A_∞ algebras and type-D structures

Lecturer: András Stipsicz, Marco Marengon

2024/2025/1

1 Review: Morse homology

Suppose X a smooth manifold, we can talk about $f : X \to \mathbb{R}$ being smooth by the chain rule. We can talk about $df : TX \to \mathbb{R}$ a 1-form.

Definition 1.1. $x \in X$ is a *critical point*, if $df|_x = 0$. In a given chart we can construct the *Hessian* by considering the second derivative matrix. f is a *Morse function*, if at every critical point this Hessian is nondegenerate

Theorem 1.2. *The Morse functions are dense in the space $C^\infty(X, \mathbb{R})$ on a compact smooth manifold X.*

From a Morse function one can determine the Homology of X. Every critical point of a Morse function admits an *index* by computing the dimension of the negative definite subspace of the Hessian at the point.

Theorem 1.3. *Suppose X to be compact without boundary dimension n, f a Morse function on X. Then $\sum(-1)^i c_i = \chi(C)$ where c_i denotes the number of index i critical points.*

Sylvester's theorem has a topological conterpart, Morse's lemma, stating that if x is a Morse critical point, then there are coordinates about x such that in those coordinates $f = -x_1^2 - ... - x_{ind(x)} + x_{ind(x)+1}^2 + ... + x_n^2$. There are multiple versions of homology that can be defined, e.g. singular, CW, and now we do Morse homology. The chain groups are generated by the index i critical points in each rank. The chain maps are a bit more complicated, first we fix a metric, which gives us ∇f by dualising the 1-form df. This vectorfield vanishes exactly at the critical points. For the coefficients consider q, an index i and p an index $i-1$ critical points and the set $\mathcal{M}(p, q) = \{\gamma : \mathbb{R} \to X\}$ smooth curves which converge to q at $-\infty$ and p at ∞ with the added assumption that $d\gamma/dt = -\nabla f|_{\gamma(t)}$. There is a natural $\mathbb{R}$ action by reparametrisation $\gamma(t) \mapsto (\gamma(t+s))$ giving as another of these flowlines, so we factor out with this relation. We denote $\tilde{\mathcal{M}} = \mathcal{M}/\mathbb{R}$. We want to count the points of this space, which would be great if it is a compact 0-dimensional manifold.

Theorem 1.4. *For a generic choice of the metric g the space $\mathcal{M}(p, q)$ is a smooth manifold of dimension $ind(q) - ind(p)$.*

If $p \neq q$, then the $\mathbb{R}$ action is free, so the quotient is a manifold of dimension $ind(q) - ind(p) - 1$. Compcatness is a bit trickier, we need to consider broken trajectories from q to p, which means a sequence of flowlines and endpoints beginning with q and ending with p, but also connecting to intermediate critical points x_i. Addin broken flowlines we get a compact space. For points with neighboring indicies this compactifying set is empty for a generic metric, thus the moduli space is already compact.

We want to see that the map we get in this way is a chain complex. Take a point q of index i, z a point of index $i-2$, what is the coefficient of z in the image of q? $M(q,z)$ is a 2-dimensional space, factoring we get a 1-dimensional space. This space may be non-compact, if there are critical points with index $i-1$ in between. These broken trajectories correspond bijectively to the compactifying points of the space $M(z,q)$. This is done by cutting off the part of the curve which breaks the trajectory and finding a nearby sequence of actual trajectories converging to the broken trajectory. Moreover there is a neighborhood of the broken trajectory which is of the form $(0,\epsilon]$, thus there is no component of the space $M(p,q)$ which is of the form $S^1 \setminus \{p\}$. Thus since the compact stick has two ends we get that $\mod 2$ there are 0 broken flowlines between q,z, thus we really get a chain complex.

Theorem 1.5. *The homology if this chain complex is just the singular homology of X with $\mathbb{Z}_2$ coefficients.*

2 Second lecture

Y is always an oriented closed 3-manifold. Note, that 3-manifolds have a unique smooth structure! To picture these, begin with an oriented surface A_g and promote it to a 3-manifold $\Sigma \times [-1,1]$.

2.1 Handle attachment

D^n is the unit disc, an n-dimensional k-handle is $D^k \times D^{n-k}$. When X is an n-manifold with ∂X an $n-1$-dim manifold, we attach a k handle to X by considering an embedding $\phi : \partial D^k \times D^{n-k} \to \partial X$. This can be encoded by $\phi_0 = \phi|_{\partial D^k \times \{0\}}$ and a framing, i.e. a trivialisation of the normal bundle of the image of ϕ_0. Since we want to present the manifolds, this information is only considered up to isotopy. The difference of 2 framings is an element of $\pi_{k-1}(GL(n-k,\mathbb{R})) = \pi_{k-1}(O(n-k))$, so after fixing one, we can describe every other framing by an element of this group. Moreover if $k \neq 1$ we are only interested in the connected component of the identity, so $\pi_{k-1}(SO(n-k))$. For $n = 2,3$ and $1 < k \leq n$ only $\pi_1(SO(2))$ is nonzero, its $\mathbb{Z}$. For the $k=1$ case we have two choices, and we always choose the orientable gluing, so for orientable manifolds it is unique as well. Consider $\beta_1 ... \beta_n$ disjoint curves in Σ such that $\Sigma \setminus \cup\beta$ is connected and planar (admits an embedding into the plane) (this happens when $n = g$ and $< [\beta_1], ..., [\beta_n] >$) are linearly independent). In this case after handle attachment the boundary will be S^2, and the self-diffeomorphisms of the sphere form a connected group, so we can attach a ball uniquely. This is seen by a simple Euler characteristic computation, each surgery, i.e. handle attachment we remove an annulus from the boundary, and add two discs, i.e. raise χ by 2. Likewise we draw the α curves.

Theorem 2.1. *Σ_g a genus g surface $\beta_1, ..., \beta_g$ disjoint closed curves in $\Sigma_g \times \{1\}$ and $\alpha_1, ..., \alpha_g$ similarly in $\Sigma_g \times \{0\}$ encode uniquely a closed 3-manifold Y. We can superimpose the two sets of curves onto the same copy of the surface Σ we get a Heegaard diagram.*

To draw these we draw circles, which get glued together along a reflection.

Theorem 2.2. *Every Y closed oriented 3-manifold can be presented by a Heegaard diagram.*
The diagram is unique up to isotopy handle slides and (de-)stabilisation.

One can obviously isotope the α, β curves and the manifold does not change. For handle slides take two α or two β curves, take a pushoff* of α_1 and connect sum this pushoff with α_2. This is an isotopy the gluing map

*inside Σ_g

of the corresponding handle. Finally stabilisation is a connected sum of the surface Σ with the torus, where α is a meridian and β is a longitude (this diagram represents the sphere).
Only S^3 has a genus 0 diagram. If $g = 1$ we get lens spaces (and $S^2 \times S^1$). Usually we pick the α curves to be standard, and picture the β curves relative to them.
Suppose $K \subset S^3$ is a knot, i.e. the image of a smooth embedding of the circle. Assume that K is framed, $\partial\nu K = T^2 \supset \mu$ and the framing gives us the longitude λ, so this torus gets trivialises as an $S^1 \times S^1$. Given $p/q \in \mathbb{Q} \cup \{\infty\}$ with $(p, q) = 1$ $(1/0 = \infty)$. $Y_{p/q}(K) = Y \setminus \nu K \cup_\phi D^2 \times S^1$ where ϕ send ∂D^2 to $p\mu + q\lambda$.
Heegaard diagrams can encode not just manifolds, but knots inside those manifolds as well. Take two points $w, z \in \Sigma \setminus \alpha \setminus \beta$. The complement of the αs is connected, so we cann draw a curve between z, w in $\Sigma \setminus \alpha$ and similarly there is a path from w to z in $\Sigma \setminus \beta$. To get an embedded curve we push the first arc "down" and the the second arc "up" into the corresponding handlebodies.
Given a knot in S^3, project it onto a plane. Take a small neighborhood of this projection in $\mathbb{R}^3$. Take the regions of the projection to be the alpha curves, i.e. the boundary of the intersection of this thickening with $\mathbb{R}^2$ except for the "outside" large curve. The β curves are attached to the crossings. This will give us $g - 1$ many β curves. The last β cuve will be a meridian, with z and w on the two sides of it.

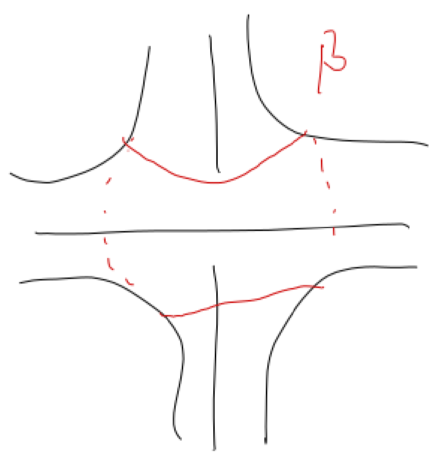


Figure 1: How to force the crossings with the β curves

Given $Y \supset K$ given by a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$ and associate to it

$$(Sym^g(\Sigma), \times\alpha, \times\beta, \{z\} \times Sym^{g-1}(\Sigma), \{w\} \times Sym^{g-1}(\Sigma)).$$

The product of the α and β curves gives us tori in the manifold $Sym^g(\Sigma)$.

Homework 2.3. $Sym^g(\mathbb{C}P^1) = \mathbb{C}P^g$.

3 Third lecture

Note, that if $\alpha \pitchfork \beta$, then the tori $\times\alpha \pitchfork \times\beta$ as well. We define from this data $\hat{CF}$ to be the vector space over $\mathbb{Z}_2$ generated by the (finitely many) points $T_\alpha \cap T_\beta$. The module CF^- is generated by the same set over $\mathbb{Z}_2[u]$, and a third one $\mathbb{CF}$ generated over the power series ring $\mathbb{Z}_2[[u]]$.

We specify now the boundary map, defined to be a module morphism with the coefficients $< \partial x, y >$ for $x, y \in T_\alpha \cap T_\beta$.
Suppose that M is a 2n-dimensional closed oriented manifold, L_1, L_2 two transversely intersecting n-dimensional submanifolds.

Definition 3.1. $\omega \in \Omega^2(M)$ is *symplectic* if $d\omega = 0$ and ω^n is a volume form for M.

Example 3.2. $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, ..., x_n, y_n$. We can take $\omega = \sum dx_i \wedge dy_i$ to be a symplectic form. In fact locally every symplectic manifold has this form!

Assume L_1, L_2 to be *Lagrangian*, which means that $\omega|_{L_i} = 0$. Note, that $[\omega] \in H^2(M; \mathbb{R})$ and also $[omega] \in H^2(M, L_1 \cup L_2; \mathbb{R})$.
The idea of Floer is to associate a chain complex to (M, L_1, L_2, ω), generated by $L_1 \cap L_2$, analogously to Morse homology. Take $V = \{\gamma : I \to M | \gamma(0) \in L_1, \gamma(1) \in L_2\}$, this will be our manifold. $T_\gamma V$ consists of vector fields along γ so that $v(0) \in TL_1$ and $v(1) \in TL_2$. For the Morse function, assume $\omega = d\alpha$ for some form α (allowing M to be non-compact, since this cannot happen in the compact case) i.e. (M, ω) is *exact symplectic*. Further assume that $L \subset (M, \omega)$ is an *exact Lagrangian*, i.e. $\alpha|_L = df$. $\mathcal{A} : V \to \mathbb{R}$ will be $\gamma \mapsto f_1(\gamma(0)) - f_2(\gamma(1)) + \int_I \gamma^*(\alpha)$.

- The critical points of $\mathcal{A}$ are in 1-1 correspondance with $L_1 \cap L_2$ (the constant paths at the intersections)
- We have to figure out gradient trajectories, so fix a metric on M, which induces a "metric" on V by $< v_1, v_2 >_V = \int_I < v_1(s), v_2(s) >_M ds$

Remark 3.3. Given ω and a metric on a manifold plus a compatibility condition we get an almost complex structure J on M as well.
Secondly the gradient flow equation $\mathbb{C} \supset \mathbb{R} \times I \to (M, \omega, L_1, L_2, g, J)$ says that u has to be pseudo-holomorphic.

Now we leave the motivation behind and return to our original Lagrangian Floer setup. Given (M, L_1, L_2, ω) choose an a.c.s. J such that $\omega(X, JY)$ is a metric. Then consider $CF(M, L_i, \omega, J) = \oplus R_{L_1 \cap L_2} < x >$ where the ring R is fixed. Then ∂ will be an endomorphism of this ring which should count

$$\#\{u : \mathbb{R} \times I \to M : \lim_{t \to \infty} u = x, \lim_{t \to -\infty} u = y, u(0, t) \in L_1, u(1, t) \in L_2, du \circ i = J \circ du.$$

But! there is an $\mathbb{R}$ action on these maps, which we mod out by. Moreover, forgetting about the pseudoholomorphic part of the requirements the maps satisfying the boundary conditions fall into homotopy classes. We prefer to consider the u maps to have domain the unit disk with two points removed from the boundary $D^2 \setminus \{(0, 1), (0, -1)\}$, since this is conformally equivalent to the strip.

Remark 3.4. The dimension of $\mathcal{M} = \{u : \mathbb{R} \times I \to M | ..., [u] = \phi\}$ is well defined, where $\phi \in \pi_2(M, L_1, L_2)$ is fixed.

Remark 3.5. There really are multiple homotopy classes,we can connect sum with a sphere representing some homology class.

Definition 3.6. $\dim \mathcal{M} = \mu(\phi)$ is called the *Maslow index*.

Now we can define the boundary map

$$\partial x := \sum_{y} \sum_{\phi:\mu(\phi)=1} \#\{u : \mathbb{R} \times I \to M | lim..., PDE, [u] = \phi\}/\mathbb{R} \cdot y.$$

We apply this to $Sym^g(\Sigma_g) = M$ for $g = 1$ we know that it has no second homotopy, so its easy to see that no sphere mentioned in Remark 3.5 exist. $Sym^2(T^2)$ is a torus bundle over the torus, applying this one can get that $Sym^2(\Sigma_2) = T^4\#\overline{\mathbb{C}P^2}$. This has second homotopy $\mathbb{Z}^\infty$. After that $Sym^g(\Sigma_g)$ has second homotopy $\mathbb{Z}$.

Compactness was problematic even in the Morse-theory case, here more things can go wrong, not just one type of broken trajectories. We can have 1-dimensional submanifolds of the strip, either from one boundary component to the other, or from one boundary to itself, or a circle inside the strip. The first type will correspond to broken trajectories. The second picture corresponds to "boundary bubbles", i.e. discs with bondary completely on one of the Lagrangians, and the third one corresponds to the far away spheres of Remark 3.5.

Floer's theory was that if $\pi_2(M) = 0$ (no sphere bubbles) and $\pi_2(M, L_1) = \pi_2(M, L_2) = 0$ (no boundary bubbles), then $\partial^2 = 0$ and we get a well defined chain complex. In our case these assumptions don't apply, but everything still works out. The sphere bubbles are ruled out by a Maslow index argument, and the boundary bubbles will come in pairs, because the Lagrangian tori are positioned "symmetrically".

What we get this way is a rather simplistic homology, its always the same for integer homology spheres, so we need to use the divisor V_w as well. So in the $\hat{CF}$ boundary map we only count pseudoholomorphic discs which are disjoint from V_w, and in the CF^- flavor we allow intersections with V_w but record it in the u variable.

For invariance, consider two different Heegaard diagrams coming from (Σ, α, β) and (Σ, α, γ), so we have three Lagrangian tori now. We want to specify a map $CF(L_1, L_2) \otimes CF(L_2, L_3) \to CF(L_1, L_3)$. The second term we hope will be simple enough to follow. Until now we counted "bigons", polygons with two sides, one side in L_1, the other in L_2. Now we want to use triangles, three points are fixed on the boundary of the circle. Since three points of a pseudoholomorphic map from the disc determines it, we will consider Maslow index 0 representatives from each relative homotopy class.

This is a chain map! We need to understand the types of degenerations once again. Boundary bubbles, sphere bubbles, but now a triagle can limit to a triangle and a bigon as well.

This "multiplication" we got looks intriguing. Is it associative? For this we have to take another step. The two different orders of multiplication gives different elements, but they will be chain homotopic! Now we take four lagrangians, but the squares have expeced dimension -1, so we have to use a family of a.c.s.'s.

4 Fourth lecture

Theorem 4.1 (Lefschetz fixed point)**.** *The number of fixed points of a map $f : X \to X$ can be expressed in terms of the Lefschetz number $\#Fix(f) = \Lambda_f = \sum_0^n (-1)^i Tr f_*|_{H_i, \mathbb{R}}$.*

If $f \sim id$, then $\Lambda_f = \chi(X)$. For $X = T^2$ we do indeed have a fixed point free map by rotation.

Arnold noticed that if (X, ω) is symplectic, and $\phi : X \to X$ is a Hamiltonian diffeomorphism homotopic to the identity, then $\#Fix(f) \geq \sum_0^n b_i$.

Remark 4.2. Given a family of functions $H : \mathbb{R} \times X \to X$ we get a vector field X_t such that $\iota_{X_t} = dH_t$. We take $\phi_t : X \to X$ a family of diffeomorphisms such that $\phi_0 = id_X$ and $\frac{d\phi_t}{dt} = \phi_t^*(X_t)$. ϕ is Hamiltonian if it is ϕ_1 for some such system.

The idea is that fixed points are intersections of $\Gamma(\phi) \subset X \times X$ with the diagonal $\Delta \subset X \times X$. If X, ω is symplectic, so is $X \times X, \omega \times -\omega$. Δ is Lagrangian in this setup, and so is the graph of a Hamiltonian. These two submanifolds are isotopic as well, that is why we take the chain complex generated by the intersection points. We need to define δ on this complex in some way and hope, that the homology of this chain complex is isomorphic to the homology of Δ. Floer considered the case when $\pi_2(X) = 0, \pi_2(X, \Delta \cup \Gamma_\phi) = 0$, in this case no degenerations can happen and the homology works.

In the Heegaard Floer case the two tori are non-isotopic, a diffeomorphism exchanges them, but not a Hamiltonian isotopy. This helps us avoid the degenerations. (Σ_g, ω), being area preserving is all there is in 2 dimensions, but in 4D this is not the case. Taking $\Sigma_g \times ... \times \Sigma_g$ with the form $\omega + .. + \omega = \eta$ we get a symplectic form, but how do we get a symplectic form on the T_α, T_β curves? we want to factor with the symmetric group, the action is not free unfortunately. We discussed that the space $Sym^g(\Sigma_g)$ is smooth despite the action being not free, but η/S_g will be singular!! We could try to work upstairs, or find another strategy. Notice, that the T_α, T_β tori are disjoint from the diagonal, where the singularities are. The another strategy comes from

Theorem 4.3 (Varouchas). *$\tilde{X} \to X$ a branched cover of complex spaces and $\tilde{\omega}$ a Kähler form on $\tilde{X}$. Then there is a Kähler form ω on X which is $\tilde{\omega}$ away from an open set containing the branch locus.*

"The symplectic camel cannot pas through the eye of a needle." I.e. is there a symplectic embedding ϕ such that $\phi(B_o^{2n}(R)) \subset B_0^2(r) \times \mathbb{R}^{2n-2}$? Gromov's theorem states, that this can only happen if $r \geq R$ for $n \geq 2$!

4.1 Surgery exact triangle

Suppose Y is a 3-manifold as before (closed oriented connected). We represent it by $H = (\Sigma_g, \alpha, \beta, w)$, and get $CF(H), \partial$, whose homology is $H_*(CF(H))$.

Theorem 4.4 (Ozsváth,Szabó). *$\partial^2 = 0$ and $HF(H)$ is an invariant of Y.*

We made a bunch of choices along the way, the symplectic form on the symmetric product, the almost complex structure. We need to show invariance from all of these, but the Heegaard diagam is also non-unique. Isotopy is simple by the Floer homology package, but handle slides and stabilisations are highly non-obvious, the genus changes!

Remark 4.5. The Lens space $L(p, q)$ is just $-p/q$ surgery on the unknot. We get $\hat{HF}(L(p, q)) = \mathbb{Z}_2^p$. We pick a toroidal Heegaard diagram, with the slope 0 and the slope p/q curve, there will be no holomorphic discs.

Theorem 4.6. *Suppose Y is a closed oriented connected 3-manifold and $K \subset Y$ with a framing f. Then we can relate $Y, Y_f(K)$ and $Y_{f+\mu}(K)$. There are maps F_1, F_2, F_3 so that the triagnle $\hat{HF}(Y) \to \hat{HF}(Y_f(K)) \to HF(\hat{Y}_{f+\mu}) \to \hat{HF}(Y)$ is exact.*

Example 4.7. Take n surgery on the trefoil. We claim, that $\hat{HF}(S^3_n(3_1)) = \mathbb{Z}_2^n$ once $n \geq 1$. We give the argument for $n \geq 5$. Apply the exact sequence. For S^3 we get $\mathbb{Z}_2$. Note, that $S^3_5(3_1) = L(5, 1)$, so we know two of the spaces. F_1 can either be 0 or an injection. In the former case we have to get $\mathbb{Z}_2^6$, since it has to surject onto $\mathbb{Z}_2$, and be surjected onto from $\mathbb{Z}_2^5$.

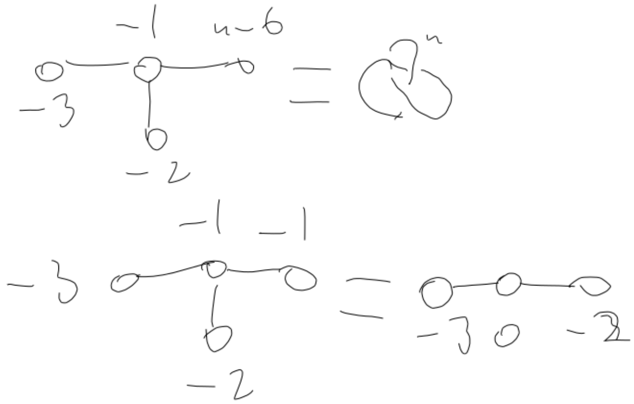


Figure 2: How 5-surgery on the trefoil is a lens-space

If F_1 is an embedding, then a 4-dimensional vector space injects into our space, and $F_3 = 0$, so we could have $\mathbb{Z}^4$ in principle, but the following excludes this possibility:
Fact is, that $HF\hat{(}Y, s)$ is of odd dimension, indeed it admits a $\mathbb{Z}_2$ grading such that $\chi(\hat{HF}(Y, s)) = 1$.

To show isomorphism, given two chain complexes $C_i\partial_i$ if there is a chain map f between them. Over the field of two elements this means that $f \circ \delta_1 + \partial_2 \circ f = 0$ we want a map g going the other way, such that $f \circ g$ and $g \circ f$ is homotopic to their respective identities.
For exact triangles things become more complicated.

Proposition 4.8. *If $C^i\partial^i$ are three chain complexes, $f^i : C^i \to C^{i+1}$ are chain maps, and $h^i : C^i \to C^{i+2}$ are module maps (all indicies are to be understood modulo 3) such that $\partial^{i+2} \circ h^i + h^i \circ \partial^i = f^{i+1} \circ f^i$. Then $\phi^i = h^{i+1} \circ f^i + f^{i+2} \circ h^i$ are chain maps. Suppose that the ϕ^i are chain homotopic to id_{C^i}. This implies that the homological triangle given by the $H(f^i)$ is exact.*

Proof. First we check that ϕ^i is a chain map. We need to see that $\phi^i\partial^i + \partial^i\phi^i = 0$, see Fig 3
Next we need inclusions between the kernels and images between the maps induced on homology. The fact that $f^{i+1}f^i = 0$ is an assumption. For the other inclusion we take an element b such that $\partial^{i+1}(b) = 0$ and $f^{i+1}(b) = \partial^{i-1}(c)$. Take $a = h^{i+1}(b) + f^{i-1} \in C^i$. We need to prove, that $[f^i(a)] = [b]$.

□

5 Fifth lecture

If we relax the requirement from ϕ^i being chain homotopic to id_{C^i} to it juts being a quasi-isomorphism (i.e. $H(\phi^i)$ is an isomorphism, not neccesarily the identity).

Definition 5.1. Two chain complexes C, C'' are quasi isomorphic if there is a C'' and maps $f : C'' \to C, f' : C'' \to C'$ such that f, f' are quasi isomorphisms.

Proposition 5.2. *If $0 \to C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \to 0$ is a SES of projective modules, then it induces an exact triangle.*

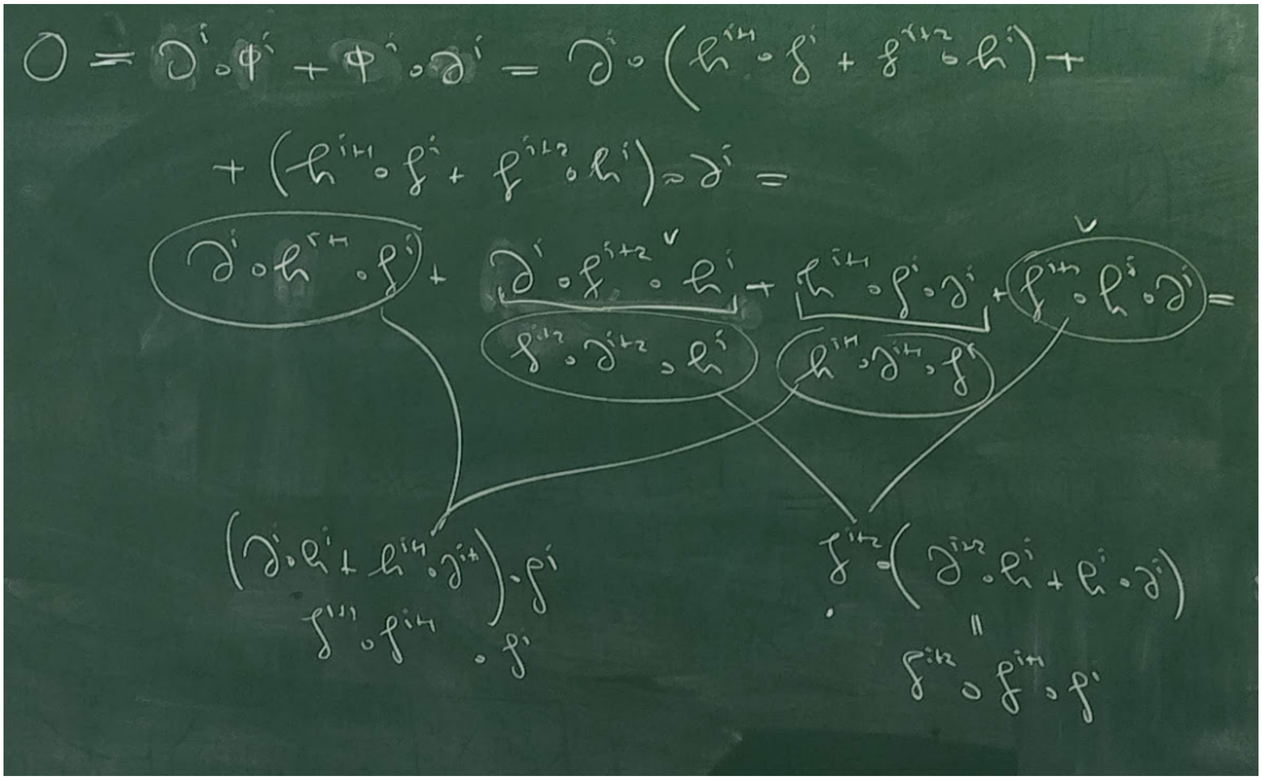


Figure 3:

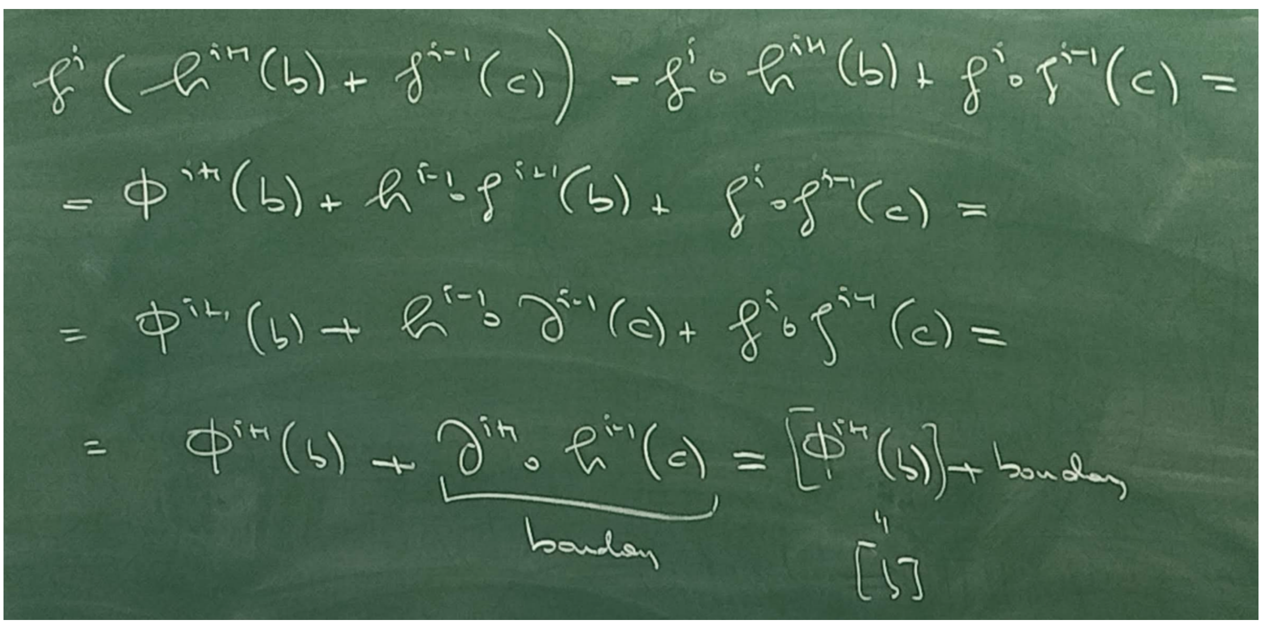


Figure 4:

Definition 5.3. P is projective if for any surjective map $\psi : M \to N$ and $\phi : P \to N$ there is a lift $\phi' : P \to M$ so that $\psi\phi' = \phi$.

Proof. C^2 surjects to C^3, so we can lift the identity of C^3 to $R : C^3 \to C^2$. Consider $\partial^2 R + R\partial^2 : C^3 \to C^2$.

This actually maps to $ker(f^2)$which by exactness and injectivity of f^1 is C^1. Thus we call this map f^3. Since f^2 is a chain map, if we compose, we get $2\partial^3 = 0$. $f^1 f^3 = \partial^2 R + R\partial^2$ by construction. We want, f^3 to be a chain map. $f^1(f^3\partial^3 + \partial^1 f^3) = f^1 f^3\partial^3 + f^1\partial^1 f^3 = (\partial^2 R + R\partial^3)\partial^3 + \partial^2 f^1 f^3 = \partial^2 R\partial^3 + \partial^2(\partial^2 R + R\partial^3) = 2(\partial^2 R\partial^3) = 0$. Since f^1 is injective, we get that f^3 is a chain map. We choose $h^1 = 0, h^2 = R$ to use the previous proposition, and now we claim, that $f^2(id_{C^2} + Rf^2) = 0$, this is trivial, since by the definition of R it equals $2f^2$. So $Im(id_{C^2} + Rf^2) \subset ker(f^2) = Im(f^1)$, so there is a map $h^2 : C^2 \to C^1$ where $f^1 h^2 = id_{C^2} + Rf^2$ by the projectivity of C^2.
We need to check the three relations for the h^i. Since $h^1 = 0$, we need $f^2 f^2$ to be zero, which is true by exactness. For the second one $f^1 f^3 = \partial^2 R + R\partial^2$, which is true by definition. Finally $\partial^1 h^2 + h^2\partial^2 = f^3 f^2$. Apply f^2 agian to get $f^1(\partial^1 h^2 + h^2\partial^2) + f^3 f^2) = \partial^2 + \partial^2 h^3 Rf^2 + f^1 h^2\partial^2 + (\partial^2 R + R\partial^3)f^2$...
Finally we need $h^{i+1} f^i + f^{i+2} h^i = id_{C^i}$. □

Pick a Heegaard diagram for Y, K, denote it $(\Sigma, \alpha, \beta, z, w)$. We can always assume, that the two points are on the two sides of a given β circle. This can be achieved by stabilisation, the meridional circle will separate the two points, and the α curve is chosen so that it is in the longitudional direction, and connect the two endpoints such that it only intersects β curves (since the complement of the α's is connected this can be done). By sliding the β curves over the new meridional β, we see that this is indeed a stabilisation. $(\Sigma_{g+1}, \alpha \cup \alpha_{g+1}, \beta) = S^3 \setminus \nu(K)$. This gives a Heegaard diagram for the knot complement, if we replace β_{g+1} with another curve winding around the knot f times, and interseting β_{g+1} once, we get the diagrams for the manifolds in the surgery triangle. For $f + \mu$ we do one extra twist.
We need to perturb the curves so that our tori become transverse, and any $\beta_i, \bar{\beta}_i$ intersect at to points without triple intersections and so on. We get $Sym^{g+1}(\sigma), T_\alpha, T_{\beta^1}, T_{\beta^2}, T_{\beta^3}$. We discussed that there is a map $CF(T_\alpha, T_{\beta^1}) \otimes CF(T_{\beta^1} T_{\beta^2}) \to CF(T_\alpha, T_{\beta^2})$ where

$$x \otimes y \mapsto \sum_z \sum_{\phi \in \pi_2(x,y,z), \mu(\phi)=0} \#\mathcal{M}(x, y, z)z$$

counting holomorphic triangles. We want to restrict this map to a specific $t \in CF(T_{\beta^1} T_{\beta^2})$. The manifold Σ, β^1, β^2 is none other than $\#_g S^2 \times S^2$ and the homology of CF will be isomorphic to $H_*((S^1)^g; \mathbb{Z}_2)$. In the chain complex the boundary map is zero, since there are two discs from a generator to its pair, and none between the other ones. We pick one intersection point on each curve, distinguished by the orientation, from which there are discs, and the intersection of the $g + 1$st curves.
By looking at degenerations of a triangle, one sees that this will be a chain map. For the h^i's we need the triple map $Y \otimes \#_g S^1 \times S^2 \times \#_g S^1 \times S^2 \to (\Sigma, \beta^3, \alpha)$. We count holomorphic rectangles, they have a one parameter family of holomorphic structures, we set $\mu = -1$, so for a one parameter family we see some discrete set of solutions. We consider degenerations of squares now, no sphere bubbles, boundary bubbles come in pairs, and two types of "bowtie" degenerations, into two triangles.

6 $\mathcal{A}_\infty$ algebras

6.1 Historical motivation

Start with (X, x_0) pointed topological space and let denote ΩX the loop space, i.e. maps $S^1 \to X$ such that $1 \mapsto x_0$. This spaces comes with an operation m_2, that is the concatenation of two loops.

Remark 6.1. This operation is *not* associative! This is only a technicality though, either the first or the last loop takes up half of the circle, these maps are homotopic.

We denote this homotopy by m_3 connecting $(\gamma_1 * \gamma_2) * \gamma_3$ with $\gamma_1 * (\gamma_2 * \gamma_3)$. We use binary trees to encode the order of the operations.
What happens if we consider 4 loops? Now we have more different ways in which we can concatenate them. We call the pentagon inside K_4, and $m_4 : K_4 \times (\Omega X)^4 \to \Omega X$ is a homotopy between all of them.

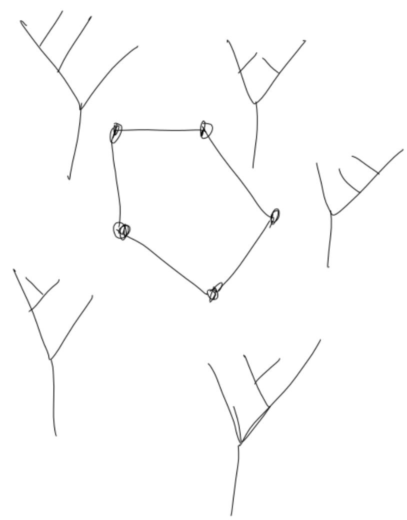


Figure 5: The 5 different ways we can put parenthesis.

In the $n = 5$ case we have a polytope K_5, with 6 pentagonal and 3 square sides.

Homework 6.2. *Check the vertices.*

The pentagonal faces represent a K_4, and the square faces are products of two interwals, i.e. $K_3 \times K_3$. There are two vertices which are not part of any square side, which correspond to $(g_1 * g_2) * (g_3 * g_4) * g_5$ and symmetrically $g_1 * (g_2 * g_3) * (g_4 * g_5)$.

Theorem 6.3 (Stasheff '64)**.** *There is a sequence of polytopes K_n of dimension $n-2$, called associohedra and maps denoted $m_N : K_n \times (\Omega X)^n \to \Omega X$ that satisfies some relations.*
Moreover a topological space Y is homotopy equivalent to ΩX iff:

- *$\pi_0(Y)$ to be a group*
- *there are maps $m_n : K_n \times Y^n \to Y$ satisfying certain compatibility and unitality conditions*

More generally Stasheff defnies an A_∞ space if it satisfies this condition.

Theorem 6.4 (Stasheff '63)**.** *If Y is an A_∞ space, then $C_*(Y)$ is an A_∞ algebra where $\mu_1 = \partial$, and $\mu_{\geq 2}$ are induced by m_i.*

Homework 6.5. *K_n can be interpreted as a polytope whose vertices are triangulations of an $n+1$-gon.*

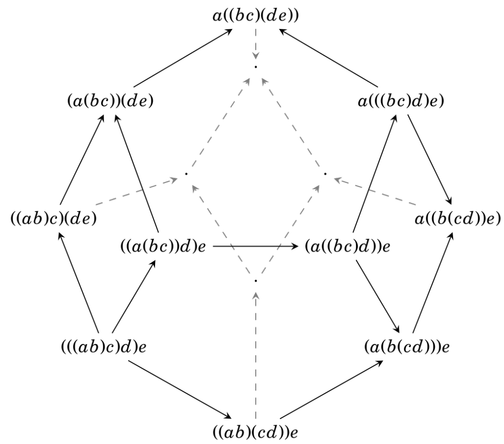


Figure 6: The associohedron from Wikipedia By Nilesj

6.2 Definition

Definition 6.6. Let k be a field. A $\mathbb{Z}$ graded A_∞ algebra over k is a $\mathbb{Z}$ graded vector space $A = \bigoplus A_p$ together with homogeneous k-linear maps $\mu_i : A \otimes_k ... \otimes_k A \to A$ subject to relations (Rn) for all $n \geq 1$.

$$Rn = (\sum_{r=0}^{n-1} \sum_{s=1}^{n-r} (-1)^{r+st} \mu_u (id^{\otimes r} \otimes \mu_s \otimes id^{\otimes t}) = 0)$$

where $r + s + t = n$, i.e. $t = n - r - s$ and $u = r + t + 1 = n - s + 1$. The μ_i are graded $2 - i$.

Remark 6.7. More generally instead of a field, we can take a commutative ring, (bi-)modules instead of vector spaces and module maps.

More generally the grading can be taken from a non-commutative group!

The action of k on A can differ on the two sides, from now on we denote the domain of μ_i by $A^{\otimes i}$, but note, that this is different from A^i!

There is a pictorial interpretation in terms of trees. Now μ_i is a tree of valence $i + 1$, i inputs are leaves "at the top" and one output "at the bottom". I.e. μ_1 is just a line with a dot in the middle, μ_2 is a Y shape with a dot at the intersction and so on. We give a graphical interpretation of the relations. We want to sum over all possible partition of the n inputs into $r + s + t$ many subsets, apply the tree corresponding to μ_s to the s subset, and the identity to the rest, and then the $\mu_{r+t+1=u}$ tree to the bottom of this tree.

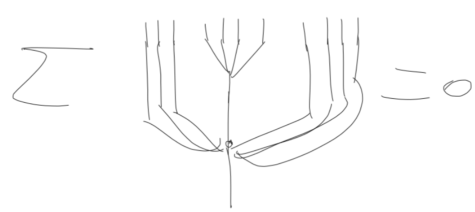


Figure 7: The relation in graphical form

Let us unpack the first few relations. $R1$ states that $\mu_1 \circ \mu_1 = 0$. The second relation can be read off to be $\mu_2 \circ (\mu_1 \otimes id + id \otimes \mu_1) + \mu_1 \circ \mu_2 = 0$, i.e. $\mu_2(\mu_1(a), b) + \mu_2(a, \mu_1(b)) = \mu_1\mu_2(a, b)$, which looks like some sort of Leibniz rule for the differential μ_1 and the "multiplication" μ_2. The third relation states that

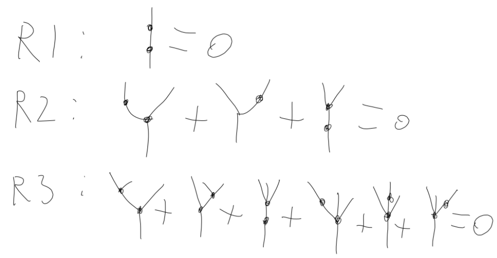


Figure 8: The first three relations in tree form

$\mu_2(\mu_2 \otimes id) + \mu_2(id \otimes \mu_2) = \mu_1\mu_3 + \mu_3(\mu_1 \otimes id^2 + id\mu_1 id + id^2\mu_1)$, here if we notice that $\mu_1 \otimes id^2 + id\mu_1 id + id^2\mu_1$ is the natural differential on $A^{\otimes 3}$, we see that μ_3 is a chain homotopy between the two different orders of the multiplication. This means, that our product is associative up to homotopy.
(A less popular pictorial repreesntation is splitting the $i+1$-gon with diagonals.)

Example 6.8. For A_∞ algebras. Firstly from Stasheff '63 $C_*(\Omega X)$ with $\mu_1 = \partial, \mu_2$ induced by loop composition, and the higher operations induced by the m_i. The associohedra encode the relations.

Example 6.9. The second example comes from deformation theory. Let B denote some k-algebra. The Hochschild cochain complex is $Hom(k, B) \to Hom(B, B) \to Hom(B^{\otimes}2, B) \to ...$ with differential $d(f)(b_0, ..., b_n) := b_0 \cdot f(b_1, ..., b_n) \sum f(b_0, ..., (b_{i-1} \cdot b_i), ..., b_n) + f(b_0, ..., b_{n-1}) \cdot b_n$, fact is that this is a differential, and it gives us the Hochschild cohomology of B. Consider a formal variable ϵ of degree $2 - N$ for some fixed $N \neq 2$ and we define $A := B[\epsilon]/(\epsilon^2), c : B^{\otimes n} \to B$ any linear map. From this we define higher maps μ_i by setting μ_2 to be induced by the multiplication of B and $\mu_N := \epsilon \cdot c$ and all other μ_i's zero.

Homework 6.10. *Check that this is an A_∞ algebra iff the linear map c is a Hochschild cocycle. (hint: The only relation to be checked is $R(N+1)$).*

Example 6.11. Differential graded algebras (DGA). This object is an A_∞ algebra with $\mu_i = 0$ for all $i \geq 3$.

6.3 Homology

Proposition 6.12. *Given a $\mathbb{Z}$ graded A_∞ algebra A there is a canonical algebra structure on $H_*(A)$.*

Proof. $\mu_2 : A \otimes A \to A$ satisfies $R2$, which means, that μ_2 is a chain map, so it descends to homology, and as we discussed. Taking the map $H(a) \otimes H(A) \to H(A \otimes A) \to H(A)$ and denoting it $\bar{\mu}_2$ we have a multiplication. $R3$ then tells us, that $\bar{\mu}_2$ is associative. □

Remark 6.13. $\bar{\mu}_1 = 0$ implies, that $\bar{\mu}_2$ is associative, which is the "wrong" reason for it to be asociative, since μ_3 should be the map measuring the non-associativity of μ_2. This means, that there can be an A_∞ structure on homology, but it will not be canonical.

6.4 Units

Definition 6.14. A *strict unit* of an A_∞ algebra is an element $1 \in A$ such that $\mu_2(a, 1) = \mu_2(1, a) = a$ for all $a \in A$, and $\mu_i(a_1, ..., a_i) = 0$ whenever any of the a_j's are 1 and $i \neq 1$ (in particular $\mu_1(1) = 0$).

Definition 6.15. A *homological unit* is $1 \in H_*(A)$ such that it is a unit of $\bar{\mu}_2$.

Theorem 6.16. *If A is homologically unital then there exists an A_∞ quasi-isomorphism with an A_∞ algebra B (with $\mu_1^B = 0$) such that B is strictly unital.*

7 Eight lecture

7.1 Modern Perspective

More motivating examples, firstly Bordered floer homology. Given Y^3, s we associate $\hat{HF}(Y, s)$ to it, a $\mathbb{Z}_2$ vector space, we wish to compute it. To do this you need a presentation of Y as a Heegaard diagram $(\Sigma, \alpha, \beta, z)$, and count holomorphic discs. For Bordered Floer homology we wish to compute the invariant locally, i.e. break the surface in two. The correct invariant turns out to be an $\mathcal{A}_\infty$ algebra A_T corresponding to the cut, which is in fact a dga. The two parts will give us an A_∞ module denoted M_{A_T}, and a type-D structure over A_T denoted ${}^{A_T}N$.

Theorem 7.1 (Lipshitz Ozsváth Thurston)**.** *$\hat{CF}(H)$is quasi isomorphic to $M_{A_T} \boxtimes^{A_T} N$*

Secondly Hanselmann Rasmussen Wattson's immersed curve invariant. Let Y^3 be a compact oriented 3-manifold with torus boundary (e.g. a knot complement). Pick a parameterasation of the boundary T^2. The previous theorem gives a type-D structure on A_T, N.

Theorem 7.2 (Hanselmann Rasmussen Wattson)**.** *N can be geometrically interpreted as a collection of curves immersed in the torus (with local systems).*

Example 7.3. For the trefoil complement the lift of such a curve on the torus is depicted on the figure.

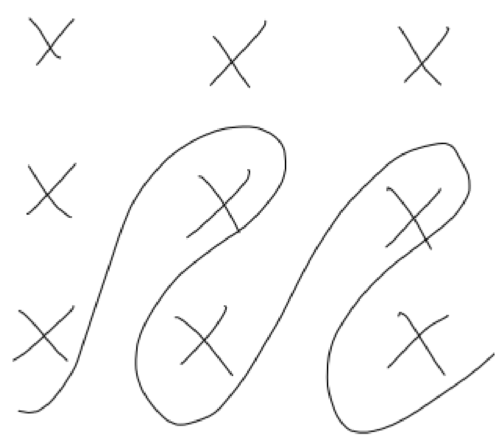


Figure 9: The curve corresponding to one of the trefoils on the universal cover $\mathbb{R}^2$.

The third example is bordered knot floer homology. Given a knot projection, slice it up such that every slice contains either one critical point of the height function or a single crossing. To each regular section they associate a dga and to each neighboring slice pair they associate a DA-bimodule.

Theorem 7.4 (Ozsváth-Szabó). *$C\hat{F}K$ is quasi isomorphic to $M_1 \boxtimes ... \boxtimes M_n$*

This definition is very efficient for computation.

7.2 The Bar Construction

Recall, k is a commutative ring for today. An A_∞ algebra over k is a k-bimodule A together with k-linear maps $\mu_i : A^{\otimes i} \to A$ for all $i \geq 1$ subject to structure relations R_n for $n \geq 1$.

$$R_n = \left(\sum \mu_{n-i+1}(id \otimes \mu_i \otimes id) = 0\right)$$

Definition 7.5. A a k-bimodule. The *bar construction* is $TA := k \oplus A \oplus A^{\otimes 2} \oplus ....$ There is an obvious algebra-structure as the tensor algebra of A, the unit map is the linear extension of the map sending the unit of k to its image in TA.

Definition 7.6. A counital coassociative coalgebra over k is a tuple (C, Δ, ϵ) where C is a k-bimodule, $\Delta : C \to C \otimes C$ is the comultiplication and $\epsilon : C \to k$ is the counit, subject to two relations:

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$ (coassociativity)

2. $(\epsilon \otimes id) \circ \Delta = id_C = (id \otimes \epsilon) \circ \Delta$ (counit)

Definition 7.7. The *deconcatenation comultiplication* on TA is a map $\Delta : TA \to \boxtimes TA$ given as $a_1 \otimes ... \otimes a_n \mapsto \sum_i a_1 \otimes ... \otimes a_i \boxtimes a_{i+1} \otimes ... \otimes a_n$ (the empty tensor product is 1).

Example 7.8. The image of $A^{\otimes n}$ is in $k \boxtimes A^{\otimes n} \oplus ... \oplus A^{\otimes n} \boxtimes k$.

Proposition 7.9. *(TA, Δ, ϵ) is a coalgebra, where $\epsilon : TA \to k$ is projection onto the first factor.*

Homework 7.10. *Prove this.*

Remark 7.11. TA with the tensor product and Δ is not a bialgebra (in particulra not a Hopf algebra).

Remark 7.12. There exists another comultiplication called the *standard comultiplication* on TA, which makes it a bialgebra (Hopf algebra even).

Remark 7.13. There is another multiplication on TA, that makes it, together with deconcatenation, that makes it a bialgebra (also a Hopf algebra).

7.3 Reduced bar construction

Definition 7.14. An *augmentation* of a unital algebra (A, ∇, η) is a map $\epsilon : A \to k$ such that $\epsilon \circ \eta = id_k$. Dually a coaugmentation of a counital coalgebra (A, Δ, ϵ) is a map $\eta : k \to A$ such that $\epsilon \circ \eta = id_k$. Coaugmented counital coalgebras are in bijection with non-counital coalgebras. Concretely $A \mapsto \bar{A} = A/im(\eta) = ker\epsilon$, and in the other direction $\bar{A} \mapsto k \oplus \bar{A}$. Moreover this is an equivalence of categories.

Remark 7.15. TA is naturally coaugmented by taking the inclusion of k inside TA, so uner the above equivalence $\overleftarrow{TA} = A \oplus A^{\otimes 2} \oplus ...$ with a comultiplication, which is no longer counital denoted $\bar{\Delta}$, called strict deconcatenation (leave out the $1 \boxtimes ...$ and $... \boxtimes 1$).

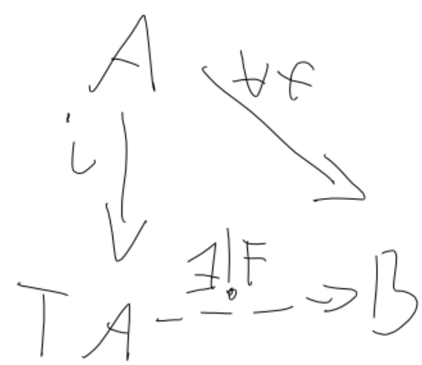


Figure 10:

7.4 Universal properties

The goal is to recast A_∞ algebras and relations as a map $M : TA \to TA$. Warmup:

Theorem 7.16. *For every k-linear map f from A to some B a unital associative algebra there is a unique map of unital algebras $F : TA \to B$, such that $F \circ i = f$ where i is the inclusion of A into TA. F is defined as the sum of $f^{\otimes i}$*

Proof. For uniqueness unitality of F forces the definition on $k \subset TA$. Commutativity of the diagram forces the definition of F on $A \subset TA$. Lastly since it is a homomorphism, it forces the definition of F on $A^{\otimes 2}$, since $F(v \otimes w) = F(v) \otimes F(w)$, and proceed inductively.
For existence define F by the formula and check the properties. □

Theorem 7.17. *Coalgebra structure of TA, we only prove for bar algebras. The map F is defined as a sum*

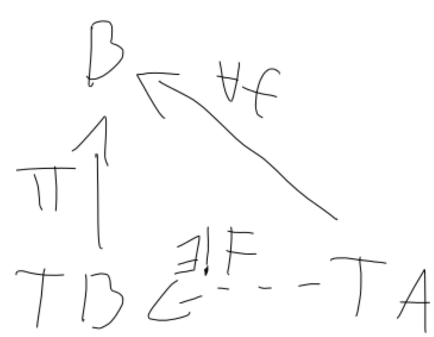


Figure 11:

of trees with some inputs combined by f for allpossible trees with n inputs and m outputs, π is the natural projection of TB, f is k-linear, F is a counital morphism of coalgebras.

Proof. Uniqueness: counitality forces the definition $F_0 = \pi_k \circ F$.
Commutativity forces the definition of $F_1 = \pi_B \circ F = f$.

Using $\Delta \circ F = (F \boxtimes F) \circ \Delta$ we show that $F_n = \pi_{B^{\otimes n}} \circ F$ is inductively determined. Consider $(TB \boxtimes TB)_n := (k \boxtimes B^n) \oplus (B \boxtimes B^{n-1}) \oplus ... \oplus (B^n \boxtimes k)$ $B^n \subset TB$ maps into this under Δ. The projections of the factors of this sum precomposed by Δ are isomorphisms. Now $F_n = \pi_{B^{\otimes n}} \circ F$ is determined by $pr_{i,n-i} \circ \pi_{B^{\otimes n}} \circ F$, which is determined by $\Delta \circ F_n = (F \boxtimes F) \circ \Delta$. Taking the output component in $B \boxtimes B^{n-1}$ we get $pr_{1,n-1} \circ \Delta \circ F_n = (F_1 \boxtimes F_{n-1}) \circ \Delta$, the right hand side is already defined for $n \geq 2$ and we are done since $pr_{1-n-1} \circ \Delta$ is an isomorphism. Thus we are done with uniqueness

For existence we define F as given and check the relations. $\pi_b F = f$ is clear. The other thing is that it is a morphism of coalgebras, i.e. $\Delta \circ F = \Delta(\sum ...)$ and the deconcatenation adds a box tensor product somwhere in between the trees, so we get the sum of all forests boxed with the sum of all forests, which is precisely $(F \boxtimes F \circ)\Delta$. □

8 Eight lecture

Definition 8.1. A *derivation* on a unital associative algebra B is a map $D : B \to B$ that satisfies

- $D \circ \eta \equiv 0$
- $D \circ \nabla = \nabla(id \otimes D + D \otimes id)$

Definition 8.2. Coderivation on a counital coassociative coalgebra C is a map $M : C \to C$ such that the dual relations hold:

- $\epsilon \circ M \equiv 0$
- $\Delta \circ M = (id \otimes M + M \otimes id) \circ \Delta$

Theorem 8.3. *Given a k-linear map $A \to TA$, then there is a unique extension of it to a map $TA \to TA$ as a unital derivation.*

Proof. Excercise. □

Theorem 8.4. *Dually, for any k-linear map $TA \to A$ there is a unique counital coderivation $TA \to TA$ that projects to A as the map.*

Proof. Excercise. □

Remark 8.5. There are universal properties for (co)derivations on $\overleftarrow{TA}$, where you drop the (co)unitality assumptions.

Definition 8.6. A derivation D is augmented if $\epsilon \circ D \equiv 0$.
A coderivation M is coaugmented if $M \circ \eta \equiv 0$.

Definition 8.7. Let A be a k-bimodule. An A_∞ structure on A is a counital coaugmented coderivation $M : TA \to TA$ that is a differential (i.e. $M \circ M \equiv 0$).

Remark 8.8. By the equivalence of categories, we could ask for just a coderivation on $\overline{TA}$.

Remark 8.9. Dropping the coaugmented condition from M we get another useful definition: curved A_∞ structures.

Proposition 8.10. *The following are equivalent*

- *An A_∞ struxture $M : TA \to TA$*
- *An A_∞ algebra $(A, \{\mu_i\})$ satisfying the Ri*

Moreover if you package the μ_i as the unique map $\mu : TA \to A$ (setting $\mu_0 = 0$) then $\pi \circ M = \mu$.

Proof. By the universal property $mu = \pi \circ M$ guarantees that μ and M determine each other. Moreover $\mu_0 = 0$ iff M is coaugmented.
Consider $M \circ M$ and 0, these are coderivations. By the universal property these two maps are the same iff the corresponding μ's are the same.
$M \circ M = \sum$ forests with two nodes, and the two nodes are on the same connected component, the other summands cancel each other.
Thus $\pi \circ M \circ M$ is the part of the sums with 1 output component and we get the A_∞ relations. □

Homework 8.11. *Find the curved A_∞ relations.*

Remark 8.12. In the literature you may find $M : T(A[1]) \to T(A[1])$, in case A is $\mathbb{Z}$ graded, this fixes the grading since μ_i has degree $2 - i$.

8.1 Morphisms

Definition 8.13. An A_∞ morphism between A_∞ algebras A, B is a counital morphism of coaugmented coalgebras $F : TA \to TB$ that is a chain map, i.e. $M_B \circ F = F \circ M_A$.

Definition 8.14. Let $F, G : C_1 \to C_2$ be counital morphisms of coalgebras. An (F, G)-coderivation is $\tilde{M} : C_1 \to C_2$ satisfying

- $\epsilon \circ \tilde{M} \equiv 0$
- $\Delta \circ \tilde{M} = (F \otimes \tilde{M} + \tilde{M} \otimes G) \circ \Delta$

Theorem 8.15. *Let $F, G : TA \to TB$ be counital morphisms of coalgebras. For every k-linear map $TA \to B$ there is a unique F, G coderivation $TA \to TB$, lifting this map.*

Proof. Exercise. We only check that $\tilde{M}$ is an (F,G) coderivation. The first condition is trivial. M is just the sum over forests with $\bar{\mu}$ somewhere, and F to the left, G to the right, the comultiplication adds a box-tensor product symbol somewhere into this sum. If this symbol is before $\tilde{\mu}$, then we get $F \boxtimes \tilde{M}$, if its after then we get $\tilde{M} \boxtimes G$, as claimed. □

Now we unpack the definition of morphisms. By the universal property of coalgebra morphisms F is determined by $\pi \circ F =: f$, which can be packaged into maps $f_i : A^{\otimes i} \to B$, F is coaugmented iff $f_0 = 0$.
$F \circ M_A$ and $M_B \circ F$ are (F, F) coderivations. This means, that they are equal iff the projections are equal.
Equating these two sums and taking homogeneous parts one gets

$$\sum_{i \in [0,n-1], j \in [1,n-i]} f_{n-j+1}(id \otimes \mu_j^A \otimes id) = \sum_{i_1 + \ldots + i_j = n} \mu_j^B (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_j}) \tag{1}$$

Look at the first few. $R1$ tells us that f_1 is a chain map from $(A, \mu_1^A) \to (B, \mu_1^B)$. Thus it descends to a map on homology. Secondly for R2 we get $f_1\mu_2 + f_2(\mu_1 \otimes id) + f_2(id \otimes \mu_1) = f_1\mu_2 + \mu_2(f_1 \otimes f_1)$ thus $\bar{f}_1$ is an algebra homomorphism.

Definition 8.16. A morphism of A_∞ algebras $f_i : A^i \to B$ is an A_∞-quasi-isomorphism if f_1 is a quasi-isomorphism in the classical sense, i.e. $\bar{f}_1 : H_*(A) \to H_*(B)$ is an isomorphism.

If $A, B$are k-bimodules, then $Hom_K(A, B)$ is also a bimodule, moreover if they are chain complexes, then the homspace is a chain complex as well with the differential $df = d_B f + f d_A$.

Remark 8.17. Chain maps $A \to B$ are cycles in this chain complex.
Two maps are homologous in $Hom_K(A, B)$ iff there exists a homotopy between them.

In particular $\mu_1|_{A^i} : A^n \to A^i$. For $F \in Hom(A^i, B)$ we get $df = \mu_1 f + f\mu_1|_{A^i}$.
Consider $\{TA \to TB : \text{k-linear counital coaugmented}\}$, which is in bijection with $Hom_K(\overleftarrow{TA}, \overleftarrow{TB})$.

Definition 8.18. Given $M_A : TA \to TB, M_B : TB \to TB$ which are A_∞ structures, and $F, G : TA \to TB$ A_∞ morphisms, an A_∞ homotopy H is a counital coaugmented (F,G) coderivation such that $F - G = M_B \circ H + H \circ M_A$.

Homework 8.19. *Unpack this definition, i.e. find the maps h_i and the A_∞ structure relations.*

Homework 8.20. *The first relation (with only 1 input) is that $f_1 - g_1 = \mu_1 h_1 + h_1 \mu_1$, ergo A_∞ homotopic A_∞ morphisms induce homotopic morphisms in their associated chain complexes.*

Theorem 8.21 (Levéfre-Hasegawa)**.** *A_∞ homotopy is an equivalence relation.*
An A_∞ quasi isomorphism always has an A_∞ homotopy inverse.

9 Ninth lecture

9.1 Chord diagrams

Consider a part of a Heegaard diagram i.e. a compact 2-manifold with boundary, with the β curves and some parts of the α curves, represented as arcs. We associate a differential algebra to this, which has two objectives. Firt in constructing the generator of $\hat{CF}(H)$ we want to remember which α curves are already occupied by some intersections on the other part of the diagram we don't see, we do this by associating idempotents. Second we need to remember how the partial domains meet at the boundary, we do this by recording strands.

Definition 9.1. A *chord diagram Z* consists of:

- a compact oriented 1-manifold P
- a finite subset $B \subset P$
- a fixed point free involution $\phi : B \to B$, called a matching

The constractible compoents of P are called linear backbones, the other ones are circular backbones.

Example 9.2.

Remark 9.3. From a chord diagram we can constuct an oriented surface with partitioned boundary $F(Z)$ by thickening up P, and attaching 1-handles according to the matching. The partition is coming from the thickening. One end of $P \times I$ is R_-, the image of the other end after attachment of the handles is called R_+, and if P has non-closed components, they give rise to the "suture" part of the boundary.

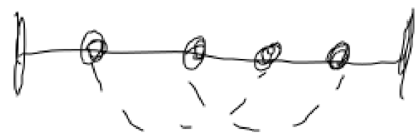


Figure 12: The standard punctured torus chord diagram

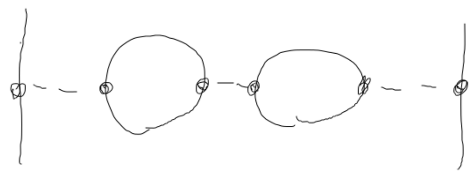


Figure 13: Motivated by Ozsváth-Szabó's bordered HFK

9.2 Pre-strands algebra

$s = \{s_1, ..., s_k\} \subset Z_s$ a collection of smooth functions $s_i : I \to P$, where the set $\{s_i(0)\}$ and $\{s_i(1)\}$ consists of k distinct points in $B \subset P$, moreover each s_i has constant non-negative constant length.

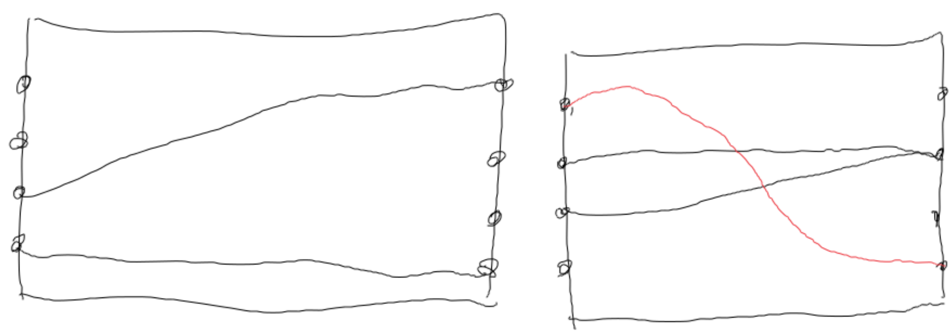


Figure 14: An example and a non-example of a strand.

Example 9.4.

Definition 9.5 (Multiplication)**.** Let $\tilde{A}(Z,k)$ be the free $\mathbb{Z}_2$ vector space generated by the $k-$strands on Z. Given s,t k-strands we define

- if $s(1) \neq t(0)$, then $st = 0$ (they are non-concatenable)
- if the concatenation contains a *bigon* after smoothing (and without rescaling!), we say $st = 0$
- otherwise st is the properly rescaled concatenation of the two strands

Example 9.6.

Definition 9.7 (Differential)**.** If s is a k-strand we degine ∂s as the sum of all k-strands obtained by smoothing a crossing without creating a bigon and properly rescaled. This extends to a linear map $\partial \tilde{A}(Z,k) \to \tilde{A}(Z,k)$.

Example 9.8.

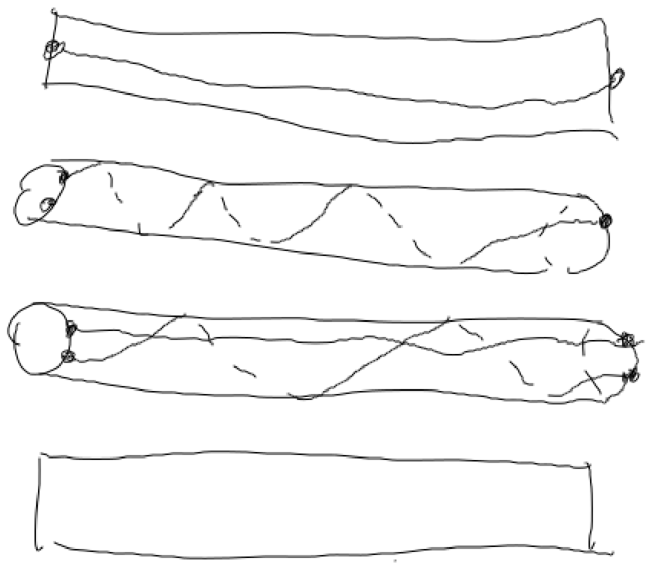


Figure 15: An example 4-strand on the HFK algebra

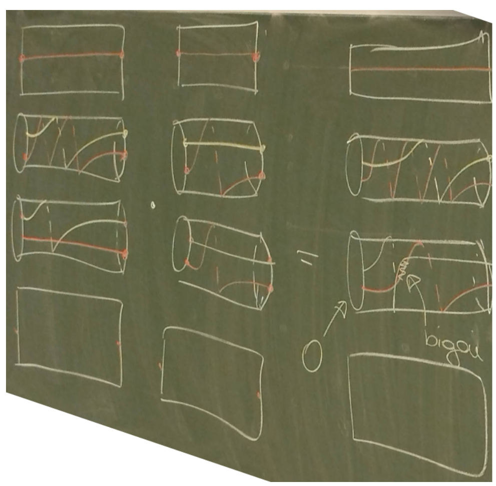


Figure 16:

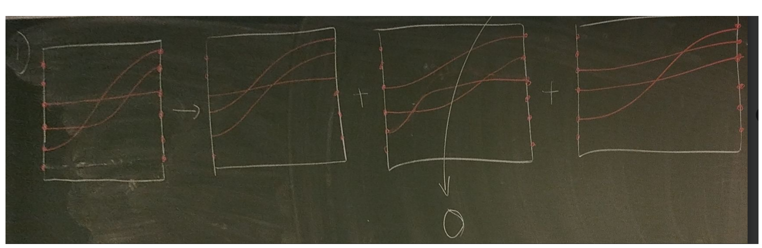
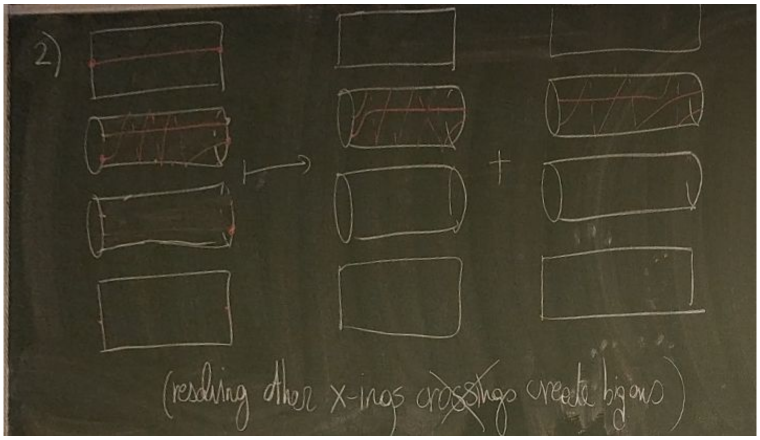


Figure 17: The boundary of a 3-strand

Example 9.8.

Lemma 9.9. $\partial^2 = 0$



Proof. If we ignore the bigon condition then we count each element in ∂^2 twice, depending on the order in which we do the two resolutions. What we need to check, is that if t contains a bigon, then $\partial t = 0$.
If t contains 1 bigon, then the two resolution at the vertices of it cancel out, if t contains at least 2 bigons, then any resolution contains bigons. □

Theorem 9.10. *$\tilde{A}(Z, k), \cdot, \partial$ is a differential algebra.*

Homework 9.11. *Check this, and find the unit.*

9.3 Strands algebras

Definition 9.12. Let s be a k-strand on Z and let I denote the set of indicies s.t. s_i is constant. For $\iota \subset I$ we define s^ι another k-strand such that

1. if $i \notin \iota$ we let $s_i^\iota = s_i$

2. if $i \in \iota$ then s_i^ι is the constant strand based at $\phi(s_i(0))$

The equaliser $E(s)$ of s is defined as $\sum_{\iota \subset I} s^\iota$ if $s(n) \cap \phi(s(n)) = \emptyset$ for $n = 0, 1$ and 0 otherwise.

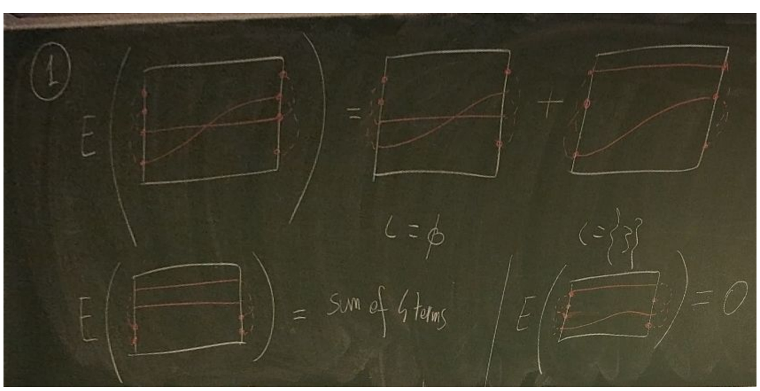


Figure 18: Equalisers

Example 9.13.

Dashed constant lines at matched means replacing the strand with the equaliser of one where we replace one (and only one) of the dashed lines with a solid strand.

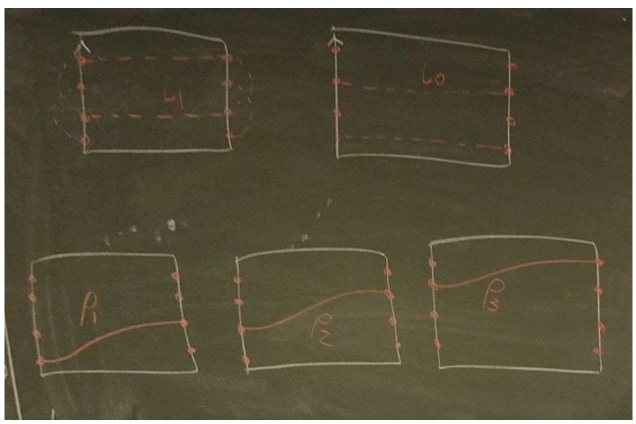
Definition 9.14. The strands algebra $A(Z,k)$ is the subspace of $\tilde{A}(Z,k)$ spanned by the Equalisers of all elements.

Lemma 9.15. *$A(Z,k)$ is closed under multiplication and differential.*

Proof. Multiplication. In $E(s)E(t)$ if there are problems with concatenation, we get zero. If $(\sum s^{\iota})(\sum t^{\iota'}) \neq 0$, then there is ι, ι' such that they are concatenable and we replace s,t with $s^{\iota}, t^{\iota'}$, since their equalisers are the same, so we can assume s,t concatenable. Expanding the product we have $\sum_{\iota \subset I \cap I'} s^{\iota} t^{\iota}$. A term of this sum contains a bigon iff st contains a bigon, since bigons cannot involve a constant strand, so the sum is equal to $\sum (st)^{\iota} = E(st)$.
Similarly one can check what happens with resolution of crossing for the differential. □

Remark 9.16. $A(Z,k)$ is not a subalgebra of $\tilde{A}(Z,k)$ because the unit is not an equaliser, but instead it has its own unit, different from the one in the free strands algebra.

Example 9.17 (The torus algebra). $A(Z,0) = \mathbb{Z}_2$ generated by the empty strand. $A(Z,1)$ is generated by 8 elemens: and the nonzero products of the ρ_i. The differential is zero, since there are no crossings.



Remark 9.18. $A(Z,1)$ is isomorphic to the path algebra of Figure 9.18

$A(Z,2)$ is generated by one idempotent ι, and $\sigma_i, \sigma_+, \sigma_{||}, \tau_-, \tau_+, \tau_\times$.
In this algebra the differential is not zero, $H_*(A(Z,2)) = \mathbb{Z}_2 = A(Z,0)$.

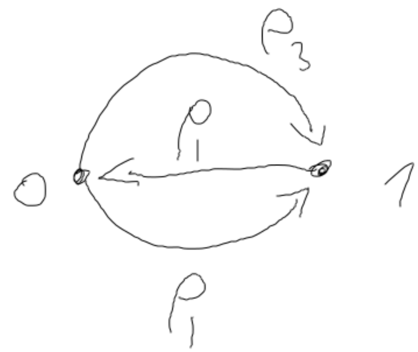
10 Tenth Lecture

10.1 Idempotents

Suppose $Z = (P, B, \phi)$ is a chord diagram, let $X \subset B/(b \sim \phi(b))$.

Definition 10.1. $I_X := E(Const_S)$ where $S \subset B$ is any lift of X.

Homework 10.2. $\{I_X : X \subset B/\phi, |X| = k\}$ *are orthogonal idempotents in* $A(Z,k)$



$$\rho_2\rho_1 = \rho_3\rho_2 = 0$$

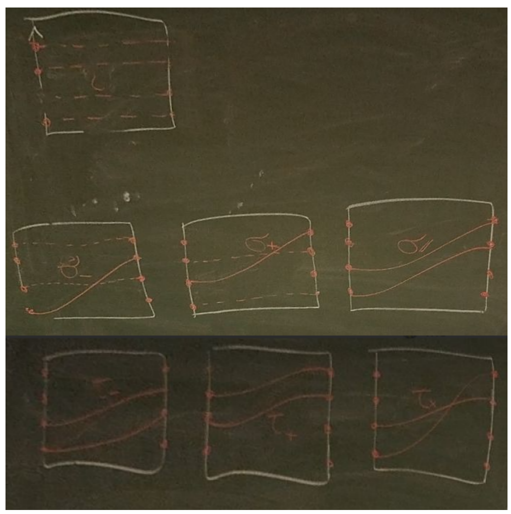


Figure 19:

Homework 10.3. *All idempotents of $A(Z,k)$form an abelian subring $I(Z,k) \subset A(Z,k)$. In fact $\{I_X\}$ is a basis of this subring, as a $\mathbb{Z}_2$ vector space.*

Definition 10.4. An idempotent $\iota \in A$ is *minimal* if it cannot be decomposed as a sum of orthogonal idempotents.

Example 10.5. $I_{0,1} + I_0$ is not minimal, but each term of this sum is minimal.

Homework 10.6. *Suppose that A is an algebra where:*

- *the idempotents form an abelian subring I*
- *$\{I_X\}$ is a basis of orthogonal idempotents.*

Then $\{I_X\}$ is the set of minimal idempotents.
In particular the set $\{I_X\}$ is determined by the algebra structure.

Definition 10.7. The unit of the strands algebra is the sum $1 = \sum_{|X|=k} I_X$.

Homework 10.8. *Verify that this is a unit.*

Theorem 10.9. $A(Z,k), \partial, \cdot, 1$ *is a unital differential algebra.*

Remark 10.10. As a vector space we have a splitting $A(Z,k) = \bigoplus_{X,Y} I_X A(Z,k) I_Y$. Moreover the differental respects the splitting. The multiplication sends $I_X A I_Y, I_Y A I_Z$ to $I_X A I_Z$, and vanishes otherwise.
In particular $\partial I_X = 0$.

Example 10.11. The torus algebra $A(Z,1)$. As seen in the previous lecture there are two idempotents ι_0, ι_1, and the other three generators ρ_1, ρ_2, ρ_3. The splitting is $\iota_0 A \iota_0 = \mathbb{Z}_2 < \iota_0, \rho_{12} >$, $\iota_0 A \iota_1 = \mathbb{Z}_2 < \rho_1, \rho_{123}, \rho_3 >$.

Homework 10.12. *Find the other two terms of the decomposition.*

10.2 DG category interpretation

We can form a category $C_{A(Z,k)}$ as follows:

- $ObC = \{I_X\}$ the set of minimal idempotents
- $Hom(I_X, I_Y) = I_X A I_Y$
- $id_{I_X} = I_X$
- composition is multiplication in the algebra

In this way we get a category where the morphism spaces are differential $\mathbb{Z}_2$ vector spaces. We call such a category a differential category.

Remark 10.13. We can also define A_∞ categories in a similar manner, but those objects will not be categories, because composition is not associative, and there are no units. Instead we get higher composition maps

$$Hom(x_{j-1}, x_j) \otimes ... \otimes Hom(x_0, x_1) \to Hom(x_0, x_j).$$

10.3 A_∞ modules

Let A be an A_∞ algebra over k (counital coassociative coaugmented coderivation M_A such that $M_A^2 = 0$). Let* $\mu = \pi \circ M_A$ and the μ_i are the various summands of μ.

Proposition 10.14 (Universal Property for Modules)**.** *Suppose X, Y are right k-modules. For any k-linear map f there is a unique lift F such that $(id \boxtimes \Delta) \circ F = (F \boxtimes id) \circ (id \boxtimes \Delta)$. In fact F is f with some extra vertical lines added.*

Proof. Uniqueness part is simple from commutativity of the diagram. The projection forces the image of F onto $Y \boxtimes k$, and the extra condition states that each projection determines the others. $F^{i \to j} : X \boxtimes A^i \to Y \boxtimes A^j$, then $\{F^{i \to 0}\}$ determines $F^{i \to j}$.
For existence we try the formula, and see that it is correct. $\square$

*$\pi : TA \to A$

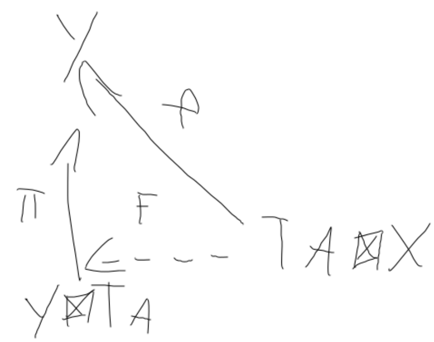


Figure 20: The universal property of modules

Definition 10.15. A *right A_∞ module* over A is a k-module X with a k-linear map $M_X : X \boxtimes TA \to X \boxtimes TA$ such that:

1. $(id \boxtimes \Delta) \circ M_X = (M_X \boxtimes id) \circ (id \boxtimes \Delta)$ (M_X commutes with the comultiplication)

2. $\tilde{M_X} := M_X + id_X \boxtimes M_A$ is a differential.

Let us unpack what this means. By the universal property of modules M_X is determined by a collection of maps $m_i : X \boxtimes A^i \to X$ for $i \geq 0$. Both $\tilde{M_X} \circ \tilde{M_X}$ satisfy the commutativity condition with the comultiplication, so they agree iff theire projections on $X \boxtimes k = X$ agree. The $(1+n)$ input relation (Rn) is

$$\sum_0^n m_{n-i}(m_i(x \otimes a_1 \otimes ... \otimes a_u) \otimes a_{i+1} \otimes ... \otimes a_n) + \sum_{i,s}^{n,n-i} m_{n-i+1}(x \otimes a_1 \otimes ... \otimes a_s \otimes \mu_i(a_{s+1} \otimes ... \otimes a_{s+i}) \otimes a_{s+i+1} \otimes ... \otimes a_n) = 0$$

Example 10.16. First few relations. $R0$ tells us that m_0 is a differential on X.
$R1$ is the Leibniz rule for the algebra action. In particular m_1 descends to a map on homology.
$R2$ measures the non-associativity of the algebra action. The difference is a boundary, and so the action is associative on homology.

Example 10.17. Let A be the torus algebra $A(Z,1)$, k will be the ring of idempotents $< \iota_0, \iota_1 >$ and $X := \mathbb{Z}_2 < x >$ with a k-action given as: $x\iota_0 = 0, x\iota_1 = x$. The only nonzero module maps are $m_1(x, \iota_1) = x$, $m_{n+2}(x, \rho_3, \rho_{2,3}, \rho_{2,3}, ..., \rho_{2,3}, \rho_2) = x$. We claim that this is an A_∞ module. We need to check the relations Ri. We only need to check strings such that you get "allowable" strings when you do Δ or μ_2. It can be of the form $(\rho_3, \rho 2, 3, \rho_{2,3}, ..., \rho_2, \rho_3, \rho 2, 3, \rho_{2,3}, ..., \rho_2)$. The only nonzero contribution comes from when we do Δ in the middle, to get x, or when we do μ_2 in the "middle" ρ_2, ρ_3 to get another ρ_{23}, so we get $x + x = 0$ and the relation is satisfied. The other strings that need to be checked are $(\iota_1, \iota_1), (\rho_3, \rho_{2,3}, \rho_{2,3}, ..., \rho_{2,3}, \iota_0, \rho_{2,3}, ..., \rho_{2,3}, \rho_2), (\iota_1, \rho_3, \rho_{2,3}, ..., \rho_{2,3}, \rho_2), (\rho_3, \rho_{2,3}, ..., \rho_{2,3}, \rho_2, \iota_1)$.

This example comes from a bordered Heegaard diagram. The conditions $m_1(x, \iota_0) = 0, m_1(x, \iota_1) = x$ can only give a generator of CF if x gets paired with a curve on "the right" of the diagram occupyng the curve α_1. The higher map $m_2(x, \rho_2, \rho_3)$ corresponds to a domain bounded by the part of α_1 and the blue curve. $m_3(x, \rho_3, \rho_{2,3}, \rho_2) = x$ corresponds to the same domain taken with multiplicity two.

Figure 21: The only nontrivial map in the torus algebra

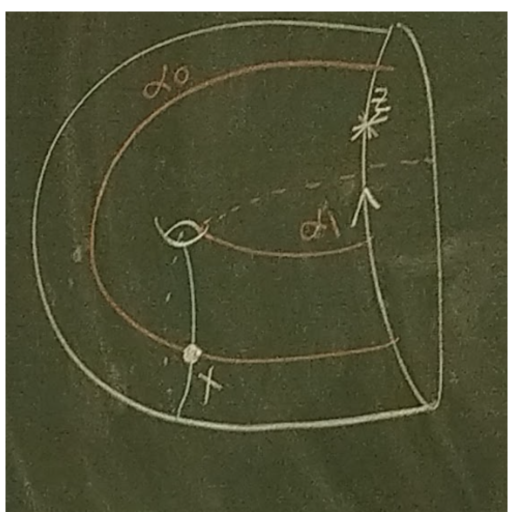


Figure 22: