Spin and Spinc structures

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1 First Lecture

1.1 Cech cohomology

Get information about the space from the combinatorics of some open cover of the space.

Definition 1.1. Let M be an n-manifold, $U = \{U_\alpha\}$ an open cover (indexed over a well ordered set A). The Chech n-cochains are real numbers $\phi_{\alpha_0 \ldots \alpha_n}$ (with $\alpha_0 < \cdots < \alpha_n$) if $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset$. $C^n(U, \mathbb{R}) = \prod_{\alpha_0 < \cdots < \alpha_n} \mathbb{R}$ if the intersection is non-empty once again. From now on we denote $U_{\alpha_0 \ldots \alpha_n} := \cap U_{\alpha_i}$. We also denote $\phi_{\beta_0 \ldots \beta_n}$ to be $sgn(\sigma)\phi_{\beta\sigma 0, \ldots \beta\sigma n}$ where σ orders the indicies, or 0, if an index occurs twice. The boundary map $(\delta\phi)_{\alpha_0 \ldots \alpha_{n+1}} := \sum (-1)^j \phi_{\alpha_0 \ldots hat\alpha_j \ldots \alpha_{n+1}}$. Its clear that $\delta^2 = 0$. From this we get the Chech cohomology groups $H^n(U; \mathbb{R})$.

In fact $C^n(U, \mathbb{R})$ can be written as a product of the sets of (locally) constant functions from each $U_{\alpha_0 \ldots \alpha_n}$ to $\mathbb{R}$.

Definition 1.2. Let X be a topological space. O_x will be the category of open sets of X, where morphisms are inclusion. A *presheaf* is a contravariant functor $F : O_X \to C$.

Definition 1.3. A *sheaf* is a presheaf with

locality if $s, t \in F(U)$ are two sections on U, and $\{U_i\}$ is an open cover of U s.t. $s|U_i = t|_{U_i}$ for all i, then $s = t$.

gluing If $\exists\{s_i\}$ on some open cover $\{U_i\}$ of U s.t. on all pairwise intersections they agree, i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then $\exists s \in F(U)$ s.t. $s|_{U_i} = s_i$.

Example 1.4. Locally constant functions into any group G. We could also take $C^0(G)$ of continous maps to a topological group G. Our most important example will be $C^\infty(G)$ of smooth functions to a Lie group G.

We need a small modification on the definition of Chech cohomology. The maps need to be restricted to the smaller intersection as well. The chain groups will be the product of the $F(U_{\alpha_0 \ldots \alpha_n})$'s. We are implicitely using commutativity to be able to add together the maps in the boundary.

Remark 1.5. If G is an abelian group, we define $H^*(U, G|C^0(G)|C^\infty(G))$ as before.

Remark 1.6. If U is an *acyclic** cover, then $H^*(U, \mathbb{R}) = H_{dR}(X, \mathbb{R})$.

Remark 1.7. Given a C^0 or C^∞ group homomorphism $G \to H$, one gets maps $H^*(U, C^?(G)) \to H^*(U, C^?(H))$.

If we have a SES of $C^?$ abelian groups $0 \to G \to H \to K \to 0$, then we get a LES in cohomology.

Actually if the sheaf has partitions of unity, the homology is $F(X)$ in degree 0, and 0 everywhere else.

*for all $\alpha_0 \ldots \alpha_n$ we have $U_{\alpha_0 \ldots \alpha_n}$ either empty or contractible

1.2 Non-abelian Chech cohomology

Let G be a (potentially) non-Abelian group, let U be an acyclic cover. C^0, C^∞ still makes sense.
We will define $H^0(X,F)$, what are the 0 cocylcles? In the Abelian case we have that ϕ_α is a cocycle if $\phi_\alpha|_{U_{\alpha\beta}} - \phi_\beta|_{U_{\alpha\beta}} = 0\ \forall \alpha,\beta$. This is the same as a global section $\phi \in F(X)$, so $H^0(X,F) = F(X)$ even in the nonabelian case.

Definition 1.8. *phi_α is a 0 cocycle if $\phi_\alpha|_{U_{\alpha\beta}} = \phi_\beta|_{U_{\alpha\beta}}$, or that $\phi_\alpha\phi_\beta^{-1} = 1$ on the appropriate set $U_{\alpha\beta}$.*

In the $H^1(X,F)$ case some more interesting things happen. In the Abelian case 1 cocycles are maps $U_{\alpha\beta} \to G$ with the condition, that restricted to $U_{\alpha\beta\gamma}$ $\phi_{\beta\gamma} - \phi\alpha\gamma + \phi_{\alpha\beta} = 0$. We generalize this now

Definition 1.9. Non-Abelian 1 cocycles are $\phi_{\alpha\beta} : U_{\alpha\beta} \to G$ maps, such that $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$, and we require $g_{\alpha\alpha} = 1$. These two conditions imply that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

Remark 1.10. This is the same condition as in the Abelian case, we just swap $\gamma\alpha$ to $\alpha\gamma$.

Now the 1 coboundaries. In the Abelian case two cocycles ϕ and ψ are cohomologous iff $\phi_{\alpha\beta} - \psi_{\alpha\beta} = \delta f_\alpha$, i.e. $\phi_{\alpha\beta} - \psi_{\alpha\beta} = f_\beta - f_\alpha$.

Definition 1.11. We say that if ϕ, ψ 1 cocycles are coohmologous if $\exists f$ a 0 cochain such that $\phi_{\alpha\beta} = f_\alpha \psi_{\alpha\beta} f_\beta^{-1}$.
From here we define $H^1(X,F) = Z^1(X,F)/\sim$, which will be a pointed set. The base point is given by $\phi_{\alpha\beta} = 1$.

Definition 1.12. $f : A \to B$ homomorphism of pointed sets one can define a kernel as the preimage of the basepoint.

Fact: If $1 \to G \to H \to K \to 1$ of possibly nonabelian groups, then we still get a LES on non-Abelian Chech cohomology up to H^1. It continues to $H^2(G)$ if $G < Z(H)$. Note, that the boundary map from H^0 to H^1 may be nonzero!

Excercise 1.13. *$X = \mathbb{R}P^2$ and take the defining S.E.S. of the spin group.*

2 Second Lecture

2.1 Principal bundles

Motivation is vector bundles. Let $E \xrightarrow{\pi} X$ be such a gadget. $E|_{U_\alpha}$ can be trivialised over U_α as $U_\alpha \times \mathbb{R}^n$. Over some $U_{\alpha\beta}$ we have two trivialising maps ϕ_α and ϕ_β, which are linear isomorphisms on each fiber, so $\phi_\alpha\phi_\beta^{-1}$ maps $(x,v) \mapsto (x, g_{\alpha\beta}(x)v)$ for some linear maps $g_{\alpha\beta}$ dependent on x.
On triple intersections $U_{\alpha\beta\gamma}$ we have three trivialisations, and three transition maps going around. It is clear that the composition of these three maps is the same as the identity by the commutativity relations of the trivialisation, which gives us the cocycle condition.

Remark 2.1. A vector bundle $E \xrightarrow{\pi} X$ induces a 1-cocycle. Moreover, isomorphic vector bundles induce the same cocycle.
Pick a trivialisation of E and E', call it U_α, the isomorphism is ψ, we get a map $\phi_\alpha^{-1}\psi\phi'_\alpha$ which maps $(x,v) \mapsto (x, f_\alpha(x)v)$. If there are two trivialisations, by commutativity we get that $g'_{\alpha\beta} = f_\alpha g_{\alpha\beta} f_\beta^{-1}$, which we wanted.

Tétel 2.2. *There is an isomorphism of pointed sets from the set of vector bundles of rank n over X modulo isomorphism, and the first Cech cohomology with coefficients $C^\infty GL_n(\mathbb{R})$.*

Definition 2.3. Let G be a topological or Lie group. A G-principal bundle is a fiber bundle $P \to X$ with a C^0 or C^∞ right action $P \times G \to G$ that is fiber preserving and transitive on each fiber.

Remark 2.4. Each fiber is a G-torsor (has a free and transitive G action).
G is also a G-torsor with right multiplication.

Example 2.5. A circle is an S^1 torsor.

Excercise 2.6. *A G-equivariant map of G-torsors is an isomorphism.*

Corollary 2.7. *A G-equivariant bundle map of G-principal bundles is an isomorphism.*

We build a 1-cocycle from $P \to X$. Over U_α we trivialise the bundle. A local section gives 1 in that trivialisation. Now we construct the same cocycle, and we get that there are transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$. The same conditions will be satisfied in this case as well, so:

Tétel 2.8. *There is an isomorphism of pointed sets from G-principal bundles over X mod isomorphism to $H^1(X; C^? G)$.*

Example 2.9. For $G = \mathbb{Z}_2$ and $X = S^1$ we have $H^1(S^1, C^\infty \mathbb{Z}_2) = H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$. We know the 0 element is the trivial bundle, the action interchanges the two copies of S^1. The other one is the boundary of the Möbius band, and the action again interchanges the two lifts.

Example 2.10. $G = \mathbb{Z}_3$ is the next example, again over the circle. $H^1(S^1; \mathbb{Z}_3) = \mathbb{Z}_3$, there are three bundles, but the connected ones are only distinguished by the action!

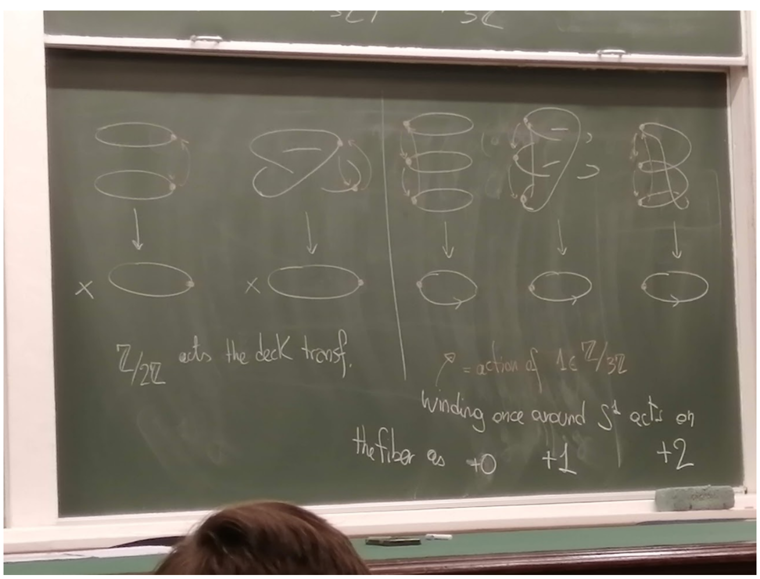

Figure 1:

Excercise 2.11. *Do the total spaces of nontrivial G-bundles need to be connected?*

Now the case of $GL_n(\mathbb{R})$. From the previous theorems we know that the rank n vectorbundles over X, and $GL_n(\mathbb{R})$ principal bundles over X are isomorphic. What is this isomorphism? In one direction, we construct the frame bundle of E, replacing every fiber with its frames, i.e. bases of E_x, i.e. linear isomorphisms $\mathbb{R}^n \to E_x$. In the other direction, given a principal bundle $P \to X$ with structure group $GL_n(\mathbb{R})$ we construct $P \times_{GL_n(\mathbb{R})} \mathbb{R}^n$, where we identify $(pg, v) \sim (p, gv)$.

Excercise 2.12. *What are $GL_n^+(\mathbb{R})$ principal bundles?*

Excercise 2.13. *What are $O(n)$ principal bundles?*

Excercise 2.14. *What are $SO(n)$ principal bundles?*

Remark 2.15. A $C^?$ homomorphism $G \xrightarrow{\phi} H$ induces a pointed set homomorphism $H^1(X, C^?G) \to H^1(X, C^?H)$. We can do from $P \to X$ to $P \times_G H \to X$, where G acts on H as $gh = \phi(g)h$.

Remark 2.16. If $H < G$ is a deformation retract of the Lie group G, then $H^1(X, C^\infty G) = H^1(X, C^\infty H)$.

Excercise 2.17. *What are $GL_n(\mathbb{C})$ principal bundles?*

Excercise 2.18. *What are $\mathbb{C}^*$ principal bundles?*

Excercise 2.19. *What are S^1 principal bundles?*

Excercise 2.20 (*)**.** *Show, that $H^1(X, C^\infty S^1) = H^2(X, \mathbb{Z})$.*

3 Third lecture

We show the last excercise.

Proof. Use the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$$

Were the first map is $2\pi \cdot$, the second is $e^{\cdot}$. Passing to the sheaves C^∞ with these groups we get a long exact sequence of homologies. The H^0 part is exact, we get

$$0 \to H^1(X, C^\infty \mathbb{Z}) \to H^1(X, C^\infty \mathbb{R}) \to H^1(X, C^\infty \S^1) \to H^2(X, C^\infty \mathbb{Z}) \to H^2(X, C^\infty \mathbb{R}) \to \dots$$

We know that $C^\infty \mathbb{R}$ has partitions of unity, and we get the desired isomorphism. □

Denote this isomorphism by c. From the Snake lemma starting from a cocycle $g_{\alpha\beta} : U_{\alpha\beta} \to S^1$ with the cocycle conditions. Lift these functions to $\mathbb{R}$. The conditions change to $\tilde{g}_{\alpha\beta} + \tilde{g}_{\beta\gamma} + \tilde{g}_{\gamma\alpha} = 2\pi K$. Pull it back to $\mathbb{Z}$ to see where is its image under the connecting homomorphism c. $C_{\alpha\beta\gamma} := \frac{\tilde{g}_{\alpha\beta} + \tilde{g}_{\beta\gamma} + \tilde{g}_{\alpha\gamma}}{2\pi}$.

Remark 3.1. $H^1(X, C^\infty S^1)$ is in bijection with complex line bundles over X modulo isomorphism. This is true because $GL_1(\mathbb{C})$ deformation retracts to S^1.

Remark 3.2. $H^1(X, C^\infty S^1)$ is a group. Multiplication is given pointwise. Given two line bundles corresponding to two elements of this group the element corresponding to the product element is the tensor product of the line bundles.

Excercise 3.3. *c is a group homomorphism, an thus an isomorphism of groups.*

Fact 3.4. *It is natural. This means, that given $f : X \to Y$, we get an induced map $H^1(X, C^\infty S^1) \to H^1(Y, C^\infty S^1)$ and $H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$, and c commutes with these maps.*

We deduce that $c = \pm c_1$.

Excercise 3.5. $c = c_1$

3.1 Spin structures

Motivation: take a 1-cocycle $(g_{\alpha\beta})$ defining a vector bundle $E \to X$ of rank n. $GL_n^+(\mathbb{R}) \leq GL_n(\mathbb{R})$, if we can lift the $g_{\alpha\beta}$ to this (or some) subgroup, we call this a reduction of the structure group. In the $GL_n^+(\mathbb{R})$ case this means that the bundle is orientable. $SO(n)$ is a strong deformaton retract of $GL_n(\mathbb{R})$, so we can always lift. This corresponds to a choice of Riemannian metric. Since $\pi_1(SO(n)) = \mathbb{Z}_2$, one can ask if there is a lift $Spin(n) \to SO(n)$ to the universal cover.

Remark 3.6. $SO(1) = \{*\},\ SO(2) = S^1,\ SO(3) = \mathbb{R}P^3$. The last one is seen be either

- $S^3 \to isom(im\mathbb{H})$ where $q \mapsto (v \mapsto qv\bar{q})$, S^3 is taken to be the unit length quaternions. The kernel of this map will be $\pm id$.
- $\mathbb{R}P^3$ is a 3-ball of radius π with antipodal points of its boundary identified. Every vector can be interpreted to be a rotation around that axis, the angle defined by the norm of the vector.

For $n \geq 3$ we really have $\pi_1(SO(n)) = \mathbb{Z}_2$.

Definition 3.7. For $n > 2$ we define $Spin(n)$ as the universal (double) cover of $SO(n)$. A Lie group of the same dimension. The group structure is given by the universal property of covering spaces by lifting the multiplication at the point $(e, e) \mapsto (e)$.
For $n = 2$ $Spin(2)$ is defined to be S^1 corresponding to the map $S^1 \xrightarrow{z^2} S^1$. For $n = 1$ $Spin(1) = \mathbb{Z}_2 \to \{*\}$.

Remark 3.8.

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

Definition 3.9. Let $E \to X$ be an orientable vector bundle of rank n, with associated principal bundle $P_{SO(n)} \to X$. A spin structure on E is a pair $(P_{Spin(n)} - \to X, H)$ where $H : P_{Spin(n)}/\mathbb{Z}_2 \to P_{SO(n)}$ is an isomorphism of $SO(n)$ principal bundles. These are taken up to isomorphism of pairs. Two pairs are isomorphic if there is an isomorphism $\Phi : P_{Spin(n)} \to P'_{Spin(n)}$ which commutes with H and H' and the factorisation maps.*

Remark 3.10. Actually $P_{Spin(n)} \to P_{Spin(n)}/\mathbb{Z}_2$ is a principal $\mathbb{Z}_2$ bundle. This happens because $\mathbb{Z}_2$ is a closed subgroup of $Spin(n)$, this does not happen always!

Definition 3.11. $Spin(E \to X)$ is the set of spin structires on E. $Spin(X) := Spin(TX \to X)$.

Remark 3.12. There is a map $\phi : Spin(E \to X) \to H^1(X, C^\infty Spin(n))$. It is not surjective in general. In the target you hit only cocycles which project to $g_{\alpha\beta}$ for $P_{SO(n)}$. ϕ is not injective in general either.

Example 3.13. There are two spin structures on the trivial principal $SO(n)$ bundle on $\mathbb{R}P^2$,but there is onl one $Spin(n)$ bundle lifting $\pi_1 : \mathbb{R}P^3 \times SO(n) \to \mathbb{R}P^3$ for some n.

*the only thing that makes sense, really

3.2 $Spin^c$ structures

Not all simply connected manifolds are spinnable :(You can lift $P^{TX}_{SO(4)}$ to $P_{Spin(4)}$ away from a surface S. The boundary of a point of the normal bundle of S has a framing, which either winds around, or it does not. If it has 0 winding, then we would be able to extend, so we have the second case. Complex numbers would help us unwind this rotation, and extend the trivialisation through S.

Definition 3.14.

$$Spin^c(n) := \frac{Spin(n) \times U(1)}{\pm(1,1)}$$

Remark 3.15.

$$1 \to U(1) \to Spin^c(n) \to SO(n) \to 1$$

$$1 \to Spin(n) \to Spin^c(n) \to U(1) \to 1$$

Excercise 3.16. *$Spin^c(n)$ is a pullback of $Spin(n+2)$ of the diagram $SO(n) \times U(1) \to SO(n+2) \leftarrow Spin(n+2)$.*

Definition 3.17. Let $E \to X$ be an orientable vector bundle of rank n, with associated principal bundle $P_{SO(n)} \to X$. A *spin*^{c} structure on E is a pair $(P_{Spin^c(n)} - \to X, H)$ where $H : P_{Spin^c(n)}/U(1) \to P_{SO(n)}$ is an isomorphism of $SO(n)$ principal bundles. These are taken up to isomorphism of pairs. Two pairs are isomorphic if there is an isomorphism $\Phi : P_{Spin^c(n)} \to P'_{Spin^c(n)}$ which commutes with H and H' and the factorisation maps.

Remark 3.18. Actually $P_{Spin^c(n)} \to P_{Spin(n)}/U(1)$ is a principal $U(1)$ bundle. This happens because $U(1)$ is a closed subgroup of $Spin^c(n)$, this does not happen always!

Tétel 3.19. *If $Spin^{(c)}(E \to X) \neq \emptyset$, then there is a free and transitive action of $H^1(X, \mathbb{Z}_2)$ or $H^1(X, C^\infty U(1)) = H^2(X, \mathbb{Z})$ on $Spin^{(c)}(E \to X)$.*

4 Fourth lecture

4.1 ACTION

Definition 4.1. A spin$^{(c)}$ structure on $P_{SO(n)} \to X$ is a $\mathbb{Z}_2$ or $U(1)$ principal bundle on $P_{SO(n)}$ such that the composition is a spin$^{(c)}$ principal bundle on X up to isomorphism (in the correct sense).

Proof. $H = \mathbb{Z}_2$ or $U(1)$. $\alpha \in H^1(X, C^\infty H)$ will be represented by $[P_H \to X]$. Given a spin$^{(c)}$ structure $P \to P_{SO(n)} \to X$ we act by α as $P_H \times_H P \to P_{SO(n)} \to X$ by projecction onto the second factor.
We need to check if it is well defined. Given $\tilde{P}_H \to X$ isomorphic to P_H, with given isomorphism ϕ its an excercise to construct an isomorphism between the associated bundles $\tilde{P}_H \times_H P \to P_H \times_H P$ by taking the produt fiberwise with the identity.
We need to check that it is free. Suppose that $P \xrightarrow{\Phi} P_H \times_H P$ are isomorphic with given isomorphism Φ. We construct a section. Take $\Phi(p) = (q, \lambda \cdot p) = (q\lambda, p)$ we can assume that the second component is always p, and can define $\phi(p) = q\lambda$. Consider $\phi(p\xi)$, by

$$(\phi(p), p\xi) = (\phi(p), p)\xi = \Phi(p)\xi = \Phi(p\xi) = (\phi(p\xi), p\xi)$$

we get that $phi(p\xi) = \phi(p)$.
Thus ϕ descends to a map $\bar{\phi} : X \to P_H$, which is a section of the H principal bundle, thus P_H is trivial, and the action is free.
Finally transitivity. $P \to P_{SO(n)} \to X$ and another structure $P' \to P_{SO(n)}$. Define $Mor(P, P')$ to be the fiber bundle on X whose fibers are the spin$^{(c)}$ equivariant maps $P_x \to P'_x$ commuting with the bundle projections. $Mor(P, P')$ is an H principal bundle over X. The action is given by pre, or postcomposition and it does not matter which, since the maps are equivariant. Now take $Mor(P, P') \times_H P \to P'$ and $(f, p) \mapsto f(p)$ provides an isomorphism. □

We can also think about this setup in terms of cocycles.

Lemma 4.2. *Fix a 1-cocycle $G_{\alpha\beta}$ for $P_{SO(n)}$, then*

1. *every spin or spinc structure on $P_{SO(n)}$ can be represented with a cocycle projecting on $g_{\alpha\beta}$*
2. *assuming this, two spin/spinc cocycles projecting on the same SO(n) cocycle define the same spin/spinc structure iff $\exists f_\alpha : U_\alpha \to H$ such that $\tilde{g}_{\alpha\beta} = f_\alpha \overline{g}_{\alpha\beta} f_\beta^{-1}$.*

Proof. For the first point consider $\tilde{g}_{\alpha\beta} : U_\alpha \to spin^{(c)}(n)$. Taking the isomorphism from the definition we get $pr \circ \tilde{g}_{\alpha\beta} = l_\alpha g_{\alpha\beta} l_\beta^{-1}$ for some $SO(n)$ 0-cochain l_α. Lift this arbitrarily to $Spin^{(c)}(n)$ to $\tilde{l}_\alpha$. Now define $\tilde{g}'_{\alpha\beta} := \tilde{l}_\alpha \tilde{g}_{\alpha\beta} \tilde{l}_\beta$. Taking its projection we see, that $\tilde{g}'_{\alpha\beta}$ projects to $g_{\alpha\beta}$ as required.
For the second part if $\tilde{g}_{\alpha\beta} = f_\alpha \overline{g}_{\alpha\beta} f_\beta^{-1}$ then the spin/spinc structures are the same because these elements project into the same element, f_α is in the kernel of the projection from spin/spinc to SO(n). In the other direction if $\tilde{g}_{\alpha\beta}$ and$\overline{g}_{\alpha\beta}$ give the same spin/spinc structure, then $\tilde{g}_{\alpha\beta} = f_\alpha \overline{g}_{\alpha\beta} f_\beta^{-1}$ where $f_\alpha : U_\alpha \to Spin^{(c)}(n)$ are chosen from an isomorphism after trivialisation. By assumption the two cocycles project to the same element, from this we want to conclude that f_α is in the kernel. □

Corollary 4.3. *The action $H^1(X, C^\infty H) \times Spin^{(c)} \to spin^{(c)}$ is given by the pointwise products of the representing cocycles. This action is free and transitive.*

Remark 4.4. This is the same action as we had for principal bundles. One can compute the transition functoions.

Remark 4.5. This action is Abelian.

5 Fifth lecture

5.1 Chern class

Consider

$$1 \to spin(n) \to spin^c(n) \to U(1)/\{\pm 1\} = U(1) \to 1.$$

This gives a LES

$$H^1(X, spin(n)) \to H^1(spin^c(n)) \xrightarrow{\det} H^1(X, U(1)) = H^2(X; \mathbb{Z})$$

where the isomorphism is given by the first Chern class.

Definition 5.1. $s \in Spin^c(E \to X)$ determines $i(s)$, a $spin^c(n)$ principal bundle. $c_1(s)$ is defined to be $c_1(\det(i(s)))$.
If $i(s)$ is represented by $[\tilde{g}_{\alpha\beta,\lambda_{\alpha\beta}}]$, then $\det i(s)$ is represented by$\lambda^2_{\alpha\beta}$.

Tétel 5.2. *$c_1(s+h) = c_1(2) + 2h$, where $s \in spin^c(E \to X)$ and $h \in H^2(X, \mathbb{Z})$.*

Proof. $i(s) = [g_{\alpha\beta}, \lambda_{\alpha\beta}]$, then $i(s+h) = [g_{\alpha\beta}, \lambda_{\alpha\beta} l_{\alpha\beta}]$ where $l_{\alpha\beta} = h \in H^1(X, C^\infty U(1))$. □

5.2 spin vs spin^c

Remark 5.3. The inclusion map $j : spin(n) \to spin^c(n)$ induces a map $Spin(E \to X) \to Spin^c(E \to X)$. This is constructed as follows. Given $P_{Spin} \to P_{Spin}/\mathbb{Z}_2 \xrightarrow{H} P_{SO(n)} \to X$ we can create $P_{Spin} \times_{Spin} Spin^c \to P_{Spin} \times_{Spin} Spin^c/U(1)$, this latter space is canonically isomorphic to $P_{Spin}/\mathbb{Z}_2$ by the inclusion, we declare this to be the imgae of the bundle under j.

Tétel 5.4. *$s \in Spin^c(E \to X)$ is in the image of j iff $c_1(s) = 0$.*

Proof. First $[P_{spin^c(n)}] \in H^1(X, Spin^c(n))$ is induced by a $Spin$ principal bundle $P_{spin} \in H^1(X, C^\infty Spin)$ iff $H^1(X, C^\infty Spin) \to H^1(X, C^\infty Spin^c) \xrightarrow{\det} H^1(X, C^\infty U(1))$, if the image is zero under the determinant iff it is trivial iff it has 0 Chern class.
Then given $s \in spin^c(E)$ is a $P_{spin^c} \to P_{SO(n)} \to X$, and we know, that there is a spin bundle, such that $P' := P_{spin} \times_{spin} spin^c \to X$ is isomorphic to $P_{spin^c} \to X$. Call this isomorphism Φ. $P'/U(1)$ is isomorphic to $P_{SO(n)}$ by the commutativity of our diagram. □

Tétel 5.5. *$j : Spin(E \to X) \to Spin^c(E \to X)$ is modelled over $\beta : H^1(X, \mathbb{Z}_2) \to H^2(X, \mathbb{Z})$. This means, that $j(s+h) = j(s) + \beta(h)$. Here β is the Bockstein map induced by $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1$ on homology.*

Proof. Let $g_{\alpha\beta}$ represent $E \to X$ and suppose that $s = \tilde{g}_{\alpha\beta}$ is a lift to $spin(n)$. If $h = (\lambda_{\alpha\beta}) \in H^1(X, C^\infty \mathbb{Z}_2)$ then $s+h$ is represented by $(\tilde{g}_{\alpha\beta} \cdot l_{\alpha\beta})$ with $l_{\alpha\beta} = e^{i\pi\lambda_{\alpha\beta}}$. We pass it to the map $j : Spin \to Spin^c$. $j(s)$ is represented by $j \circ \tilde{g}_{\alpha\beta}$ and $j(s+h)$ is represented by $(j(\tilde{g}_{\alpha\beta} \cdot l_{\alpha\beta})) = [\tilde{g}_{\alpha\beta} \cdot l_{\alpha\beta}, 1] = [\tilde{g}_{\alpha\beta}, l_{\alpha\beta}] = j(s) + c_1([l_{\alpha\beta}])$. Our goal is now to check $\beta(\lambda_{\alpha\beta}) = c_1([e^{i\pi\lambda_{\alpha\beta}}])$. We get a commutative square

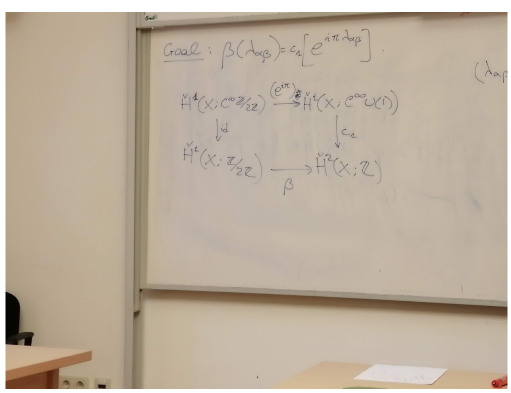

Figure 2:

Take integral lifts $\tilde{\lambda}_{\alpha\beta}$ its boundary is $\tilde{\lambda}_{\alpha\beta} + \tilde{\lambda}_{\beta\gamma} + \tilde{\lambda}_{\gamma\alpha}$. We can lift to $\frac{\tilde{\lambda}_{\alpha\beta} + \tilde{\lambda}_{\beta\gamma} + \tilde{\lambda}_{\gamma\alpha}}{2} = b_{\alpha\beta\gamma}$. Consider $\pi\tilde{\lambda}_{\alpha\beta} \in \mathbb{R}$. The coboundary will be $\pi\tilde{\lambda}_{\alpha\beta} + \tilde{\lambda}_{\beta\gamma} + \tilde{\lambda}_{\gamma\alpha}$, and pulling back to $\mathbb{Z}$ coefficients we have to divide by 2π, so the two connecting homomorphisms are the same. □

Example 5.6. $S^1 \times S^2$ has 2 spin structures and $\mathbb{Z}$ spin^c structures. Both $s, s' \mapsto t_0$.

MISSING

6 Seventh lecture

6.1 Functoriality

Let $f : X \to Y$ be a smooth map. We can pull back smooth bundles, and similarly we can pull back $E \to X$ if it is a G-bundle. On the levels of cocycles we can pull back an acyclic cover U_α of Y to *some* cover $f^{-1}(U_\alpha)$ of X. This will not be acyclic usually, but contains some $V_{\bar{\alpha}}$ acyclic cover of X. If $g_{\alpha\beta}$ is a cocycle then $(g_{\alpha\beta} \circ f)_{\bar{\alpha}\bar{\beta}}$ is a cocycle. We get a map $f^* : H^1(Y, C^\infty G) \to H^1(X, C^\infty G)$ of pointed sets.
Everything mentioned thus far extends to spin and spin^c structures as well: Take $P_{spin^{(c)}} \to P_{SO(n)} \to Y$, and pull back the $SO(n)$ bundle first by f, and pull back the spin/spin^c bundle using the "upper" side of the square (or directly $X \to Y \leftarrow P_{spin^{(c)}}$), which gives the same). This way we get a spin or spin^c structure over X from Y, and since we can pull back morphisms, we get a well defined map between the Chech cohomologies. The induced map is $(x, p) \mapsto (x, h(p))$.

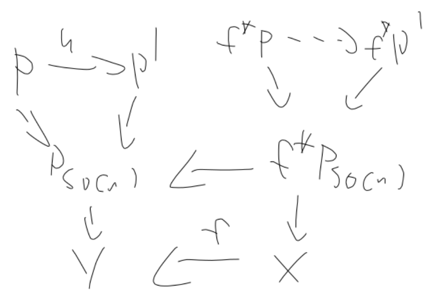

Figure 3:

Remark 6.1. If $X^n \to Y^n$ is an immersion of n-dimensional manifolds, then $f^*TY = TX$. This gives a map $Spin^c(Y) \to Spin^c(X)$. The actions of the corresponding H^2 groups are also related by the pullback map of cohmologies.

Tétel 6.2. *For $f : X \to Y$ a smooth map and $E \to Y$ a vector bundle of rank n, the map $f^* : Spin^c(E) \to Spin^c(f^*E)$ satisfies $f^*(s + h) = f^*(s) + f^*(h)$ for $h \in H^2(Y; \mathbb{Z})$.*

Excercise 6.3. *Can check it using $\mathbb{Z}_2$ or $U(1)$ principal bundles, or by cocycles.*

Excercise 6.4. $c_1(f^*(s)) = f^*(c_1(s))$ *(should be immediate from the cocycle description, or a line bundle computation).*

6.2 Restriction map

There is a square If $n = 2$, then the map is reduction mod 2, and the inclusion of S^1 into S^3.

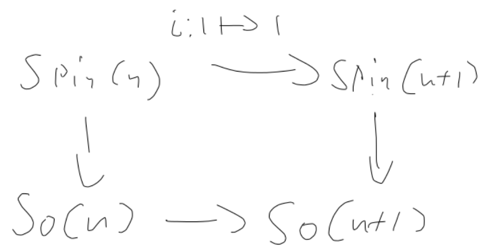

Figure 4:

Take the product of the square with $U(1)$, and we get something that passes to the quotient by $\mathbb{Z}_2$ a similar square with $Spin^c(n)$ and $SO(n) \times U(1)$. The maps are inclusion and identity, and the factor map, determinant.

Tétel 6.5. *i induces a map $i_* : Spin^c(E \to X) \to Spin^c(E \oplus \mathbb{R} \to X)$.*

Given $P \to P_{So(n)} \to X$ the map induces $P \times_{Spin^c(n)} Spin^c(n+1) \to P_{SO(n)} \times_{SO(n)} SO(n+1) \to X$.

Remark 6.6. $c_1(s) = c_1(i_*(s))$ because of the square, *det* is the same on both sides.

- i_* is well defined? yes, check is homework
- on cocycles $g_{\alpha\beta} \mapsto (i \circ g_{\alpha\beta})$
- i_* commutes with the action of $H^1(X, C^\infty H)$ for $H \in \{\mathbb{Z}_2, U(1)\}$. Check that twisting by an H-principal bundle commutes with i_*, or do it on the cocycles

All of this gives that i_* is an isomorphism of $H^1(X, C^\infty H)$ torsors.

Tétel 6.7. *Let $Z \to X$ be a smooth map that is a framed* immersion. Then the composition*

$$Spin^c(TX \to X) \xrightarrow{f^*} Spin^c(TX|_Z = TZ \oplus \mathbb{R}^k \to Z) \to Spin^c(TZ \to Z)$$

is called the restriction map. Denote this composition r. This is a map of pointed sets, and is modelled over the map $f^* : H^1(X, C^\infty H) \to H^1(Z, C^\infty H)$, $r(s+h) = r(s) + f^*(h)$.

Remark 6.8. If X^n is orientable, then ∂X is oriented and framed, so we get a restriction map.
Observe, that if $Z^{n-1} \to X^n$ both orientable, then Z is framed. In paticular we have a restriction map.

6.3 Alternative definition

Suppose $n \geq 3$, then $H^1(SO(n), C^\infty H) = \mathbb{Z}_2$,i.e. there is a unique nontrivial $\mathbb{Z}_2$ or $U(1)$ bundle, denoted $Spin(n)$ and $Spin^c(n)$.

Definition 6.9. A spin^c structure is an element of $H^1(P_{SO(n)}, C^\infty H)$ that restricts to the nontrivial element on each fiber of $P_{SO(n)} \to X$.

The action in this definition is even simpler to describe now. Take $\pi : P_{SO(n)} \to X$, some cocycle in $H^1(X, C^\infty H)$ acts as follows: pull it back to the $P_{SO(n)}$ bundle, and multiply the cocycles.

* $\nu(Z \subset X)$ is trivialised

7 Eight lecture

7.1 Classifying spaces

Let G be a topological or Lie group

Definition 7.1. A *Universal bundle* $\pi_G : EG \to BG$ is a G-principal bundle, with BG weakly contractible (i.e. $\pi_n(BG) = 0$). We call BG the *classifying space* for G.

Tétel 7.2 (Milnor)**.** *For every group G a classifying space exists.*

Proof. The join of two spaces is defined as:

$$X \star Y := (X \times Y \times [0,1])/\{((x,y,0) \sim (x',y,0), (x,y,1) \sim (x,y',1))\}.$$

Now $EG = G \star G \star \ldots$. This space has an obvious G action, the factor will be BG. □

Example 7.3. $BSO(n) = Gr^+(n, \infty)$.

Tétel 7.4. *Let $\pi_G : EG \to BG$ be a universal G-principal bundle, then for any G-principal bundle $\pi : E \to X$ there exists a continous function $f : X \to BG$ such that $f^*EG = E$. Moreover if $f_1^*EG = F_2^*EG$ iff f_1 is homotopic to f_2.*

Proof. We choose a CW decomposition of X and apply induction. Suppose $f|_{\partial D}^* EG = E|_{\partial D}$. Since D is contractible $E|_D = D \times G$. Restricting this to the boundary we get a map from the boundary to EG. Since EG is weakly contractible, this map extends to the whole of D.
Similar reasoning shows the homotopy part. □

Fact 7.5. *A group homomorphism $\psi : G \to H$ induces a map $B\psi : BG \to BH$.*
From $EG \to BG$ we canconstruct an H principal bundle $EG \times_H H \to BG$, and take its classifying map $BG \to BH$ for $B\psi$.

Remark 7.6. A vector bundle or $SO(n)$ principal bundle corresponds to a map $f : X \to BSO(n)$.

Definition 7.7. A spin or spin^c structure on f is a lift of f to $BSpin(n)$ or $BSpin^c(n)$.

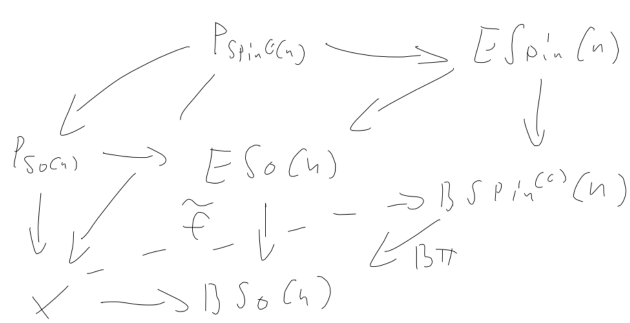

Figure 5: Equivalence of the two definitions

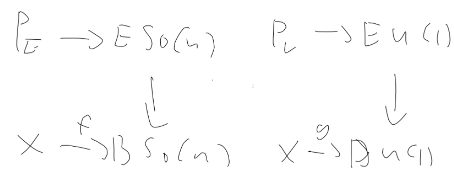

Figure 6:

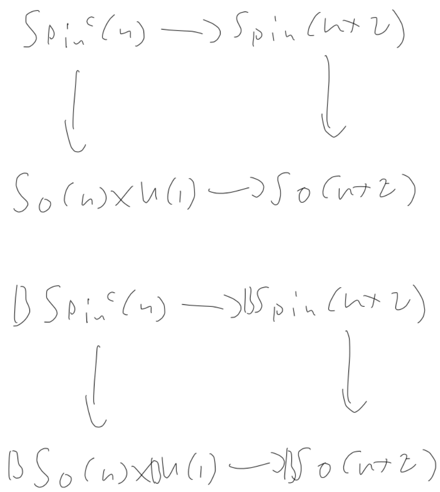

Figure 7:

Definition 7.8. A *Spin*^{c} structure on $E \to X$ is a complex line bundle $L \to X$ together with a spin structure on $E \oplus L_{\mathbb{R}}$.

Proof. From $(f,g) : X \to BSO(n) \times BU(1)$ we can push forward to $BSO(n+2)$ to get the classifying map of $E \oplus L_{\mathbb{R}}$. A spin structure on this bundle is a lift of this map to $BSpin(n+2)$, and we can just pull back to $BSpin^c(n)$ and vice versa. □

Definition 7.9. A spin^c structure is an almost complex structure is an almost complex structure on sk_2X that extends to sk_3X up to homotopy.

Definition 7.10. An a.c.s. is a $J : E \to E$ map of vector bundles with the condition, that $J \circ J = -id_E$.

Proof. An a.c.s. is a lift of $f : X \to BSO(2n)$ to $BU(n)$. Every unitary matrix induces an orthogonal transofrmation on the underlying real space $U(n) \to BSO(2n)$, and this induces the map $BU(n) \to BSO(2n)$, over which we want to lift.

Fact 7.11. *this map lifts to* $BJ : BU(n) \to spin^c(2n)$

Fact 7.12. *The homotopy fiber of* Bj *is 2-connected*

We can lift the map s to the 3 skeleton because the fiber is 2-connected. Given homotopic maps $s, s' : X \to BSpin^c(2n)$ we can lift the whole homotopy between them to the 3-skeleton of $X \times I = sk_2X \times I$.
Other direction??? □

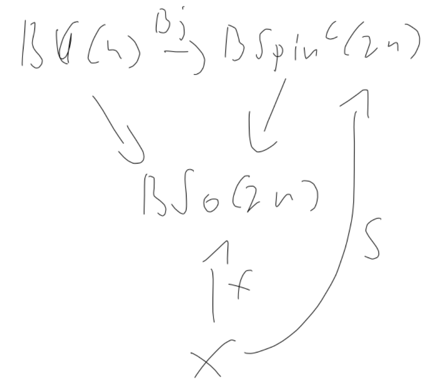

Figure 8:

8 Ninth lecture

8.1 Other definitions of spin structures

Recall, that a spin structure on $P_{SO(n)}$ is an element s of $H^1(P_{SO(n)}, \mathbb{Z}_2)$ such that $\forall x \in X$ the map $i_x : SO(n) \to P_{SO(n)}$ (fiber inclusion) such that $i_x^*(s)$ is the nontrivial element of $H^1(SO(n), \mathbb{Z}_2)$ for $n \geq 2$. The action of $H^1(X, \mathbb{Z}_2)$ is by pullback and addition.

Tétel 8.1. *A spin structure on X is a trivialisation of $P_{SO(n)}$ on the 1-skeleton $sk_1(X)$ that extends to the 2-skeleton, considered up to homotopy ($n \geq 3$).*

For principal bundles a trivialisation is the same as a section. Given $s : X \to E$ for some G-bundle E, we define $s(x) = 1_G$, and this trivialises the bundle.

Proof. Take two sections $s_1, s_2 : sk_2X \to P_{SO(n)}$. Define $s_1/s_2 \in Hom(\pi_1(X, \mathbb{Z}_2))$. Since $s_1(x) = s_2(x) \cdot g(x)$, we get a map $g : sk_2X \to SO(n)$, and $Hom(H_1(X), \mathbb{Z}_2) = H^1(X, \mathbb{Z}_2)$.

- We need to check, that s_1 is homotopic to s_2 iff $s_1/s_2 = 0$.
- The $H^1(X, \mathbb{Z}_2)$ action is transitive. For this we need to check, that defining $s_1 = s_1/s_2 \cdot s_2$ works

Define a map $H^1(P_{SO(n)}, \mathbb{Z}_2) > Spin(X) \to Triv(P_{SO(n)}|_{sk_2X})/\sim$. Over each 1-cell we have two choices to lift the loop to $P_{SO(n)}$, one of them gives 0 under $\phi : \pi_1(P_{SO(n)}) \to \mathbb{Z}_2$.
????? □

8.2 Euler structures

$Eul(X^n)$ with $\chi(X) = 0$ are an $H_1(X, \mathbb{Z})$ torsor. If X^3 is closed, then $H_1(X) = H^2(X)$ by Poincaré duality, and we get that $Spin^c(X) = Eul(X)$.

Definition 8.2 (Combinatorial Euler structure)**.** Suppose, that A is a CW complex with geometric realisation X^n. A singular 1-chain θ on A is an *Euler chain* if

$$\partial\theta = \sum_{e \in \text{cells of A}} (-1)^{|e|} x_e$$

, where x_e is the "center"* of the cell† e. Two Euler chains are homologous if they differ by a boundary of some 2-chain. $Eul(A) = Euler\ chains/\sim$.

Definition 8.3. v, w non-singular vector fields tangent to X, a connected closed manifold. We say that $v \sim w$ are homologous if $\exists D^n \subset X^n$ such that $w|_{X \setminus D}$ is homotopic through non-vanishing (nowhere 0) vector fields to $v|_{X \setminus D}$.
$Vect(X)$ is defined as the set of non singular vector fields up to this homotopy. These are called *vectorial Euler structures*. $Vect(X)$ is a torsor over $H_1(X, \mathbb{Z})$ (if $\chi(X) = 0$).

9 Tenth lecture

Tétel 9.1. *$vect(X)$ is an $H_1(X, \mathbb{Z})$ torsor.*

We need some obstruction theory (cf. Steenrod). Let v, w be nonsingular vector fields on X, i.e. sections of $SX \to X$ of the unit sphere bundle of the tangent bundle. Since the fiber is $n-2$ connected, v can be homotoped to w on $sk_{n-2}X$. There is an obstruction class $d(v, w) \in H^{n-1}(X, \pi_{n-1}(S^{n-1}))$.

Tétel 9.2 (Steenrod)**.** *v is homotopic to w on $sk_{n-1}X$ iff $d(v, w) = 0$.*

v and w point in opposite directions on a 1-submanifold of X generically. Restricting to a small ball intersecting this 1-dimensional submanifold, we get a map $\phi_\gamma : S^{n-1} \to S^{n-1}$. We take the 1-cycle $\frac{v}{w} := \sum (deg\phi_\gamma)\gamma$, where the sum runs over connected components of the 1-manifold $\{v = -w\}$. This cycle is equal to $PD(d(v, w))$.

Tétel 9.3 (Steenrod)**.** $\frac{v}{w} + \frac{w}{z} = \frac{v}{z}$

Now we can prove theorem 9.1

Proof. For $h \in H_1(X, \mathbb{Z})$ and $[w] \in vect(X)$ we define $[w] + h$ as the class of a vector field w such that $\frac{v}{w} = h$. How do we construct such a w? Assume that w points in the opposite to h along h. This can be achieved by a small homotopy. Using *Reeb turbulance*
Well definedness: if w_1, w_2 are homotopic on $sk_{n-1}X$ and they pairwise have the same obstruction class with some v_1, v_2, then we use Steenrod's theorem $\frac{v_1}{v_2} = h + 0 - h = 0$, thus v_1 is homotopic to v_2.
Action is transitive by definition, the obstruction class gives the element required.
It is also free, since if v is homotopic to w, then their obstruction class vanishes, thus they represent the same class. ☐

Tétel 9.4. *Given a manifold X, equipped with a handle structure A, then there is a map $Eul(A) \to vecet(X)$, which is an isomorphism of H_1 torsors.*

*some point in the cells interion
†the sum runs over cells of every dimension, that's why $|e|$ is there

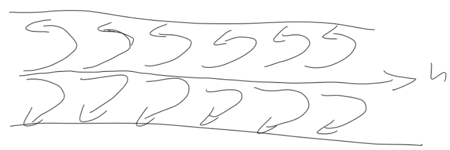

Figure 9:

Proof. Define the map, let θ be an Euler class, pick a Morse function. θ connects critical points of opposite parity, call the tubular neighborhood of this curve ν. On $\partial\nu$ the Morse function has degree 0, and we can extend the gradient vectorfield inside ν in a nonsingular manner.
If $[\theta - \eta] = h \in H_1(X, \mathbb{Z})$. We have 3 vector fields, $\nabla f, v_\theta, v_\eta$. Take a neighborhood of a critical point p, here $deg\nabla d = 1$. We can modify in a tubular neighborhood to get a degree 1 vector field representing $\frac{v_\eta}{v_\theta}$, which is also $\theta - \eta$... ☐

Definition 9.5. A normal Euler structure on X^n a cohomology class $\xi \in H^{n-1}(SX, \mathbb{Z})$ that restricts to the canonical*generator of $H^{n-1}(S^{n-1}, \mathbb{Z})$ on each fiber. $nor(X)$ denotes the set of normal Euler structures on X.

Tétel 9.6. *$nor(X)$ is an $H^{n-1}(X, \mathbb{Z})$ torsor.*

Tétel 9.7 (Leray-Hirsch)**.** *$F \to E \to B$ a fiber bundle such that $H^*(F)$ is freely generated and $\exists c_j \in H^*(E)$ such that $i_x^*(c_j)$ is a basis for the cohomology of the fiber. Then $H^*(F) \otimes H^*(B) \to H^*(E)$ is an isomorphism, where the map is $i_x^*(c_j) \otimes b_i \mapsto c_j \cup \pi^*(b_i)$.*

Proof of the theorem using Leray-Hirsch. If ξ is a normal Euler structure, then $\{1, \xi\}$ are global classes, which restrict to a basis on each fiber. This means, that $H^{n-1}(SX)$ elements can be represented as $1 \otimes b + \xi \otimes n \cdot 1_{H^*(X)}$. Normal Euler structures are classes where $n = 1$, the action of H^{n-1} is by pullback and addition, this leaves the $n = 1$ condition intact.
We can also give a more abstract definition of this action. $(\xi, b) \mapsto \xi + \pi^*(b)$. Since thecomposition of fiber inclusion andbundle projection is the constant map, we see that the pullback restricts as 0 on the fiber, giving what we wanted. From Leray Hirsch however we get more, namely that this action is free and transitive. ☐

Tétel 9.8. *$\exists H_1(X, \mathbb{Z})$ equivariant isomorphism between $vect(X)$ and $nor(X)$.*

Proof. Suppose that $[v] \in Vect(X)$, so $v : X \to SX$ defines a submanifold $\Sigma_V = Imv \subset SX$. $[\Sigma_v] \in H_n(SX)$. The Poincaré dual of this surface is in $H^{n-1}(SX)$, and we will prove, that it is a normal Euler structure. Consider a point $x \in X$, the fiber over it and the fiber inclusion i. $i^*(PD([\Sigma_v])) = PD([i^{-1}(\Sigma_v)])$. Since the section intersects the fiber in exactly one point, we get the canonical generator as required.
Now for equivariance. For $a \in H_n(SX)$, $PD(a)$ is the cohomology class represented by $\phi_a \in C^{n-1}(SX)$ such that $< \phi_a, e >= a \cap e$ for each $n-1$ cell e. (We are implicitly assuming a CW structure on SX, induced from such a structure on X.) Let $v = w + h$, i.e. $\frac{v}{w} = h \in H_1(X) = H^{n-1}(X)$.

☐

*note, that this needs X to be oriented