., t_n]] \to k$ *where* $t_i \mapsto 0$
- $k[x_1, ..., x_n] \to k[x'_1, ..., x'_n, x_1, .., x_n]$ *where* $x_i \mapsto x_i + x'_i$
- $k[x_1, ..., x_n] \to k[t, x_1, ..., x_n]$ *where* $x_i \mapsto tx_i$

6.4 Modules as linear fiber spaces

If k is a field, V is a k-module, how do we get the affine space geometrically realizing V? If $v_1, ..., v_n$ is a basis, we can think about $k[v_1, .., v_n]$, but this depends on the choice of basis :(

Definition 6.11. Let R be a ring, M a module. The symmetric algebra of M over R is

$$Sym(M) = \bigoplus_0^\infty M^{\otimes \ell}/(a \otimes b - b \otimes a).$$

Example 6.12. If $M = R^{\oplus n}$, then $Sym(M) = R[e_1, ..., e_n]$.

Example 6.13. $R = \mathbb{C}[x, y], M = (x, y) = \frac{Re_1 \oplus Re_2}{(ye_1 - xe_2)}$, then $Sym(M) = \mathbb{C}[x, y, e_1, e_2]/(ye_1 - xe_2)$.

A (sort of) answer, is going to be $specSym(V)$.

7 Quasicoherent Sheaves

7.0 Wrap-up

Definition 7.1. R is a ring, $M \in R - mod$. The spectrum of M is defined to be $specSymM$ with an LFS structure over $specR$ given by R-algebra maps $R \leftarrow Sym(M)$ $m \mapsto 0$, addition is defined as $m \otimes 1 + 1 \otimes m \leftarrow m$ and $m \mapsto tm$ is the induced scalar multiplication.

Homework 7.2. *If $R = k$ and $M = Rx_1 \oplus ... \oplus Rx_n$ show that these agree with the LFS structure on $\mathbb{A}^n_k$ from last week.*

Homework 7.3. *Verify that these make spec M a LFS in general.*

Homework 7.4. *If R is a ring $\phi : M \to N$ an $R-mod$ map, show that ϕ induces a map Spec $N \to$ Spec M contravariantly functorially. Bonus: Show that every map of linear fiber spaces from Spec $N \to$ Spec M comes about this way.*

How do we retrieve M from this construction?

Definition 7.5. If S is any scheme and X is a linear fiber space over S a linear form on X is a map of linear fiber spaces over S, $X \to \mathbb{A}^1_S$. $L(X) :=$ {linear forms on X}. If $\Phi : X \to Y$ is a map of linear fiber spaces over S, then we can pull back linear forms as one would expect, i.e. there is $\Phi^* : L(Y) \to L(X)$ under which $\lambda \mapsto \lambda \circ \Phi$.

Proposition 7.6. *If S is a scheme, X is a linear fiber space over S, then there is a map $L(X) \times L(X) \to L(X)$ summing two forms in each fiber ($X \xrightarrow{\lambda \times_S \mu} \mathbb{A}^2_S \xrightarrow{+} \mathbb{A}^1_S$), this makes $L(X)$ an abelian group. If $S =$ spec R, then we can also define a scalar multiplication $R \times L(X) \to L(X)$ using the composition ($X \xrightarrow{\lambda} \mathbb{A}^1_S \xrightarrow{\cdot r} \mathbb{A}^1_S$). This makes $L(X)$ an R-module, the pullback map is an R-module map in this setting.*

Homework 7.7. *If R is a ring, M an R-module, then show that $M = L(spec\ M)$. Given another R-module N and tsutsutsutsu let's see here, and I have a map of R-modules $M \xrightarrow{\phi} N$ show that the induced map $L(spec\ M) \to L(spec\ N)$ is ϕ (under the previous natural identifications).*

Definition 7.8. If X is a scheme, $x \in X$ is a point, $m \subset \mathcal{O}_{X,x}$ the maximal ideal. The *Zariski tangent space* to X at x is $Spec(m/m^2)$.

7.1 Shaves of Modules

Definition 7.9. $(X, \mathcal{O}_X)$ is a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups F such that for every $U \subset X$ open $F(U)$ is an $\mathbb{O}(U)$ module such that it is compatible with the restrictions: $\rho(r \cdot m) = \rho(r) \cdot \rho(m)$.

Proposition 7.10. *If R is a ring, M is an R-module, then there is a corresponding sheaf of $\mathbb{O}_{spec\ R}$ modules specified by $\tilde{M}(specR_f) := R_f \otimes_R M$.*

Definition 7.11. If X is a scheme, F is a sheaf of $\mathcal{O}_X$ modules then F is *quasicoherent* if for every affine open *spec* $R \subset X$ we have that $F|_{spec\ R} = \tilde{M}$ for some R-module M. Equivalently we can check it on some affine cover.
If X is locally Noetherian, F is called coherent if all of the M's in the definition of a quasicoherent sheaf are finitely generated/presented.

We define (most) module things affine-locally (direct sum, tensors kernels, cokernels, exactness).

Remark 7.12. *Hom* behaves wierdly! If F is coherent, then $Hom_{\mathcal{O}_X}(F, -)$ is fine though. In particular the dual sheaf of a coherent sheaf is coherent.

Proposition 7.13. *$X \xrightarrow{\phi} Y$ a map of schemes, F is a quasicoherent sheaf of modules over X, and G similarly over Y, then $\phi_* F(U) := F(\phi^{-1})(U)$ is the pushforward sheaf (of F along ϕ) is quasicoherent if X is Noetherian.*
Pullback is a bit more nice, $\phi^* G := \phi^{-1}(G) \otimes_{\phi^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Affine locally if Spec S is an open subset of X contained in some $\phi^{-1}(Spec\ R)$, then $G|_{Spec\ R} = \tilde{M}$ by the quasicoherent assumption. Then $\phi^* G(spec\ S)) = S \otimes_R M$ and the pullback is always quasi coherent.
If X, Y are locally Noetherian and G is coherent, then the pullback is coherent as well.

Remark 7.14. Locally given a point $(R, m) \to (S, n)$ the stalks with be what we expect, namely $Mm \to S \otimes_R Mm$.

Given a map $R \to S$ of rings and $M \in S - mod$, then the pushforward of M along this map is the same module but considered as an R module, induced by the map.
If X is Noetherian, then $Hom_{\mathcal{O}_X}(\phi^* G, F) = Hom_{\mathcal{O}_Y}(G, \phi_* F)$.

Proposition 7.15. *Pullback is right exact, i.e. given $0 \to A \to B \to C \to 0$ a short exact sequence of quasicoherent sheaves, then the corresponding sequence of pullback $\phi^* A \to \phi^* B \to \phi^* C \to 0$ is exact.*

7.2 Relative spectra

Definition 7.16. If X is a scheme and A is a quasicoherent sheaf of $\mathcal{O}_X$ algebras. *spec* $A \to X$ is called the relative spectrum of A, and is given locally for an affine open *spec* $R \subset X$, $A(spec\ R) = a$, the algebra structure is given by $R \to a$, by *spec* $(a) \to$ *spec* R.

Definition 7.17. A map of schemes $Y \to X$ is called affine if either of the following equivalent things is true:

1. $Y \to X$ is the structure morphism of *spec* A for some quasicoherent sheaf of algebras
2. the inverse image of every affine open is affine

Definition 7.18. If X a scheme, I is a quasioherent sheaf of ideals, or just ideal sheaf if it is a quasicoherent sheaf of O_X modules with an inclusion $I \to O_X$.
Over affine opens this gives back just ideals of the affine ring.
We can talk about the closed subscheme of X cut out by I, it is defined as $spec(O_X/I)$.

Definition 7.19. This gives a bijection between ideal sheafes and closed inclusions.

8 More on Quasicoherent Sheaves

Recall the exercise of constructing the projective line by gluing together two affine lines $Spec\mathbb{C}[x], Spec\mathbb{C}[y]$ along $\mathbb{C}[x]_x \to \mathbb{C}[y]_y$ by mapping $x \mapsto y^{-1}$. The point given by $x = 2$ on the other affine chart looks like $y = 1/2$, let I be the ideal sheaf cutting it out. $I|_{spec\mathbb{C}[x]} = (x-2)$ as a module, clearly on the other chart we get that $I|_{spec\mathbb{C}[y]} = (y-1/2)$, and these clearly get identified under our gluing map. Let us try and compute the global sections of this sheaf. These will look like (f, g) such that $f \in (x-2) \subset \mathbb{C}[x]$ and $g \in (y-1/2) \subset \mathbb{C}[y]$, and under the gluing map $f/1 \mapsto g/1$. If $f(x) = a(x-2)(x-r_1)...(x-r_n)$ for some $a, r_i \in \mathbb{C}$, then $g(y) = a(1/y-2)(1/y-r_1)...(1/y-r_n) = a/y^{n+1}(1-2y)(1-r_1y)...(1-r_ny)$. This implies that $a = 0$, otherwise we cannot clear denominators.

Definition 8.1. If X is a scheme, Y_1, Y_2 are closed subschemes with I_i their respective ideal sheaves, the union $Y_1 \cup Y_2 := spec\ \mathcal{O}_X/(I_1 \cap I_2)$.

If R is a ring, $I, J \triangleleft R$ then $\frac{I \cap J}{IJ} = Tor_1^R(R/I, R/J)$.

Homework 8.2. *Let q be the origin of $spec\mathbb{C}[y]$. Describe the ideal sheaf I cutting out q in $\mathbb{P}^1_{\mathbb{C}}$, i.e. compute the restrictions to the two affine charts, and the global sections. Next do the same thing for the sheaf of $p \cup q$ where p is the point $x = 2$ from the previous example.*

8.1 Quasicoherent Sheaves as Linear Fiber Spaces

Definition 8.3. If X is a scheme, F is a quasicoherent sheaf over X, then the symmetric algebra sheaf of F over X is $Sym(F) := (\bigoplus_0^\infty F^{\otimes l})/(a \otimes b - b \otimes a)$ where a, b are sections of F ($F^{\otimes 0} = \mathcal{O}_X$, multiplication is tensor product).

Remark 8.4. Locally this gives the construction of symmetric algebra we saw before.

Definition 8.5. Let X be a scheme, F a quasicoherent sheaf on X, then the relative spectrum of F is defined as *spec* $Sym(F)$, endowed with the structure of a linear fiber space using the same operations from the affine case affine locally.

Remark 8.6. The relative spectrum construction gives a contravariant functor.

Definition 8.7. If X is a scheme, V a linear fiber space, the sheaf of linear forms on V to be the sheaf (of $\mathcal{O}_X$ modules) on X (!!!) given by $L(U) := L(V|_U)$ (remember, that $V|_U = V \times_X U \to \mathbb{A}^1_U$). Given a map $\Phi : V \to W$ of linear fiber spaces, there is an induced pullback map $\Phi^* : L(W) \to L(V)$ given by precomposition.

Proposition 8.8. *Given a scheme X, and a quasicoherent sheaf F over X, then $L(spec\ F)$ is naturally isomorphic to F.*
The $spec(-), L(-)$ functors give an anti-equivalence of abelian categories between quasicoherent sheaves over X anda full abelian subcategory of linear fiber spaces over X.*

Proposition 8.9. *$\phi : X' \to X$ is a map of schemes, F a quasicoherent sheaf over X, then there is a natural isomorphism of linear fiber spaces between $X' \times_X spec\ F$ and $spec(\phi^* F)$.*

Remark 8.10. Pullback is not exact! A short exact sequence of quasicoherent sheaves $0 \to A \to B \to C \to 0$ gives an exact sequence $0 \to specC \to specB \to specA \to 0$, but for $x \in X$ we only have that $0 \to specC|_x \to specB|_x \to specA|_x$ is exact.

Example 8.11. $0 \to (x, y) \to \mathbb{C}[x, y] \to \mathbb{C}[x, y]/(x, y) \to 0$. in the middle we see the trivial line bundle over the plane. On the right side we get the trivial line bundle over the origin. On the left side we get a linear fiber space which has rank two over the origin, and a line bundle on the complement of the origin. This shows us the faliure of exactness at the level of fibers.

Definition 8.12. If X is a scheme and V is a linear fiber space over X and $n \in \mathbb{N}$. We call V a vector bundle of rank n if $\forall x \in X$ there is $x \in U \subset X$ an open neighborhood such that $V|_U = \mathbb{A}^n_U$ as linear fiber spaces.

Proposition 8.13. *If $V = specF$ for F a locally free sheaf of rank n (there is an open neighbrohood around every point such that $F|_U = \mathcal{O}_X^{\oplus n}$) on X, then V is a vector bundle, and the converse also holds.*

Homework 8.14. *If X is a locally noetherian scheme, show that the coherent sheves on X are identified under this correspondance with finite rank vector bundles..?*

8.2 Nakayama's lemma

Theorem 8.15 (Nakayama's lemma)**.** *Let R be a ring, M a finitely generated R-module and I an ideal contained in all maximal ideals. Then*

- *if $IM = M$, then $M = 0$*
- *if the images of $m_1, ..., m_n \in M$ in M/IM generate M/IM, then they generate M*

What is the geometric meaning of this theorem? If I is contained in all maximal ideals, then $specR/I$ contains all closed points of $specR$. In particular since $IM = M$ is equivalent to saying $M/IM = 0$, this means that $\tilde{M}|_{specR/I} = 0$.

Corollary 8.16. *If X is a locally Noetherian scheme and $x \in X$ is a point, F a coherent sheaf on X, then $specF|_x = 0$ implies that there is an open neighborhood U of x such that $specF|_U = 0$.*

*every morphism is in the image

Proof. Apply Nakayama's lemma in $\mathcal{O}_{X,x}$ with $I = m$, then lift from local ring to open neighborhood by starting with $specR \ni x$, and using the standard argument of lifting local information to a neighborhood by noetherianity.
For the second statement, it is equivalent to having $\mathbb{R}^{\oplus n} \to M \to M/I$ surjective. This implies, that $R^{\oplus n} \to M$ is surjective. □

Corollary 8.17 (Upper Semicontinuity of Fiber Dimension)**.** *If X is a locally noetherian scheme and $x \in X$ is a point, F a coherent sheaf over X, and define $n := \dim specF|_x$, then there exists an open subscheme $x \in U$ such that $specF|_U$ embeds as a closed sub-linear fiber space of $\mathbb{A}^n_U$.*

Homework 8.18. *If X is a locally noetherian integral scheme, F a coherent sheaf on X with $n :=$ dim of fiber of $specF$ over the generic point of X. Use Nakayama's lemma to show that there is an open dense subscheme U such that $specF|_U$ is a trivial rank n vector bundle.*

Homework 8.19. *M is a complex matrix with entries depending algebraically on some parameter t. Let r be the rank of M considered as a matrix over $\mathbb{C}(t)$. Use the previous exercise to show that for all but finitely many values of $t \in \mathbb{C}$, M is a rank r matrix (over $\mathbb{C}$). (hint: M is a module map $\mathbb{C}[t]^p \to \mathbb{C}[t]^q$, and take its cokernel)*

9 Calculus on Schemes

Example 9.1. $V(y - x^2) \hookrightarrow \mathbb{A}^2$. It is mostly clear what the tangent bundle of the affine plane should be, corresponding to $\mathbb{C}[x, y, dx, dy] \leftarrow \mathbb{C}[x, y]$, pushing forward along the map to the algebra of the parabola we get the tangent bundle should be living in $\mathbb{C}[x, y]/(y - x^2)[dx, dy]$, and from differential geometry we know that it should be obtained by factoring with $(dy - xdx)$.

9.1 Motivation

If X is a smooth manifold, a smooth vector field is a smooth section of the tangent bundle. If V is a smooth vector field on X, and $f : X \to \mathbb{R}$ is a smooth function, we get a new function $Vf : X \to \mathbb{R}$ by taking the derivative of f in the direction of V at each point. Since we can do this locally, V gives us an $\mathbb{R}$ linear map of sheaves $\mathcal{O}_X \to \mathcal{O}_X$, which obeys the Leibniz rule.
In fact this correspondance is a bijection, the set of all smooth vector fields is identified with the set of $\mathbb{R}$-linear sheaf maps with the Leibniz rule.

9.2 Definitions

Definition 9.2. Given a map of rings $\phi : R \to S$, $M \in S - mod$. An R-linear derivation of S into M is a map $d : S \to M$ of abelian groups obeying the Leibniz rule $d(fg) = fdg + gdf$, and $\forall r \in R : d(\phi(r)) = 0$ (or equivalently d is a morphism of R-modules).
$Der_R(S, M)$ denotes the set of all such derivations, which we consider as an S-module by pointwise multiplication.

Definition 9.3. $\phi : R \to S$ a ring map, the module of Kähler differentials of S over R is an S-module denoted

$$\Omega_{S/R} := \frac{\oplus_{f\in S} Sdf}{(d(fg) - fdg - gdf | f, g \in S) + (d(\phi(a)f + \phi(b)g) - \phi(a)df - \phi(b)dg | a, b \in R, f, g \in S)}$$

The map $d : S \to \Omega_{S/R}$ sending $f \mapsto df$ is called the universal derivation of S over R.

Remark 9.4. The same definitions can be given with sheaves of rings and modules.

Proposition 9.5. *The universal derivation is a derivation. Moreover if M is an S-module, $d_M : S \to M$ is an R-linear derivation, then there is a unique map of R-modules $\psi : \Omega_{S/R} \to M$ such that $d_M = \psi \circ d$. (Equivalently there is a natural isomorphism $Der_R(S, M) \simeq Hom_S(\Omega_{S/R}, M)$.)*

Proposition 9.6. *$R \to S$ a ring map, $U \subset S$ a multiplicatively closed subset. Then $\Omega_{U^{-1}S/R} = U^{-1}S \otimes_S \Omega_{S/R}$.*

Proof. Use the quotient rule algebraically. □

Corollary 9.7. *$\Omega_{S_f/R} = S_f \otimes_S \Omega_{S/R}$ for any $f \in S$.*
$\Omega_{S_p/R} = S_p \otimes_S \Omega_{S/R}$ for any prime ideal $p \lhd S$.

Definition 9.8. If there is $\Phi : X \to Y$ a map of schemes, then there is a unique quasi-coherent sheaf of $\mathcal{O}_X$ modules denoted $\Omega_{X/Y}$ or Ω_Φ on X such that $\forall spec\, R \subset Y, spec\, S \subset \Phi^{-1}(spec\, R)$ opens $\Omega_{X/Y}|_{spec\ S} = \Omega_{S/R}$, and the gluing maps are compatible with this identification. We call this sheaf $\Omega_{X/Y}$ the sheaf of (relative) Kähler differentals of X over Y. The (relative) tangent scheme of X over Y is $T_{X/Y} := spec\, \Omega_{X/Y}$ (sometimes also denoted T_Φ).

Remark 9.9. If we understand to be working in the context of schemes over Y (e.g. $Y = spec\ \mathbb{C}$, and the map is the structure map) we drop Y from the notation and just talk about Ω_X and T_x, etc.

Proposition 9.10. *Let $Phi : X \to Y$ be a map of schemes. Sections of $T_{X/Y}$ over X are canonically identified with $Der_{\Phi^{-1}\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$.*

Proof of the affine case. Let $Y = spec\ R, X = spec\ S$. The set of sections of $\pi : T_{X/Y} \to X$ are canonically identified with maps $Sym\Omega_{S/R} \to S$ such that the composition $S \to Sym\Omega_{S/R} \to S$ is the identity. It is enough to see the images of the degree 1 elements, since they generate the symmetric algebra. So the set of such maps gets identified with S-module maps $\Omega_{S/R} \to S$. These maps are by definition make up $Der_R(S, S)$. □

Proposition 9.11. *A map of rings $R \to S$ and another map of rings $R \to R'$. Define $S' := S \otimes_R R'$, then we have $\Omega_{S'/R'} = R' \otimes \Omega_{S/R}$.*
For schemes this means, that given a fiber product square $X \times_Y Y'$ we have that $T_{X'/Y'} = T_{X/Y} \times_Y Y'$ as linear fiber spaces over X'.

9.3 Computing Kähler differentials

Proposition 9.12. *If R is a ring, $n \geq 0$ and $S := R[x_1, .., x_n]$, the map is the natural inclusion. Then $\Omega_{S/R} = \bigoplus_1^n Sdx_i$.*
So if Y is a scheme, $X = \mathbb{A}^n_Y$, then $T_{X/Y} = \mathbb{A}^n_X = \mathbb{A}^n_{\mathbb{A}^n_Y} = \mathbb{A}^{2n}_Y$.

Example 9.13. If $Y = spec\mathbb{C}$, then the tangent bundle of C^n is $\mathbb{C}^n \times \mathbb{C}^n$.

Proposition 9.14. *Given ring maps $R \to S \to T$, we can compose derivations $S \to T$ with the universal derivation $T \to \Omega_{T/R}$, this composition induces an S-module map $\Omega_{S/R} \to \Omega_{T/R}$, which gives us a T-module map $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$.*
Hence $\Phi : X \to Y$ is a map of Z-schemes, there is a natural map of quasicoherent schemes over X $\Phi^*\Omega_{Y/Z} \to \Omega_{X/Z}$, giving us $D_Z\Phi : T_{X/Z} \to \Phi^*T_{Y/Z}$, which we call the differential of Φ over Z.

Example 9.15. $X = \mathbb{A}^2_{\mathbb{C}}$, $Y = \mathbb{A}^1_{\mathbb{C}}$ are two $\mathbb{C}$ schemes. $\Phi : X \to Y$ is given by $t \mapsto xy$. We have some maps $\mathbb{C} \to \mathbb{C}[t] \to \mathbb{C}[x, y]$, what is the map $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$. Firstly $\Omega_{S/R} = \mathbb{C}[t]dt$, so tensoring over R gives us $\mathbb{C}[x, y]dt$. Secondly $\Omega_{T/R} = \mathbb{C}[x, y]dx \oplus \mathbb{C}[x, y]dy$. The map Φ^* will send $dt \mapsto ydx + xdy$.

Proposition 9.16 (Relative (co)tangent sequence)**.** *Let $R \to S \to T$ be ring maps, $d : T \to \Omega_{T/S}$ is R-linear. This gives us a map $\Omega_{T/R} \to \Omega_{T/S}$ as R-modules. The following sequence is exact:*

$$T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0$$

This means in particular, that given a map of Z-schemes $\Phi : X \to Y$, we get an exact sequence of quasicoherent sheaves:

$$\Phi^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

$$0 \to T_{X/Y} \to T_{X/Z} \to \Phi^*T_{Y/Z}$$

i.e. $T_{X/Y} = kerD_Z\Phi$*.*

Remark 9.17. One can continue this exact sequence to get the André-Quillen homology.

Example 9.18. From the previous example we get that $\Omega_{\mathbb{C}[x,y]/\mathbb{C}[t]} = \frac{\mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy}{(ydx+xdy)}$. Let $p = V(t-1)$ and i its inclusion map. The fiber over p is $\mathbb{C}[x, y]/(xy - 1)$, we will compute the tangent bundle of this fiber. $i^*\Omega_{X/Y} = \Omega_{\Phi^{-1}(p)/Y} = \Omega_{\mathbb{C}[x,y]/(xy-1)/\mathbb{C}}$ by base change. Thus we get $\frac{Udx \oplus Udy}{ydx+xdy}$ where $U = \mathbb{C}[x, y]/(xy - 1)$, and this tangent space is isomorphic to $\frac{Udx \oplus Udy}{dy + \frac{y}{x}dx} = \frac{Udx \oplus Udy}{(dy + \frac{1}{x^2}dx)}$

Homework 9.19. *Let o be the origin in $\mathbb{A}^1_{\mathbb{C}}$. Compute $T_{\Phi^{-1}(o)/o}$, what are the dimensions of its fibers over the closed points of $\Phi^{-1}(o) = V(x, y)$.*

10 Calculus on Schemes 2: Electric Boogaloo

10.1 Infinitesimal Neighborhoods and the Zariski Normal Scheme

Recall, that given $x \subset X$, the local ring $R = \mathcal{O}_{x,X}$ with maximal ideal m, we had $specR/m^{k+1}$, and thought of it as the k'th order infinitesimal neighborhood of x in X.

Definition 10.1. If Y is a scheme, $X \subset Y$ is a closed subscheme, it has a corresponding ideal sheaf I. We define for any $k \geq 0$ the k'th order infinitesimal neighborhood of X in Y as $V(I^{k+1}) \subset Y$, i.e. $spec(\mathcal{O}_Y/I^{k+1})$.

Example 10.2. $Y = \mathcal{A}^2_{\mathbb{C}} = spec\mathbb{C}[x, y]$ and $X = V(y) = spec\mathbb{C}[x, y]/(y)$, the kth order infinitesimal neighborhood is $spec\mathbb{C}[x, y]/(y^{k+1})$.

Definition 10.3. $X \subset Y$ is a closed subscheme of a scheme Y with $I = I(X)$ is the corresponding ideal sheaf, then I/I^2 (viewed as a quasicoherent sheaf on X) is called the conormal sheaf of X in Y. $N_{X/Y} := Spec(I/I^2)$ is called the Zariski normal scheme (a linear fiber space over X).

10.2 Relative (Co)normal Sequence

Recall that given a map Φ of Z-schemes $X \to Y$, we had the exact sequence $0 \to T_{X/Y} \to T_{X/Z} \to \Phi^* T_{Y/Z}$, or in the other direction on the level of sheaves.

Proposition 10.4. *Y is a scheme, $X \subset Y$ is a closed subscheme. Then $T_{X/Y} = 0$ (i.e. $\Omega_{X/Y} = 0$).*

Proposition 10.5 (relative (co)normal sequence)**.** *If $R \to S$ is a ring map, $I \triangleleft S$ is an ideal with $T = S/I$. The map $I \to \Omega_{S/R} \to T \otimes_S \Omega_{S/R}$ factors through the map $I/I^2 \to T \otimes_S \Omega_{S/R}$ composed with the natural projection $I \to I/I^2$.*
The sequence $I/I^2 \to T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to 0$ is exact. Correspondingly on the level oflinear fiber spaces over X we have that the sequence $0 \to T_{X/Z} \to i^* T_{Y/Z} \to N_{X/Y}$ is exact.

Corollary 10.6. *If R is a ring, $n, r \geq 0$ integers, and $S = R[x_1, ..., x_n]$, $f_1, .., f_r \in S$ with $T := S/(f_1, .., f_r)$. Then $\Omega_{T/R} = \frac{\bigoplus_1^n T dx_i}{(df_1, ..., df_r)}$ where $df = \sum \partial_i f dx_i$. Implying that $T_{specT/specR} = spec \frac{R[x_1, ..., x_n, dx_1, ..., dx_n]}{(f_1, ..., f_n, df_1, ..., df_n)}$.*

Example 10.7. $Z = spec\mathbb{C}, Y = spec\mathbb{C}[x, y], X = V(xy - 1)$, we get that $T_{X/Z} = \frac{\mathbb{C}[x,y,dx,dy]}{(xy-1, xdy+ydx)}$.

Example 10.8. $Z = spec\mathbb{C}, Y = spec\mathbb{C}[x, y, z], X = V(xz, yz)$, now $T_X = spec \frac{\mathbb{C}[x,y,z,dx,dy,dz]}{(xz, yz, zdx+xdz, zdy+ydz)}$.

10.3 Smoothness

Proposition 10.9. *$X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} Z$ such that $i, j \circ i$ are closed inclusions of schemes. Then we have an exact sequence $0 \to i^* T_{Y/Z} \to N_{X/Y} \to N_{X/Z}$ of linear fiber spaces over X.*

Proof. See Stacks Project $065V$. □

Corollary 10.10. *If k is a field, X is a k-scheme and $x \in X$ is a k-point (i.e. $\kappa(x) = k$), then $T_X|_x$ is the Zariksi tangent space to X at x.*

Proof. $speck = x \hookrightarrow X \to speck$ the closed inclusion of the point into X and the structure map, so we can apply the prevous proposition. Since $N_{speck/speck} = 0$, we get an isomorphism $0 \to i^* T_X \to N_{x/X} \to 0$. □

Definition 10.11. If k is a field, X is a k-scheme, $n \geq 0$ an integer, we say that X is smooth of dimension n (over k) if

- X is locally finite type
- X has pure dimension n (i.e. every irreducible component has dimension n)
- T_X is a vector bundle of rank n

Proposition 10.12. *If $i : X \hookrightarrow Y$ is a closed inclusion of smooth k-schemes, then $0 \to T_X \to i^* T_Y \to N_{X/Y} \to 0$ is exact.*

Proposition 10.13. *If k is a field, then every smooth k-scheme is regular.*
If k is a perfect field then every regular locally finite type k-scheme is smooth.*

Proposition 10.14. *$K|k$ a field extension, there is a corresponding map spec $K \to$ spec k. $T_{spec\ K/spec\ k}$ may be nonzero.*

*$char(k) = 0$, or the Frobenius map is an automorphism

Theorem 10.15 (generic smoothness)**.** *If k is a perfect field and X is an integral finite type k-scheme, then there is a dense open $U \subset X$ such that U is smooth (of dimension $dim_k X$) over k.*

Theorem 10.16 (Jacobian Criterion - special case)**.** *If k is a field and $n, r \geq 0$ integers, $X = spec\ R \subset \mathbb{A}_k^n$ of pure dimension d with $R = \frac{k[x_1,...,x_n]}{(f_1,...,f_r)}$ taking the matrix $M = [\partial_i f_j]_{i,j=1,1}^{n,r} \in M_{n\times r}(R)$ we define $J_X :=$ $((n-d)\times(n-d)$ minors of $M)$. Then J_X is independent of $X \hookrightarrow \mathbb{A}_k^n$ and $V(J_X) \subset X$ is previsely the locus where X is not smooth.*